

On a Geometric Derivation of the Einstein Equations from the Causal Action Principle

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Curvature, Optimal Transport, and Gravity

Universität Wien, 23 September 2026

A few general words on **causal fermion systems**

- approach to *fundamental physics*
 - Limiting cases:
 - classical field theory, general relativity
 - relativistic quantum field theory
 - From mathematical perspective:
 - general framework for **non-smooth spacetime geometry**
 - analytic backbone: causal action principle
more generally: causal variational principles
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A few general words on **causal fermion systems**

- The approach is **fully quantum**
 - basic object is a family of wave functions (*physical wave functions*, complex-valued or spinorial)
 - geometry is encoded in this family of wave functions
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How to get into the setting of causal fermion systems?

- Let $S\mathcal{M}$ vector bundle over a smooth manifold $\mathcal{M}$ (spinor bundle, tangent bundle, complex line bundle, ...)
- Assume that each fiber $S_x\mathcal{M}$ is endowed with an inner product of signature (n, n) ,

$$\prec | \cdot \succ_x : S_x\mathcal{M} \times S_x\mathcal{M} \rightarrow \mathbb{C}.$$

- Consider a family (ψ_n) of sections (for example wave functions, vector fields, ...)
- Assume that sections form a Hilbert space $\mathcal{H}$, endowed with scalar product $\langle \cdot | \cdot \rangle_{\mathcal{H}}$

$$\langle \cdot | \cdot \rangle_{\mathcal{H}} : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{C}.$$

How to get into the setting of causal fermion systems?

$$\begin{aligned} \prec \cdot | \cdot \succ_x \circ |_{(x,x)} &: \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{C} \\ \langle \cdot | \cdot \rangle_{\mathcal{H}} &: \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{C} \end{aligned}$$

- For any $x \in \mathcal{M}$ introduce the local correlation operator $F(x)$ by

$$\langle \psi | F(x) \phi \rangle_{\mathcal{H}} := \prec \psi(x) | \phi(x) \succ \quad \forall \psi, \phi \in \mathcal{H}.$$

- Varying the base point x , we obtain the **local correlation map**

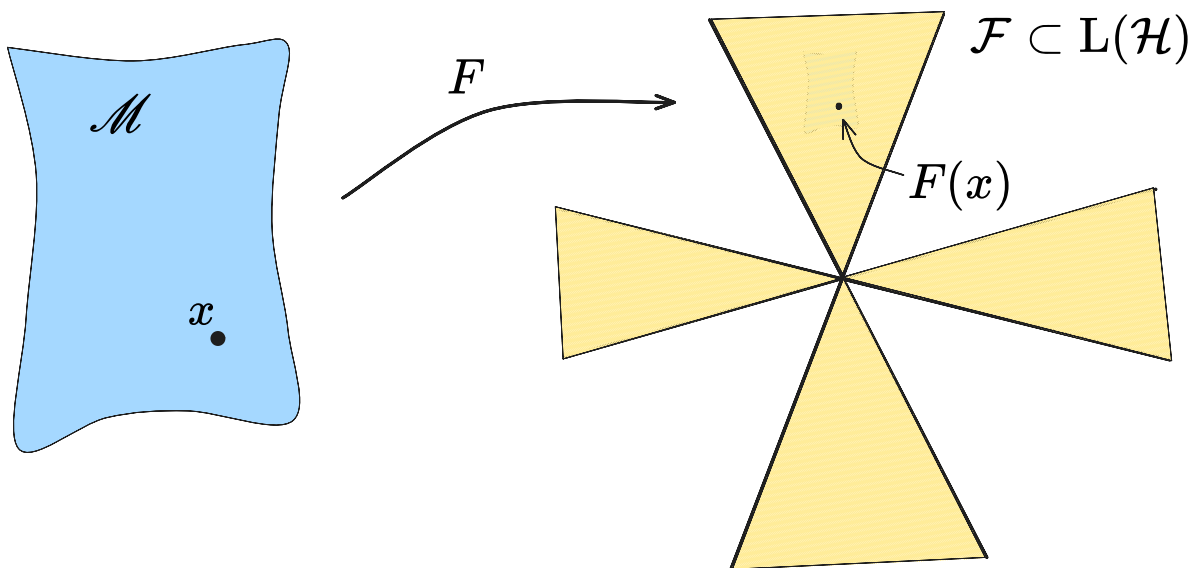
$$F : \mathcal{M} \rightarrow \mathcal{F} \subset \mathbf{L}(\mathcal{H}).$$

where $\mathcal{F}$ are all the symmetric operators of rank at most $2n$ which have at most n positive and at most n negative eigenvalues.

- Assume a volume measure $\mu_{\mathcal{M}}$ on $\mathcal{M}$. Introduce the push-forward measure,

$$\rho := F_*(\mu_{\mathcal{M}}) \quad \text{i.e. } \rho(\Omega) := \mu_{\mathcal{M}}(F^{-1}(\Omega))$$

How to get into the setting of causal fermion systems?



- general idea: work exclusively with objects on the right

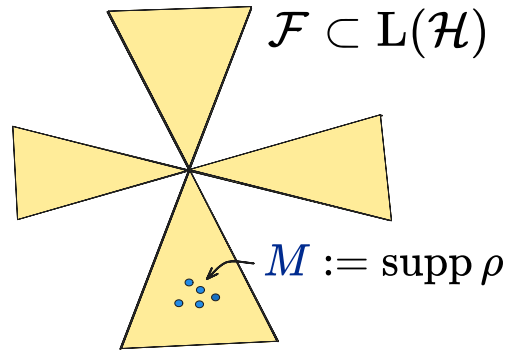
The abstract definition

Definition (causal fermion system)

- Hilbert space $(\mathcal{H}, \langle \cdot | \cdot \rangle_{\mathcal{H}})$
 - $n \in \mathbb{N}$ (*spin dimension*)
 - $\mathcal{F} \subset \mathcal{L}(\mathcal{H})$ symmetric linear operators of rank at most $2n$, with at most n positive and at most n negative eigenvalues
 - ρ a measure on $\mathcal{F}$
- Then $(\mathcal{H}, \mathcal{F}, \rho)$ is a **causal fermion system**

- $M := \text{supp } \rho$ is **spacetime**.
- Is much more general than spacetime manifolds.

In particular, spacetime could be discrete:



The causal action principle

Definition (causal Lagrangian and causal action)

$$\mathcal{L} : \mathcal{F} \times \mathcal{F} \rightarrow \mathbb{R}_0^+$$

$$\mathcal{S}(\rho) := \int_{\mathcal{F}} d\rho(x) \int_{\mathcal{F}} d\rho(y) \mathcal{L}(x, y)$$

- $\mathcal{L}(x, y)$ *formed of eigenvalues* of operator product xy .
- Is continuous and symmetric: $\mathcal{L}(x, y) = \mathcal{L}(y, x)$

Definition (causal action principle)

Minimize $\mathcal{S}$ under variations of ρ under constraints:

- *volume constraint*: $\rho(\mathcal{F}) = \text{const}$
- *trace constraint* and *boundedness constraint*: here built into the Lagrangian using Lagrange multipliers.

The causal structure

- $M := \text{supp } \rho$ is *spacetime*.

Definition

$x, y \in M$ are *timelike separated* if $\mathcal{L}(x, y) > 0$.

The Euler-Lagrange (EL) equations

Euler-Lagrange equations

Let ρ be a minimizer. Then the function ℓ defined by

$$\ell(x) := \int_M \mathcal{L}(x, y) d\rho(y) - \mathfrak{s} : \mathcal{F} \rightarrow \mathbb{R}$$

satisfies for a suitable parameter $\mathfrak{s} > 0$ the equation

$$\ell|_M \equiv \inf_{\mathcal{F}} \ell = 0 .$$

Geometric structures

Geometric structures of causal fermion system

- *spacetime* $M := \text{supp } \rho$ (topological space, subset of $\mathcal{F} \subset L(\mathcal{H})$)

Definition (with Andreas Grotz, 2012)

$S_x := x(\mathcal{H})$	spin space
$\prec \cdot \cdot \succ_x := \langle \cdot x \cdot \rangle_{\mathcal{H}}$	spin inner product
$\nabla_{x,y} : S_y \rightarrow S_x$	spin connection
$T_x M$	tangent space
$g_x : T_x M \times T_x M \rightarrow \mathbb{R}$	Lorentzian metric

Corresponding holonomoy gives **curvature**

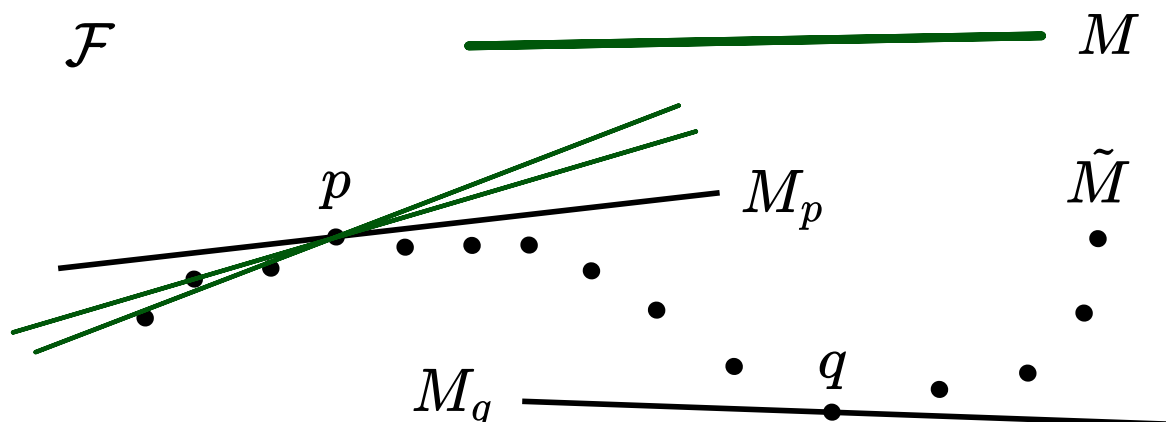
- **Disadvantage:** these structures do not seem to harmonize with the causal action principle

Connection to the Einstein equations

- Derivation using the **continuum limit analysis** (in blue book 2016)
- Derivation using connection between **matter flux and area change**, following **Ted Jacobson's** thermodynamic arguments (2019)
- **Geometric derivation** (2026):
 - F.F., C. Krpoun, "A geometric derivation of the Einstein equations from the causal action principle", arXiv:2607.13871 [math-ph]

Osculating vacua

- Let $\tilde{\rho}$ be a minimizing measure with $\tilde{M} := \text{supp } \tilde{\rho}$ the **interacting spacetime**; can be arbitrary.
- Let ρ be a minimizing measure describing the **vacuum** with $M := \text{supp } \rho \simeq \mathbb{R}^4$.
- Identify the corresponding Hilbert spaces $\tilde{\mathcal{H}}$ and $\mathcal{H}$ via a unitary transformation.
- Idea: "Approximate $\tilde{M}$ locally by M as good as possible":

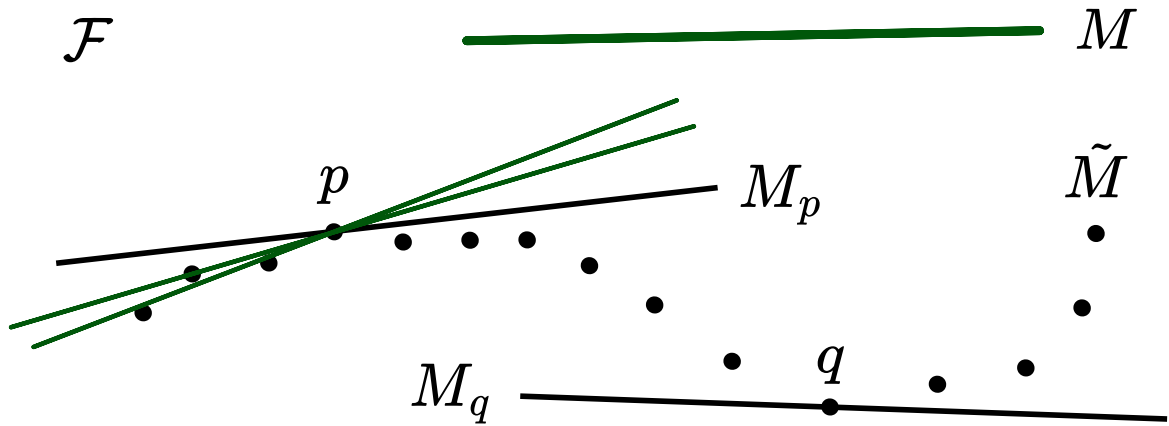


Osculating vacua

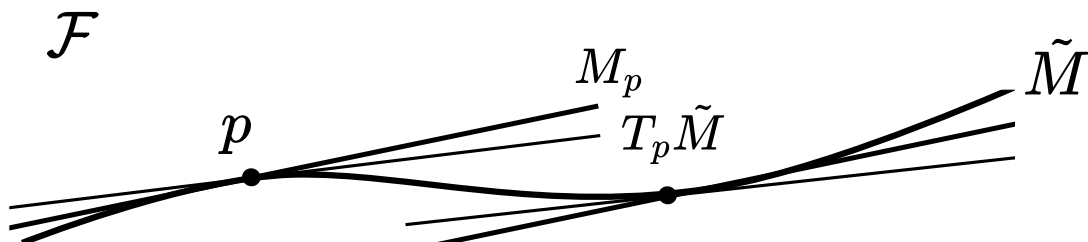
- Idea is made precise by **variational principle for osculation**:

$$\text{minimize } \mathcal{S}_p(\mathcal{U}) := \sum_{i=0}^3 D_{e_i, e_i}^2 \int_{\tilde{M}} \mathcal{L}(x, y) d\tilde{\rho}(y),$$

where derivatives are tangential to M .



Osculating vacua

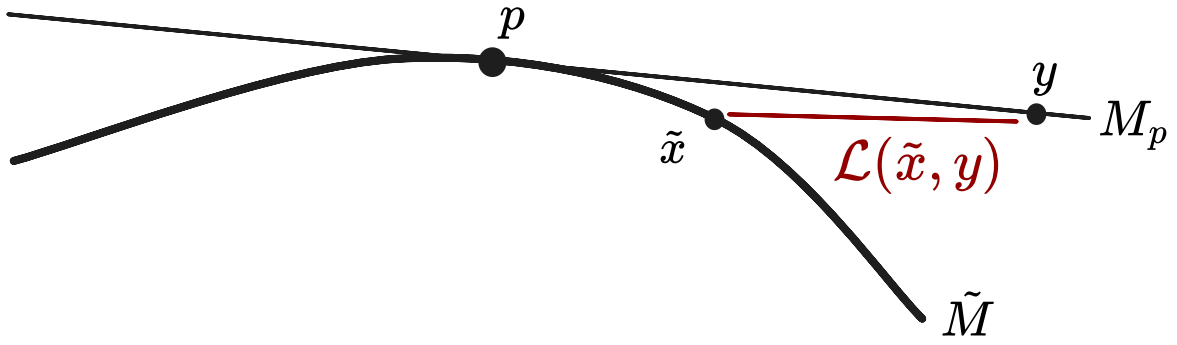


- One needs to distinguish the **geometric tangent space** $T_p \tilde{M}$ from the **osculating vacuum** M_p .
- But the difference is very small, gives **Planck scale corrections**.
- Therefore in the talk: Identify $T_p \tilde{M}$ with M_p :
Approximation of **optimal osculations**

The $\mathcal{L}$ -induced charts

$$\phi_p : \tilde{M} \rightarrow M_p, \quad \phi_p(\tilde{x}) := \frac{1}{s} \int_{M_p} \mathcal{L}(\tilde{x}, y) y d\rho_p(y)$$

- Defines a **distinguished local chart** in a neighborhood of $p \in \tilde{M}$.
- Plays a similar role as Gaussian charts in Riemannian geometry.



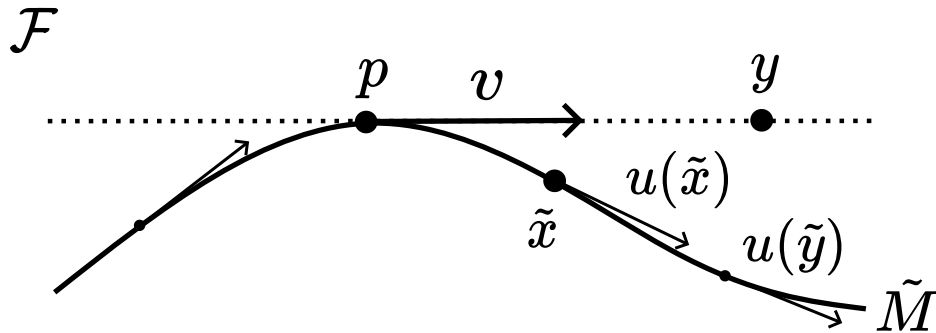
The connection $\nabla^{\mathcal{L}}$

- The resulting connection can also be described more directly:
- Given a vector field $u \in \Gamma(\tilde{M}, T\tilde{M})$ and a vector $v \in T_p\tilde{M}$ we want to define

$$\nabla_v^{\mathcal{L}} u|_p \in T_p\tilde{M} = M_p$$

- To this end set

$$\nabla_v^{\mathcal{L}} u|_p := v(u\phi_p)|_{\tilde{x}=p} \quad (\phi_p : \tilde{M} \rightarrow M_p)$$



The curvature of $\nabla^{\mathcal{L}}$

- The Christoffel symbols take the form

$$\Gamma_{ik}^j(p) = \frac{\partial^2}{\partial x^i \partial x^k} \phi_p^j(x) \Big|_{x=p}$$

- Curvature can be computed with the usual formula

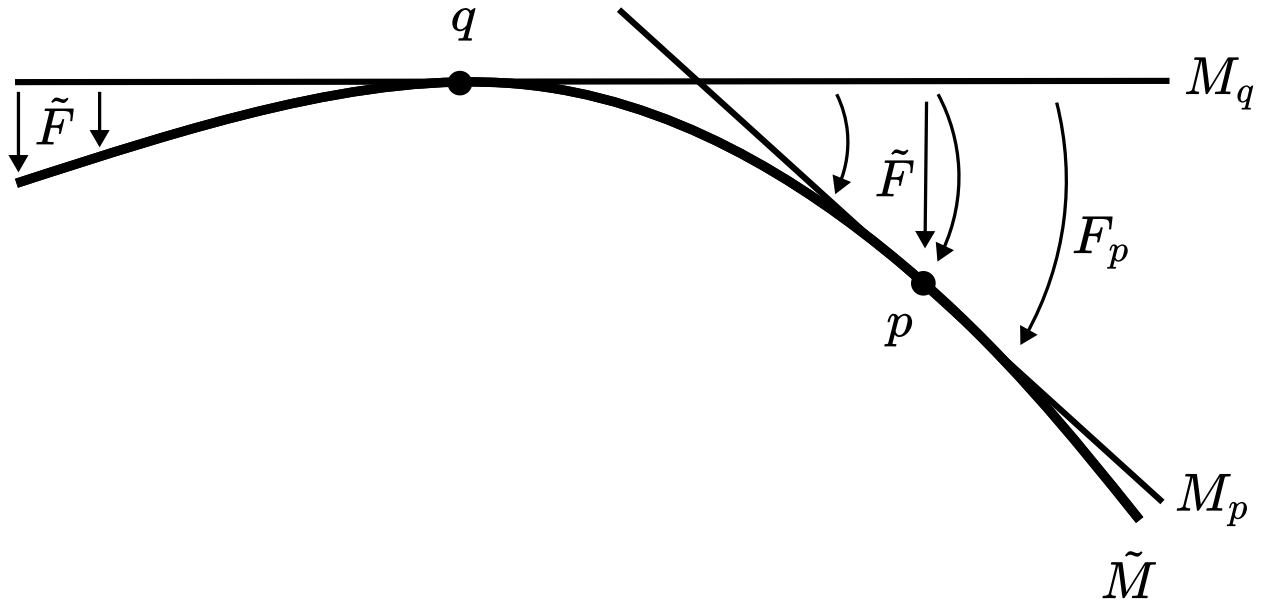
$$R_{ijl}^k = \partial_i \Gamma_{jl}^k - \partial_j \Gamma_{il}^k + \Gamma_{ia}^k \Gamma_{jl}^a - \Gamma_{ja}^k \Gamma_{il}^a$$

- Note: One needs to also *differentiate with respect to the base point* of the osculation

The curvature of $\nabla^{\mathcal{L}}$

- Convenient formalism: Work with so-called **osculation maps**

$$\tilde{F} : M \rightarrow \tilde{M}, \quad F_p : M \rightarrow M_p$$



Properties of the curvature of $\nabla^{\mathcal{L}}$

- Is **torsion-free** (not metric, also we have no metric for the moment)
- The Ricci tensor has a **divergence structure**

$$R_{il}(q) := R_{ikl}^k(q) = \left(\frac{\partial}{\partial p^i} \frac{\nabla^2}{\partial x^l \partial x^k} \phi_p^k(x) - \frac{\nabla^2}{\partial x^i \partial x^l} \frac{\partial}{\partial p^k} \phi_p^k(x) \right) \Big|_{x=p=q}$$

How to relate the Ricci tensor to the causal action principle?

Euler-Lagrange equations

Let ρ be a minimizer. Then the function ℓ defined by

$$\ell(x) := \int_M \mathcal{L}(x, y) d\rho(y) - \mathfrak{s} : \mathcal{F} \rightarrow \mathbb{R}$$

satisfies for a suitable parameter $\mathfrak{s} > 0$ the equation

$$\ell|_M \equiv \inf_{\mathcal{F}} \ell = 0.$$

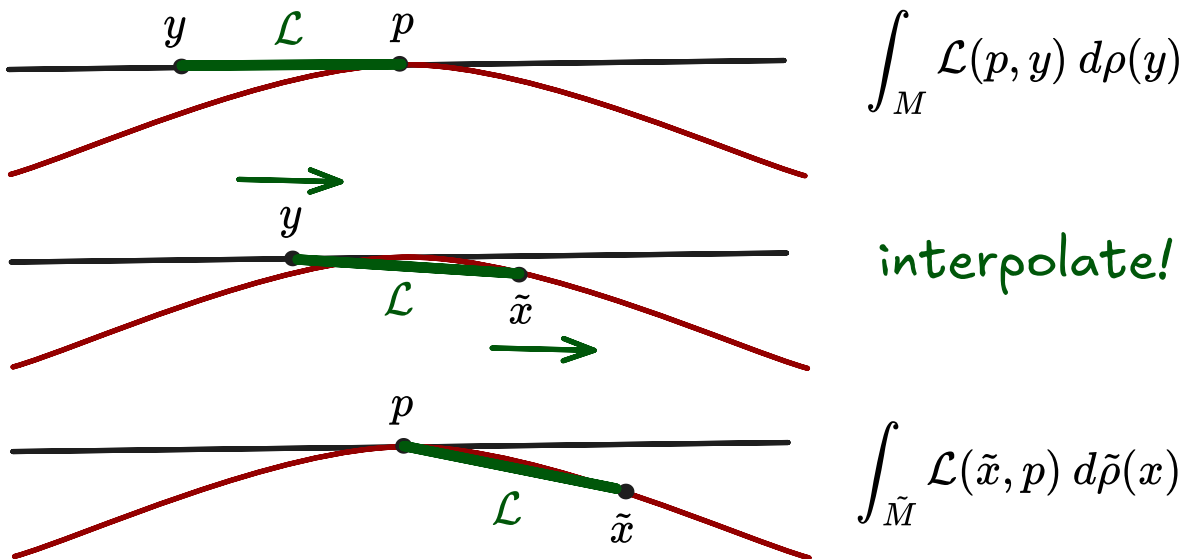
How to relate the Ricci tensor to the causal action principle?

- We know that

$$\int_M \mathcal{L}(p, y) d\rho(y) = \mathfrak{s} = \int_{\tilde{M}} \mathcal{L}(\tilde{x}, p) d\tilde{\rho}(\tilde{x})$$

- We want to compute the x -divergence of ϕ_p ,

$$\frac{\partial}{\partial x^k} \int_M \mathcal{L}(\tilde{x}, y) (y - x)^k d\rho(y)$$



How to relate the Ricci tensor to the causal action principle?

- In formulas: Interpolation parameter $s \in [0, 1]$,

$$x_s = \zeta - (1 - s)\xi, \quad y_s = \zeta + s\xi$$

$$\varphi(s, \zeta) := \int_M \tilde{f}(x_s) \mathcal{L}(\tilde{F}(x_s), y_s) d\rho(\xi)$$

(where $\tilde{f}$ is the Radon-Nikodym weight of $\tilde{\rho}$). Then

$$0 = \phi(1, x) - \phi(0, x) = - \sum_{r=1}^{\infty} \frac{(-1)^r}{r!} \frac{d^r}{ds^r} \varphi(1, x)$$

$$= \frac{\partial}{\partial x^k} \int_M \mathcal{L}(\tilde{x}, y) (y - x)^k d\rho(y) + \dots$$

How to relate the Ricci tensor to the causal action principle?

- For subsequent x -derivatives. Work with *alignment vector field*.

$$\begin{aligned}\phi_p^k(x) &= \frac{1}{\mathfrak{s}} \int_M \mathcal{L}(\tilde{F}(x), F_p(y)) (y - p)^k f_p d\rho(y) \\ &= \frac{1}{\mathfrak{s}} \int_M \mathcal{L}(\tilde{F}(x), F_p(y)) (\mathbf{x} - p)^k f_p d\rho(y) \\ &\quad + \frac{1}{\mathfrak{s}} \int_M \mathcal{L}(\tilde{F}(x), F_p(y)) (y - \mathbf{x})^k f_p d\rho(y)\end{aligned}$$

- Factor $x - p$ can be pulled out of the integral; apply EL equations
- Method for the *p-divergences* is of similar style

We can **expand the Ricci tensor** in powers of δ , where

- δ is the *range* of the Lagrangian; can be thought of as the *Planck length*.
-

A Riemannian metric and a time direction functional

- There is a **Riemannian metric**

$$g_p^*(\phi, \phi') := \frac{1}{\mathfrak{s} \delta^2} \int_{M_p} \mathcal{L}(p, y) \phi(y) \phi'(y) d\rho_p(y)$$

The Lorentzian metric

- There is a **time direction functional**

$$\mathcal{C} : M \times M \rightarrow \mathbb{R}, \quad \mathcal{C}(x, y) := i \text{tr}(y x \pi_y \pi_x - x y \pi_x \pi_y)$$

- This gives rise to the **regularizing vector field**

$$u(p) := \int_{M_p} \mathcal{L}(p, y) \mathcal{C}(p, y) y d\rho_p(y)$$

- The **Lorentzian metric** η is introduced as the flip metric

$$\eta^{jk} = 4 u^j u^k - g^{jk}$$

This metric is *compatible with the causal structure* of the causal fermion system.

The Lorentzian Einstein equations

- The connection $\nabla^{\mathcal{L}}$ is *metric* w.r.t. η *up to corrections of order δ^2* (with δ the Planck length).
- This gives the **usual Einstein equations**

$$R_{il}^{\eta} - \frac{1}{2} R^{\eta} \eta_{il} = T_{il} \quad \text{with} \quad T_{il} = \mathcal{O}(\delta^2).$$

- T_{il} has an *explicit expansion* in powers of δ .

Outlook: The Einstein equations in **non-smooth spacetimes**

- I showed in the talk:

$$\text{EL equations} \xrightarrow{\tilde{M} \text{ smooth}} \text{Einstein equations} + (\text{corrections})$$

- Next: Work out **corrections** to the Einstein equations.
- Some of the corrections seem to come from a *Lorentz-Finsler structure*.
- Left side needs no smoothness! Therefore, **intermediate steps** are possible

$$\text{EL equations} \xrightarrow{\tilde{M} \text{ is } C^{*,*}} (???) \xrightarrow{\tilde{M} \text{ smooth}} \text{Einstein equations}$$

- Remark: The *exterior calculus* (outer differentiation, Stokes' theorem) have been worked out recently *in the non-smooth setting* (dubbed the *$\mathcal{L}$ -calculus*, arXiv:2608.08811 [math-ph]).

Thank you for your attention

www.causal-fermion-system.com

Abstract of the talk

On a geometric derivation of the Einstein equations from the causal action principle

Wien, September 2026

I will report on a recent geometric derivation of the Einstein equations from the causal action principle for causal fermion systems. The starting point is a minimizing measure whose support is assumed to have the structure of a smooth four-dimensional manifold. Based on

the concept osculating vacua, I will outline how to get a Riemannian and Lorentzian metric. I will explain in detail how the resulting curvature can be related via the Euler-Lagrange equations of the causal action to an energy-momentum tensor. Extensions to non-smooth and "quantum" spacetimes will be discussed.
