
Causal Fermion Systems and the Einstein Equations

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OSMU Lecture Series

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What is a causal fermion system?

- approach to fundamental physics
- physical equations are formulated in **generalized spacetimes**
- Different **limiting cases**:
 - *Continuum limit*: Quantized fermionic fields interacting via classical bosonic fields
 - Last year: *QFT limit*: fermionic and bosonic quantum fields
 - This year: Geometric derivation of the *Einstein equations*
- For overview, more details (papers, books, videos, online course, physical applications,...)

www.causal-fermion-system.com

A few general words on **causal fermion systems**

- The approach is **fully quantum**
 - basic object is a family of wave functions (*physical wave functions*, complex-valued or spinorial)
 - all other spacetime structures are encoded in this family of wave functions
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How to get into the setting of causal fermion systems?

- Let $\mathcal{M}$ be four-dimensional Minkowski space.
Spacetime points $x, y \in \mathcal{M}$
- Let $\psi_1(x), \dots, \psi_f(x)$ be spinorial wave functions in spacetime (four component complex, can but need not satisfy the Dirac equation)
- Assume scalar product $\langle \cdot | \cdot \rangle_{\mathcal{H}}$. Orthonormalize: $\langle \psi_a | \psi_b \rangle_{\mathcal{H}} = \delta_{ab}$

- Introduce the **local correlation operator**: Given $x \in \mathcal{M}$,

$$F(x)_b^a := \overline{\psi_a(x)} \psi_b(x)$$

(where $\overline{\psi} := \psi^\dagger \gamma^0$ is the adjoint spinor)

- diagonal entries: density of a^{th} wave function
- off-diagonal entries: correlation of ψ_a and ψ_b at spacetime point x .

How to get into the setting of causal fermion systems?

- Local correlation operator can also be defined in a basis-independent way:

$$\langle \psi | F(x) \phi \rangle_{\mathcal{H}} := \prec \psi(x) | \phi(x) \succ \quad \forall \psi, \phi \in \mathcal{H}.$$

(with $\prec \psi(x) | \phi(x) \succ := \overline{\psi(x)} \phi(x)$)

- $F(x) : \mathcal{H} \rightarrow \mathcal{H}$ linear, symmetric (= selfadjoint), has *finite rank*, has *at most two positive* and *at most two negative eigenvalues*.

How to get into the setting of causal fermion systems?

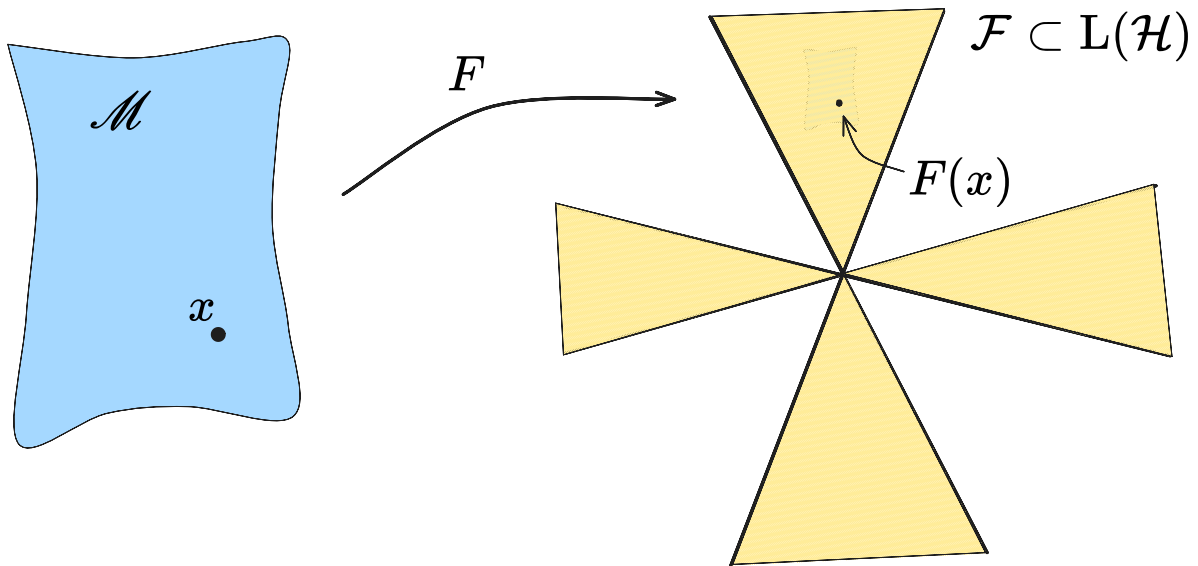
- Varying the base point x , we obtain the **local correlation map**

$$F : \mathcal{M} \rightarrow \mathcal{F} \subset \mathcal{L}(\mathcal{H}).$$

where $\mathcal{F}$ are all the symmetric operators of rank at most four which have at most two positive and at most two negative eigenvalues.

- Assume a volume measure $\mu_{\mathcal{M}}$ on $\mathcal{M}$. Introduce the push-forward measure,

$$\rho := F_*(\mu_{\mathcal{M}}) \quad \text{i.e. } \rho(\Omega) := \mu_{\mathcal{M}}(F^{-1}(\Omega))$$



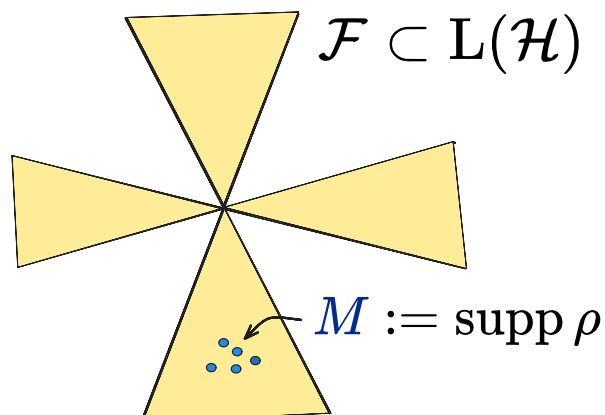
- general idea: work exclusively with objects on the right

The basic definition

Definition (causal fermion system)

- Hilbert space $(\mathcal{H}, \langle \cdot | \cdot \rangle_{\mathcal{H}})$
 - $n \in \mathbb{N}$ (*spin dimension*)
 - $\mathcal{F} \subset \mathcal{L}(\mathcal{H})$ symmetric linear operators of finite rank, with at most n positive and at most n negative eigenvalues
 - ρ a measure on $\mathcal{F}$
- Then $(\mathcal{H}, \mathcal{F}, \rho)$ is a **causal fermion system**

- $M := \text{supp } \rho$ is **spacetime**.
 - Is much more general than spacetime manifolds.
- In particular, spacetime could be discrete:



The causal action principle

- Let $x, y \in \mathcal{F}$. Then x and y are linear operators. $x \cdot y \in \mathcal{L}(H)$
 - $\text{rank} \leq 2n$
 - in general not symmetric: $(x \cdot y)^* = y \cdot x \neq x \cdot y$
thus non-trivial **complex** eigenvalues $\lambda_1^{xy}, \dots, \lambda_{2n}^{xy}$

The causal action principle

- Nontrivial eigenvalues of xy : $\lambda_1^{xy}, \dots, \lambda_{2n}^{xy} \in \mathbb{C}$

 **Definition** (causal Lagrangian and causal action)

$$\text{Lagrangian} \quad \mathcal{L}(x, y) = \frac{1}{4n} \sum_{i,j=1}^{2n} \left(|\lambda_i^{xy}| - |\lambda_j^{xy}| \right)^2 \geq 0$$

$$\text{causal action} \quad \mathcal{S} = \iint_{\mathcal{F} \times \mathcal{F}} \mathcal{L}(x, y) d\rho(x) d\rho(y) \in [0, \infty]$$

The causal action principle

 **Definition** (causal action principle)

Minimize $\mathcal{S}$ under variations of ρ , with constraints:

$$\begin{aligned} \text{volume constraint:} & \quad \rho(\mathcal{F}) = \text{const} \\ \text{trace constraint:} & \quad \int_{\mathcal{F}} \text{tr}(x) d\rho(x) = \text{const} \\ \text{boundedness constraint:} & \quad \iint_{\mathcal{F} \times \mathcal{F}} \sum_{i=1}^{2n} |\lambda_i^{xy}|^2 d\rho(x) d\rho(y) \leq C \end{aligned}$$

- Gives mathematically well-defined variational principle:
 - F.F., "Causal variational principles on measure spaces," arXiv:0811.2666 [math-ph], J. Reine Angew. Math. 646 (2010) 141--194

The Euler-Lagrange (EL) equations

Euler-Lagrange equations

Let ρ be a minimizer. Then the function ℓ defined by

$$\ell(x) := \int_M \mathcal{L}(x, y) d\rho(y) - \mathfrak{s} : \mathcal{F} \rightarrow \mathbb{R}$$

satisfies for a suitable parameter $\mathfrak{s} > 0$ the equation

$$\ell|_M \equiv \inf_{\mathcal{F}} \ell = 0 .$$

The causal structure

- $M := \text{supp } \rho$ is *spacetime*.

Definition

$x, y \in M$ are **causally related** (timelike or lightlike) if $\mathcal{L}(x, y) > 0$.

Inherent structures

- Spacetime points are linear operators.
- Every spacetime point $x \in M \subset \mathcal{F}$ comes with a distinguished subspace of $\mathcal{H}$

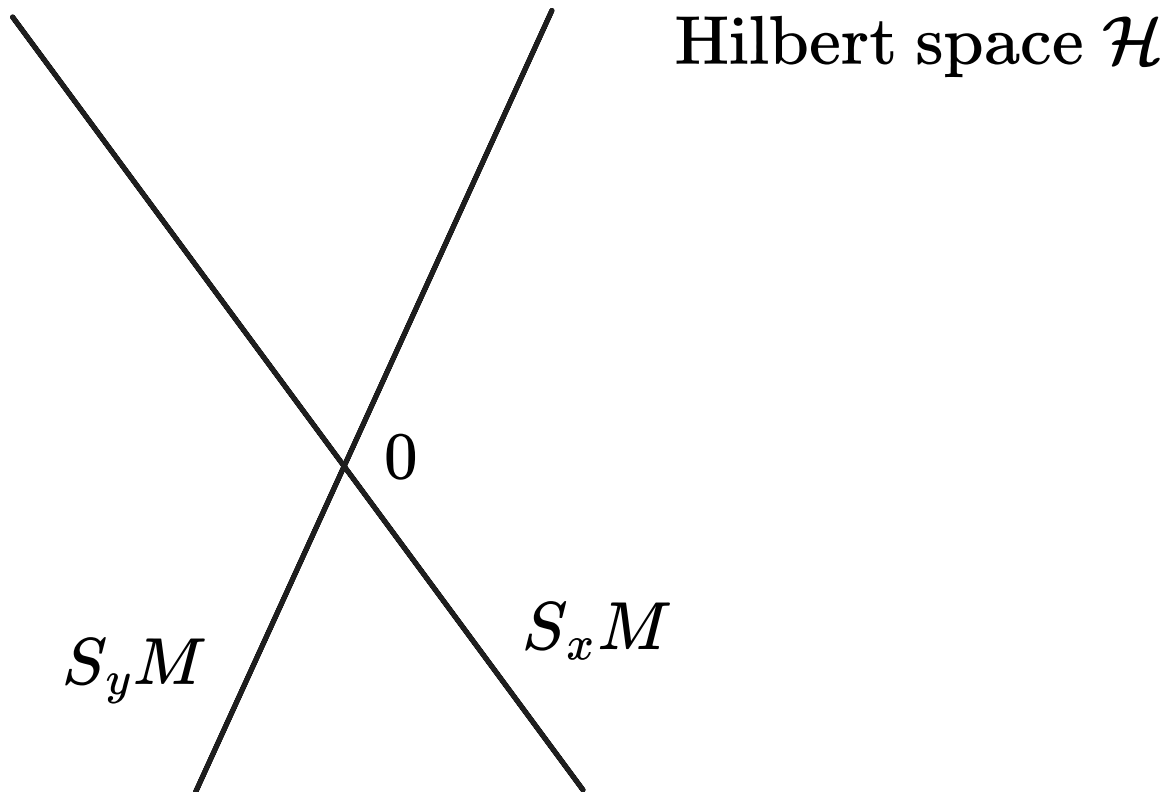
$$S_x M := x(\mathcal{H}) \subset \mathcal{H}$$

- Similar to **vector bundle** (topological vector bundle)

Inherent structures

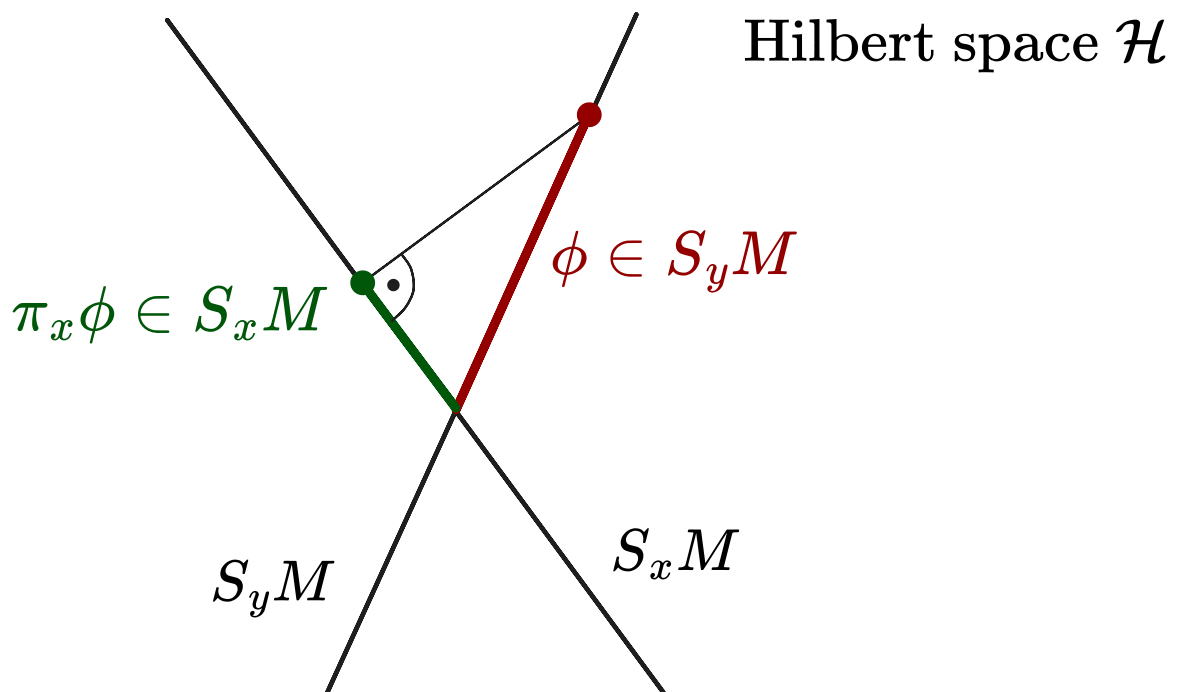
$$S_x M := x(\mathcal{H}) \subset \mathcal{H}$$

- But specific feature: All fibers are subspaces of $\mathcal{H}$



Inherent structures

- Projecting these subspaces onto each other gives relations in spacetime



Geometric structures

Geometric structures of causal fermion system

- *spacetime* $M := \text{supp } \rho$ (topological space, subset of $\mathcal{F} \subset L(\mathcal{H})$)

 **Definition** (with Andreas Grotz, 2012)

$S_x := x(\mathcal{H})$	spin space
$\prec \cdot \cdot \succ_x := \langle \cdot x \cdot \rangle_{\mathcal{H}}$	spin inner product
$\nabla_{x,y} : S_y \rightarrow S_x$	spin connection
$T_x M$	tangent space
$g_x : T_x M \times T_x M \rightarrow \mathbb{R}$	Lorentzian metric

Corresponding holonomy gives **curvature**

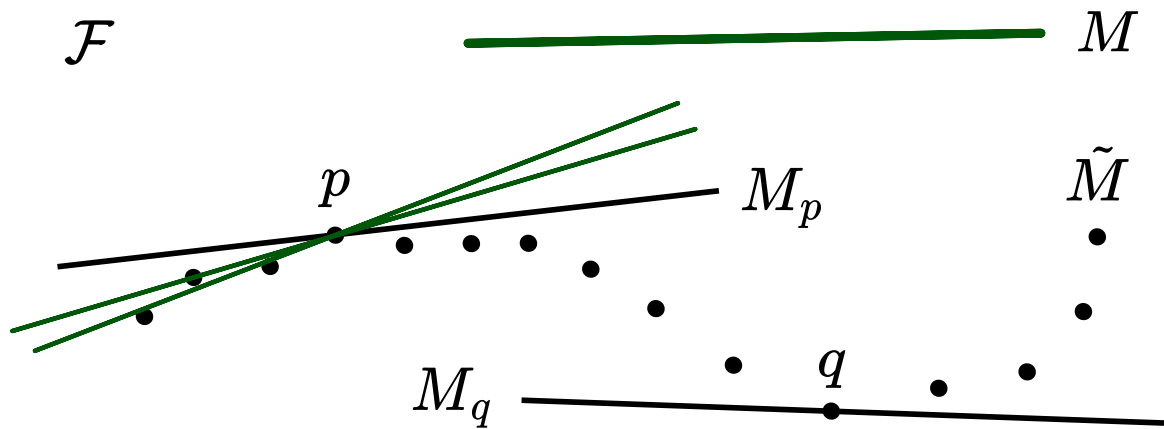
- *Disadvantage*: these structures do not seem to harmonize with the causal action principle

Connection to the Einstein equations

- Derivation using the **continuum limit analysis** (in blue book 2016)
- Derivation using connection between **matter flux and area change**, following *Ted Jacobson*'s thermodynamic arguments (2019)
- *Geometric derivation* (2026):
 - F.F., C. Kroup, "A geometric derivation of the Einstein equations from the causal action principle", arXiv:2607.13871 [math-ph]

Osculating vacua

- Let $\tilde{\rho}$ be a minimizing measure with $\tilde{M} := \text{supp } \tilde{\rho}$ the *interacting spacetime*; can be arbitrary.
- Let ρ be a minimizing measure describing the *vacuum* with $M := \text{supp } \rho \simeq \mathbb{R}^4$.
- Identify the corresponding Hilbert spaces $\tilde{\mathcal{H}}$ and $\mathcal{H}$ via a unitary transformation.
- Idea: "Approximate $\tilde{M}$ locally by M as good as possible":

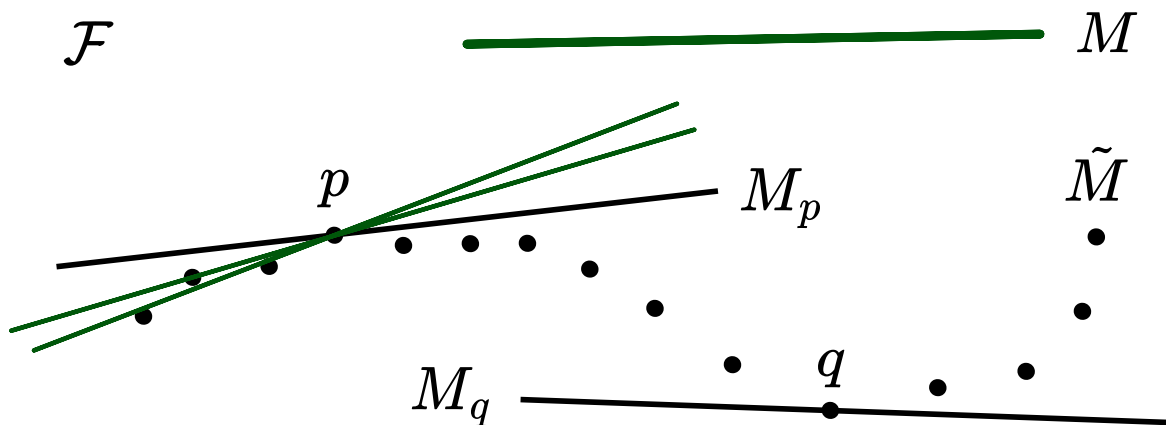


Osculating vacua

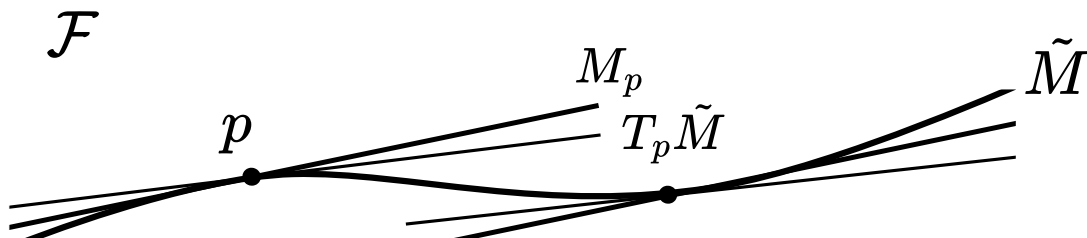
- Idea is made precise by **variational principle for osculation**:

$$\text{minimize } \mathcal{S}_p(\mathcal{U}) := \sum_{i=0}^3 D_{e_i, e_i}^2 \int_{\tilde{M}} \mathcal{L}(x, y) d\tilde{\rho}(y),$$

where derivatives are tangential to M .



Osculating vacua



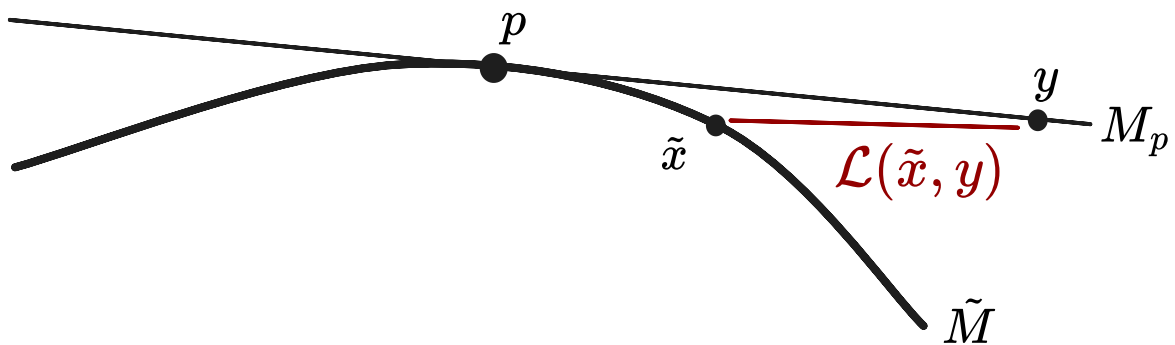
- One needs to distinguish the **geometric tangent space** $T_p M$ from the **osculating vacuum** M_p .
- But the difference is very small, gives **Planck scale corrections**.

- Therefore in the talk: Identify $T_p\tilde{M}$ with M_p :
Approximation of **optimal osculations**

The $\mathcal{L}$ -induced charts

$$\phi_p : \tilde{M} \rightarrow M_p, \quad \phi_p(\tilde{x}) := \frac{1}{5} \int_{M_p} \mathcal{L}(\tilde{x}, y) y \, d\rho_p(y)$$

- Defines a **distinguished local chart** in a neighborhood of $p \in \tilde{M}$.
- Plays a similar role as Gaussian charts in Riemannian geometry.



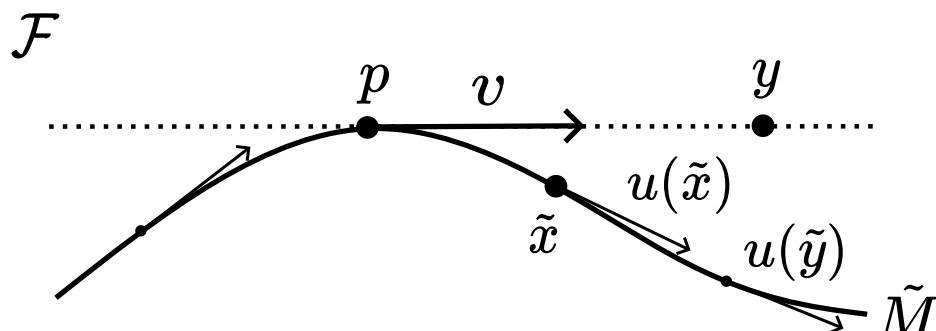
The connection $\nabla^{\mathcal{L}}$

- The resulting connection can also be described more directly:
- Given a vector field $u \in \Gamma(\tilde{M}, T\tilde{M})$ and a vector $v \in T_p\tilde{M}$ we want to define

$$\nabla_v^{\mathcal{L}} u|_p \in T_p\tilde{M} = M_p$$

- To this end set

$$\nabla_v^{\mathcal{L}} u|_p := v(u\phi_p)|_{\tilde{x}=p} \quad (\phi_p : \tilde{M} \rightarrow M_p)$$



The curvature of $\nabla^{\mathcal{L}}$

- The Christoffel symbols take the form

$$\Gamma_{ik}^j(p) = \frac{\partial^2}{\partial x^i \partial x^k} \phi_p^j(x) \Big|_{x=p}$$

- Curvature can be computed with the usual formula

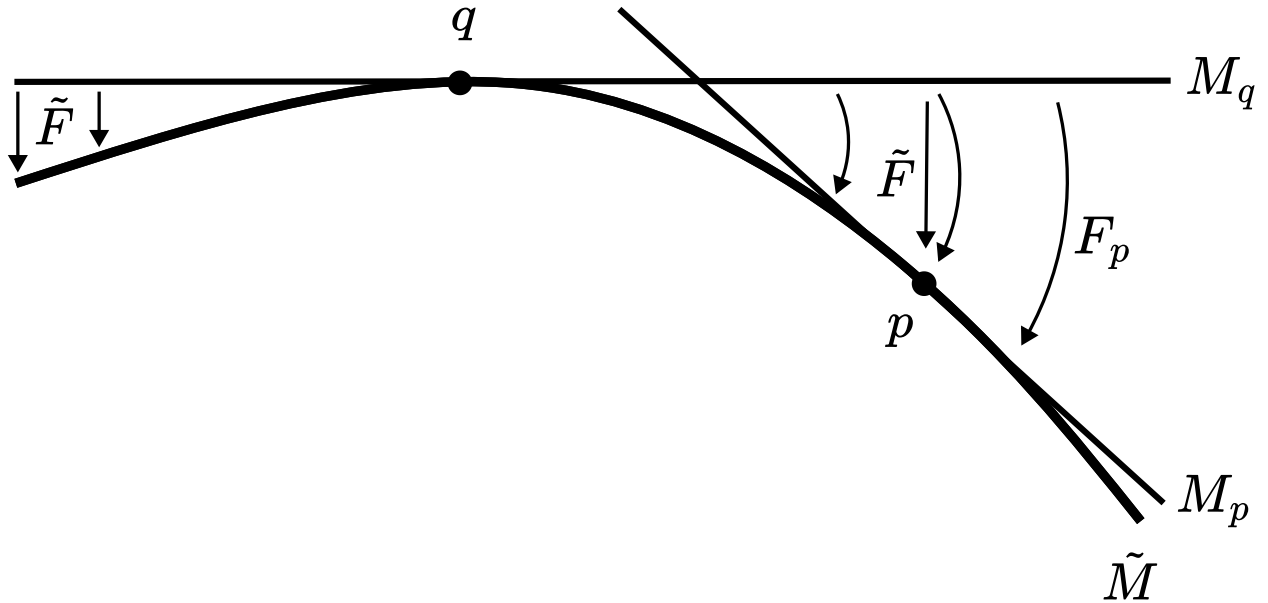
$$R_{ijl}^k = \partial_i \Gamma_{jl}^k - \partial_j \Gamma_{il}^k + \Gamma_{ia}^k \Gamma_{jl}^a - \Gamma_{ja}^k \Gamma_{il}^a$$

- Note: One needs to also *differentiate with respect to the base point* of the osculation

The curvature of $\nabla^{\mathcal{L}}$

- Convenient formalism: Work with so-called **osculation maps**

$$\tilde{F} : M \rightarrow \tilde{M}, \quad F_p : M \rightarrow M_p$$



Properties of the curvature of $\nabla^{\mathcal{L}}$

- Is *torsion-free* (not metric, also we have no metric for the moment)
- The Ricci tensor has a **divergence structure**

$$R_{il}(q) := R_{ikl}^k(q) = \left(\frac{\partial}{\partial p^i} \frac{\nabla^2}{\partial x^l \partial x^{\bar{k}}} \phi_p^{\bar{k}}(x) - \frac{\nabla^2}{\partial x^i \partial x^l} \frac{\partial}{\partial p^{\bar{k}}} \phi_p^{\bar{k}}(x) \right) \Big|_{x=p=q}$$

How to relate the Ricci tensor to the causal action principle?

Euler-Lagrange equations

Let ρ be a minimizer. Then the function ℓ defined by

$$\ell(x) := \int_M \mathcal{L}(x, y) d\rho(y) - \mathfrak{s} : \mathcal{F} \rightarrow \mathbb{R}$$

satisfies for a suitable parameter $\mathfrak{s} > 0$ the equation

$$\ell|_M \equiv \inf_{\mathcal{F}} \ell = 0 .$$

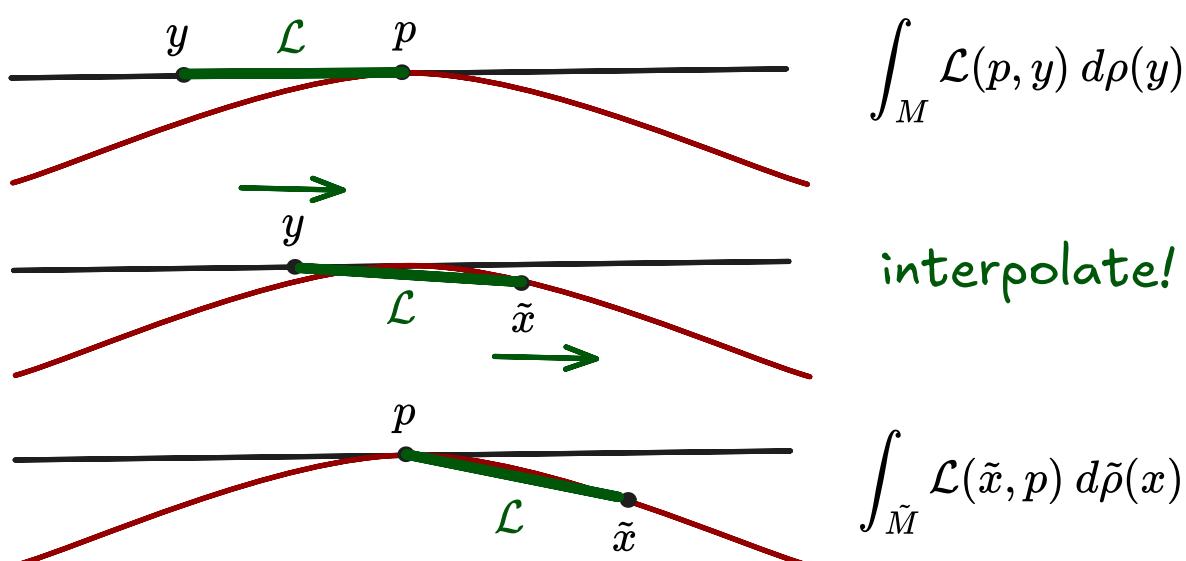
How to relate the Ricci tensor to the causal action principle?

- We know that

$$\int_M \mathcal{L}(p, y) d\rho(y) = \mathfrak{s} = \int_{\tilde{M}} \mathcal{L}(\tilde{x}, p) d\tilde{\rho}(\tilde{x})$$

- We want to compute the x -divergence of ϕ_p ,

$$\frac{\partial}{\partial x^k} \int_M \mathcal{L}(\tilde{x}, y) (y - x)^k d\rho(y)$$



How to relate the Ricci tensor to the causal action principle?

- In formulas: Interpolation parameter $s \in [0, 1]$,

$$x_s = \zeta - (1 - s)\xi, \quad y_s = \zeta + s\xi$$

$$\varphi(s, \zeta) := \int_M \tilde{f}(x_s) \mathcal{L}(\tilde{F}(x_s), y_s) d\rho(\xi)$$

(where $\tilde{f}$ is the Radon-Nikodym weight of $\tilde{\rho}$). Then

$$0 = \phi(1, x) - \phi(0, x) = - \sum_{r=1}^{\infty} \frac{(-1)^r}{r!} \frac{d^r}{ds^r} \varphi(1, x)$$

$$= \frac{\partial}{\partial x^k} \int_M \mathcal{L}(\tilde{x}, y) (y - x)^k d\rho(y) + \dots$$

How to relate the Ricci tensor to the causal action principle?

- For subsequent x -derivatives. Work with *alignment vector field*.

$$\begin{aligned} \phi_p^k(x) &= \frac{1}{s} \int_M \mathcal{L}(\tilde{F}(x), F_p(y)) (y - p)^k f_p d\rho(y) \\ &= \frac{1}{s} \int_M \mathcal{L}(\tilde{F}(x), F_p(y)) (\textcolor{red}{x} - p)^k f_p d\rho(y) \\ &\quad + \frac{1}{s} \int_M \mathcal{L}(\tilde{F}(x), F_p(y)) (y - \textcolor{red}{x})^k f_p d\rho(y) \end{aligned}$$

- Factor $x - p$ can be pulled out of the integral; apply EL equations
- Method for the *p -divergences* is of similar style

We can *expand the Ricci tensor* in powers of δ , where

- δ is the *range* of the Lagrangian; can be thought of as the *Planck length*.

A Riemannian metric and a time direction functional

- There is a *Riemannian metric*

$$g_p^*(\phi, \phi') := \frac{1}{s \delta^2} \int_{M_p} \mathcal{L}(p, y) \phi(y) \phi'(y) d\rho_p(y)$$

The Lorentzian metric

- There is a **time direction functional**

$$\mathcal{C} : M \times M \rightarrow \mathbb{R}, \quad \mathcal{C}(x, y) := \text{itr} \left(y x \pi_y \pi_x - x y \pi_x \pi_y \right)$$

- This gives rise to the **regularizing vector field**

$$u(p) := \int_{M_p} \mathcal{L}(p, y) \mathcal{C}(p, y) y d\rho_p(y)$$

- The **Lorentzian metric** η is introduced as the flip metric

$$\eta^{jk} = 4 u^j u^k - g^{jk}$$

This metric is *compatible with the causal structure* of the causal fermion system.

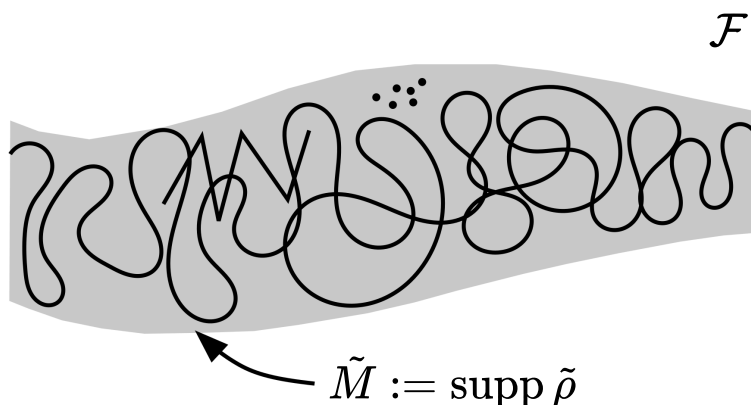
The Lorentzian Einstein equations

- The connection $\nabla^{\mathcal{L}}$ is *metric* w.r.t. η *up to corrections of order δ^2* (with δ the Planck length).
- This gives the **usual Einstein equations**

$$R_{il}^{\eta} - \frac{1}{2} R^{\eta} \eta_{il} = T_{il} \quad \text{with} \quad T_{il} = \mathcal{O}(\delta^2).$$

- T_{il} has an *explicit expansion* in powers of δ .

Outlook: The Einstein equations in **quantum spacetimes**



- causal action principle, Euler-Lagrange equations, ... still well-defined
- notion of osculating vacua might be problematic (clear because M can no longer be approximated by a smooth manifold).
- This is what I would call quantum gravity.

- Connection to other approaches to quantum gravity at present unclear.

Thank you for your attention



www.causal-fermion-system.com

Abstract of the talk

Causal Fermion Systems and the Einstein Equations

Online Lecture Series "Octonions, Standard Model, and Unification"

16 October 2026

The theory of causal fermion systems is an approach to fundamental physics. It gives quantum mechanics, general relativity and quantum field theory as limiting cases and is therefore a candidate for a unified physical theory. The dynamics of a causal fermion system is described by a nonlinear variational principle, the causal action principle.

The aim of the talk is to give a simple introduction, with an emphasis on the underlying concepts. At the end of the talk, I will explain how the Einstein equations of General Relativity can be derived from the causal action principle.