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UNIVERSITÀ DI PAVIA
Department of Physics
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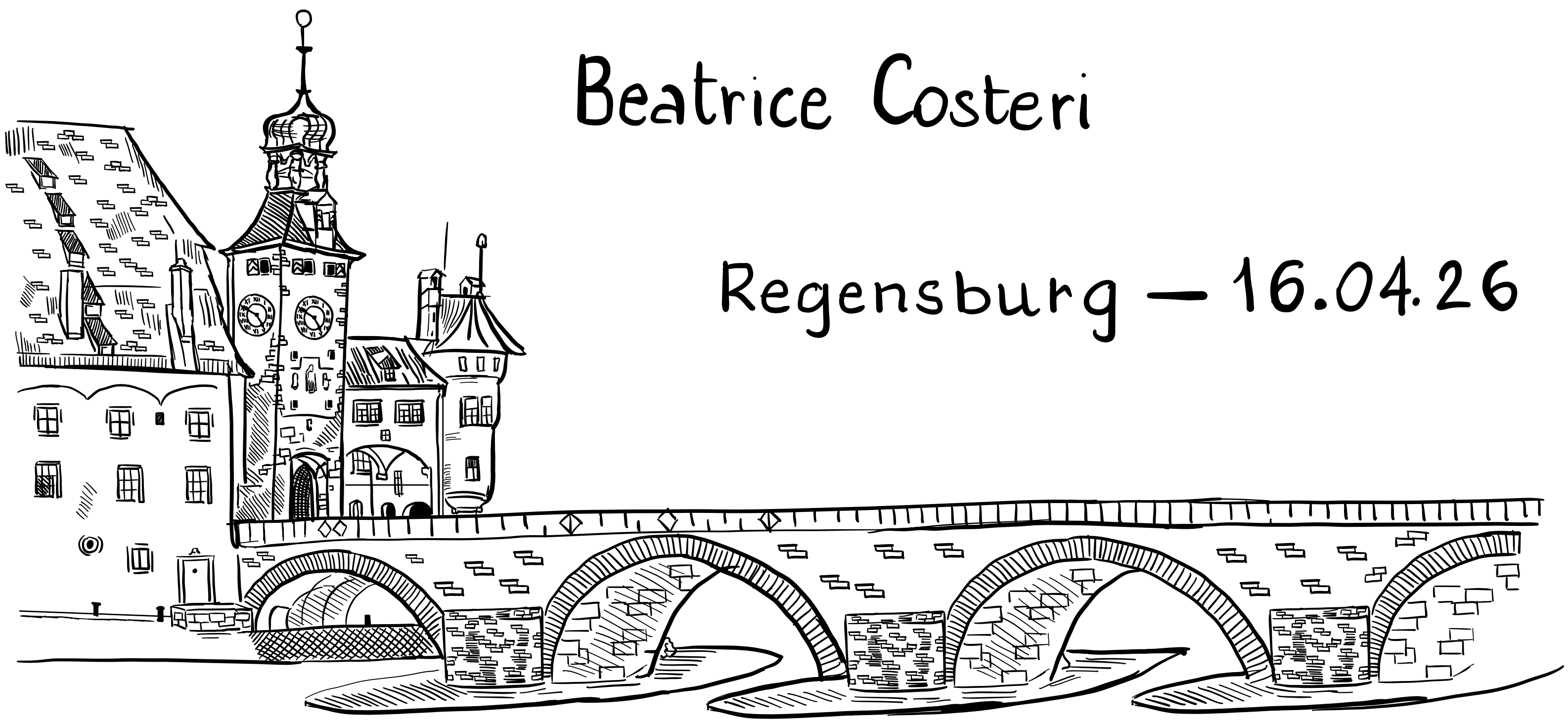
Hadamard parametrix on half-Minkowski with Robin boundary conditions

Beatrice Costeri



Regensburg — 16.04.26

The Team[1]

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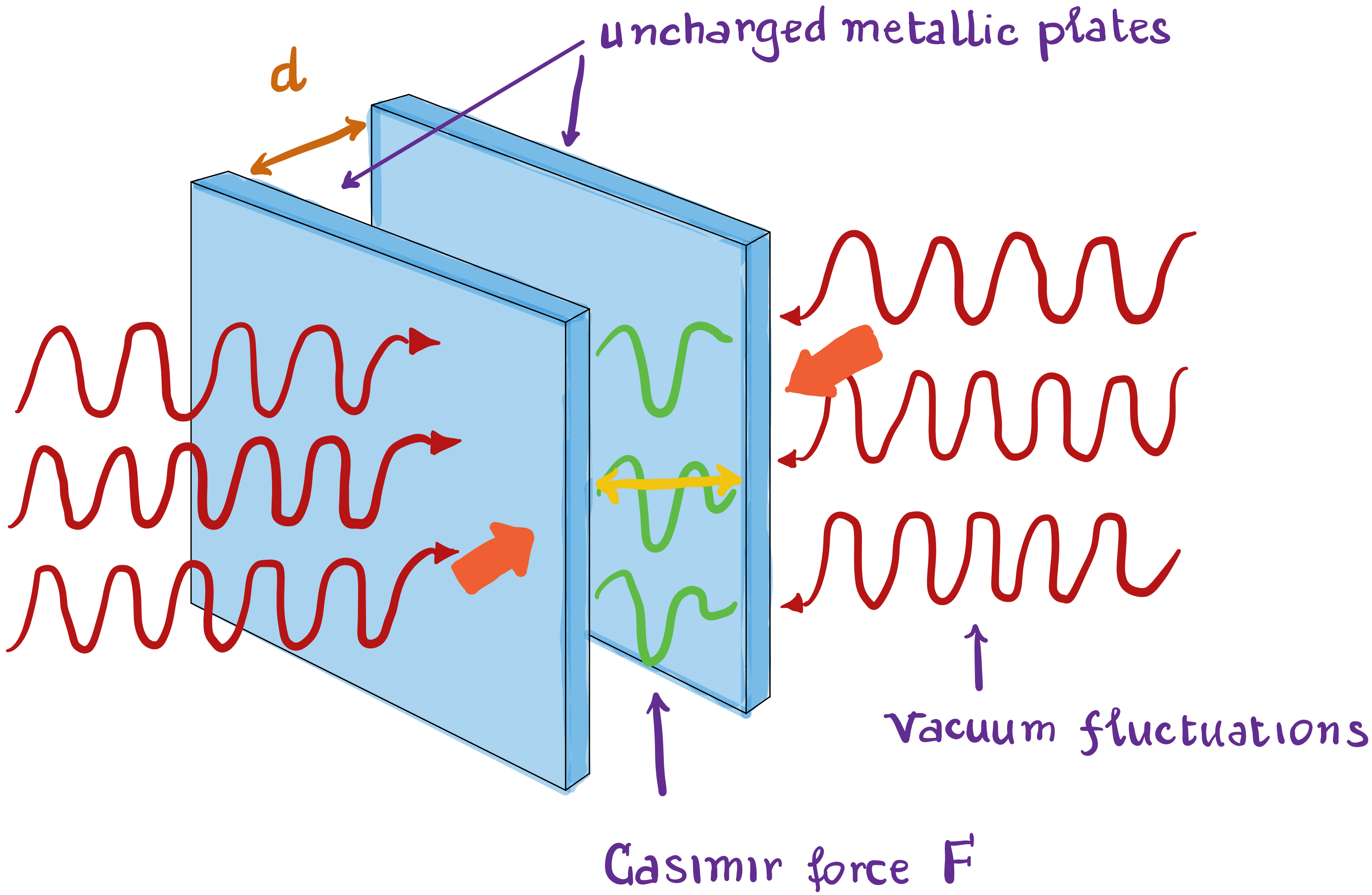
(1) B.C., C. Dappiaggi, B.A. Juárez-Aubry, R.D. Singh, Accepted for publication in <u>Annales Henri Poincaré</u>, (2026)

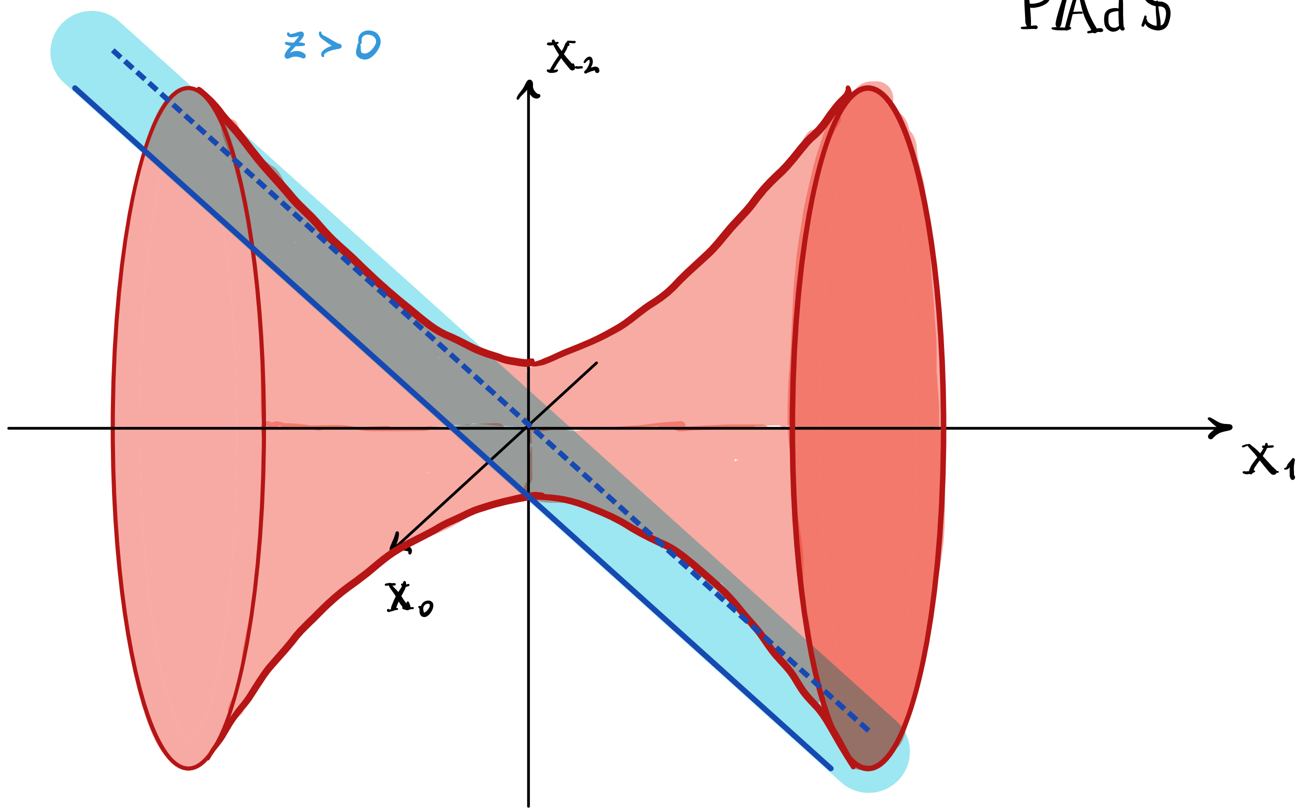
Boundaries in Physics

Casimir effect



Boundaries in Physics

Anti-de Sitter spacetime



$$ds^2 = \frac{\ell^2}{z^2}\left(dt^2 - \sum_{i=1}^{d-2} dx_i^2 - dz^2\right), \quad \ell^2 = -\frac{(d-1)(d-2)}{2\Lambda}$$

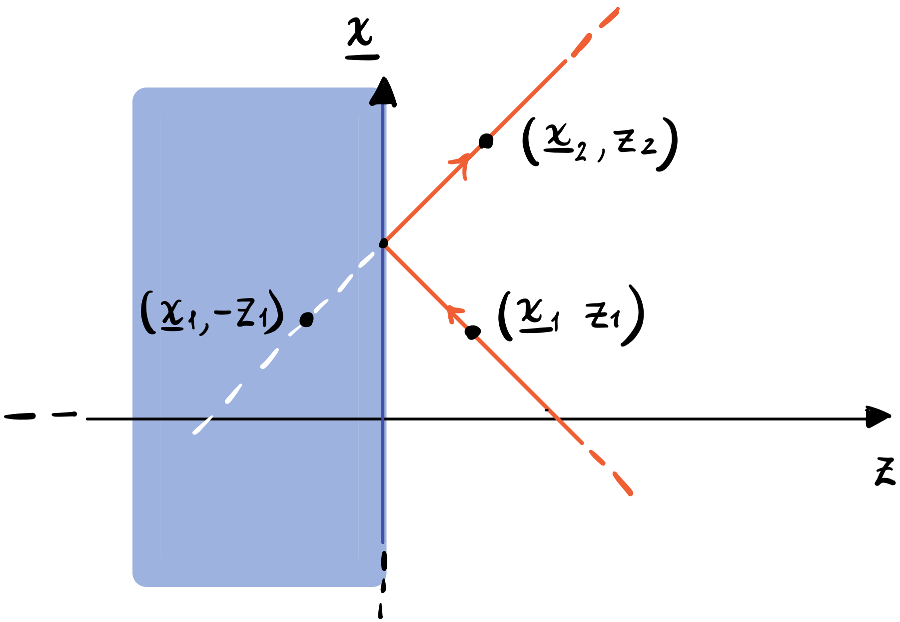
→ AdS/CFT correspondence

Geometric setting

$d \geqslant 2$ - dimensional half-Minkowski spacetime

$(\mathbb{H}^d, \eta)$ signature $(-, \underbrace{+, \dots, +}_{d-1})$

$\mathbb{H}^d := \{ (\underline{x}, z) \in \mathbb{R}^d \mid z \geqslant 0 \}$ $\underline{x} = (t, \overbrace{x_1, \dots, x_{d-2}}^{\mathrm{x}})$



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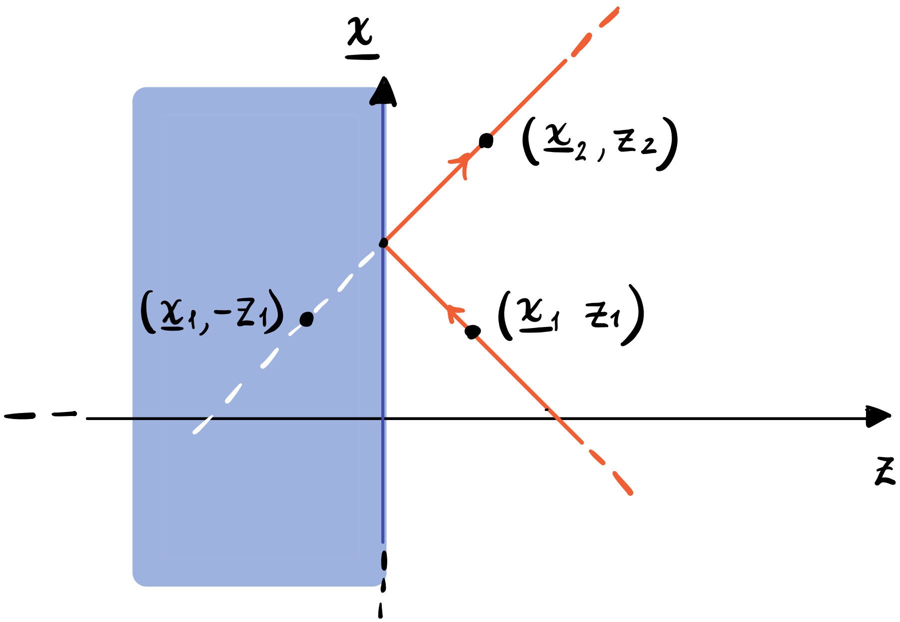

Synge's function

$$\sigma := \frac{1}{2} \left[(\underline{x}_1 - \underline{x}_2)^2 - (z_1 - z_2)^2 \right\}$$

Reflected Synge's function

$$\sigma_- := \frac{1}{2} \left[(\underline{x}_1 - \underline{x}_2)^2 - (z_1 + z_2)^2 \right\}$$

The Problem

$\Phi : \mathbb{H}^d \to \mathbb{R}$, scalar field

Klein - Gordon operator : $P = \Box_\eta + m^2$, $m^2 \geq 0$

$$\begin{cases} P\Phi = f & \text{Dynamics} \\ \Phi\big|_{\Sigma} = \Phi_0 , \quad \nabla_n \Phi\big|_{\Sigma} = \Phi_1 & \text{Initial conditions} \\ (\partial_z + \kappa)\Phi\big|_{z=0} = 0 , \quad \boxed{\kappa \leq 0} & \text{Robin boundary conditions} \end{cases}$$

Algebraic QFT

1. Solution theory

2. Algebra of observables $\rightarrow \mathcal{A}$

   * deformation quantization :

$$\boxed{G := G^{-} - G^{+}}$$

retarded - advanced propagator

Klaus Fredenhagen

3. Physical states : $\omega : \mathcal{A} \longrightarrow \mathbb{C}$ positive, normalized, linear

   * Gaussian
   * quasi-free $\Rightarrow \omega_2$: two-point correlation
   * Hadamard

local Hadamard form

A brief digression: No boundary (I)

$(\mathcal{M}, g)$ globally hyperbolic spacetime, $\partial\mathcal{M} = \emptyset$

- existence & uniqueness of advanced $(+)$ / retarded $(-)$ propagators

$$G^{\pm} \in \mathcal{D}'(\mathcal{M} \times \mathcal{M}) \quad \text{and} \quad \mathrm{supp}(\mathcal{G}^{\pm}(f)) \subseteq J^{\mp}(\mathrm{supp}(f)),\ f \in \mathcal{D}(\mathcal{M})$$

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- causal propagator $G := G^{-} - G^{+}$

$$\begin{cases} (P \otimes \mathbb{1})\, G = (\mathbb{1} \otimes P)\, G = 0 \\ G\big|_{t=t'} = 0\,, \quad \partial_t G\big|_{t=t'} = -\partial_{t'} G\big|_{t=t'} = \delta_{\Sigma} \end{cases}$$

initial data

$$\Rightarrow \quad G^{-}(x,x') = \Theta(t-t')\, G(x,x')\,, \qquad G^{+}(x,x') = -\Theta(t'-t)\, G(x,x')$$

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$$\Rightarrow G^-(x,x') = \Theta(t-t')G(x,x'), \quad G^+(x,x') = -\Theta(t'-t)G(x,x')$$

••• distinguished propagators: $G^{\pm}$, $G^{F\backslash\overline{F}}$ (1)

(1) J.J. Duistermaat, L. Hörmander, Acta Mathematica (1972)

A brief digression: No boundary (II)

Hadamard two-point function:

$$\omega_2 \in \mathcal{D}'(\mathcal{M}\times\mathcal{M}) \quad \text{s.t.}$$

1. $(P \otimes \mathbb{1})\,\omega_2 = (\mathbb{1} \otimes P)\,\omega_2 = 0$

   $\forall f, f' \in \mathcal{D}(\mathcal{M})$:

2. [CCR] $\omega_2(f,f') - \omega_2(f',f) = i\,G(f,f')$

3. $\Im\left(\omega_2(f,f')\right) = \frac{1}{2}\left(G(f,f')\right)$

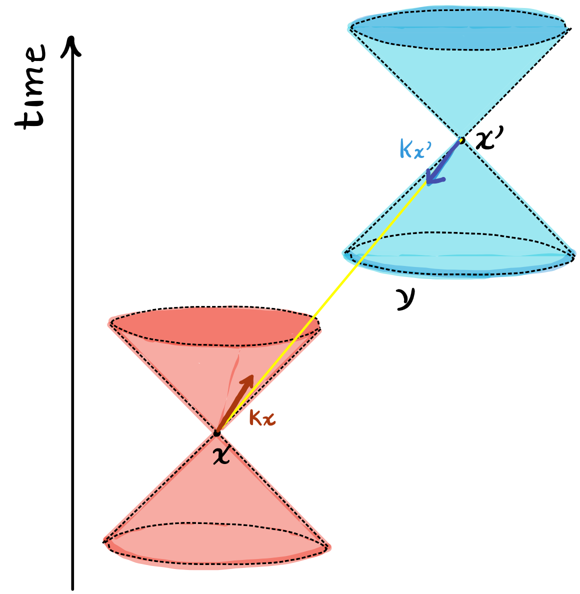
4. $|G(f,f')|^2 \leq 4\,\omega_2(f,f')\,\omega_2(f',f)$

A brief digression: No boundary (II)

Hadamard WF set:

$$\mathrm{WF}(\omega_2) = \{(x, k_x, x', -k_{x'}) \in T^*(\mathcal{M} \times \mathcal{M}) \setminus \{0\} \mid (x, k_x) \sim (x', k_{x'}),\ k_x \triangleright 0\}$$

$\Rightarrow$ ω_2 is of global Hadamard form



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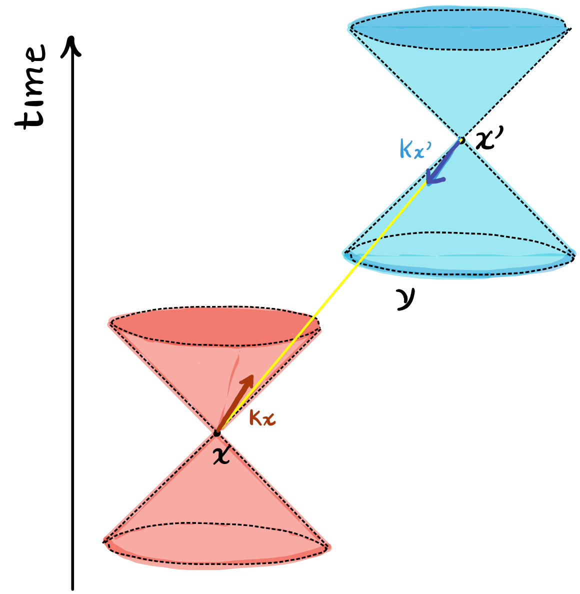

Local Hadamard form: $\mathcal{O} \subseteq \mathcal{M}$

$$\omega_2(x,x') := \lim_{\varepsilon \to 0^+} \left\{ \frac{U(x,x')}{4\pi \sigma_\varepsilon^{\frac{d-2}{2}}(x,x')} + \delta_d V(x,x') \log \frac{\sigma_\varepsilon(x,x')}{\lambda^2} \right\} + \underbrace{W(x,x')}_{C^\infty(\mathcal{O}\times\mathcal{O})}, \quad \lambda > 0$$

$$\sigma_\varepsilon(x,x') := \sigma(x,x') + i\varepsilon\,(t(x) - t(x')) + \varepsilon^2, \qquad \delta_d = \begin{cases} 1 & d \text{ even} \\ 0 & d \text{ odd} \end{cases}$$

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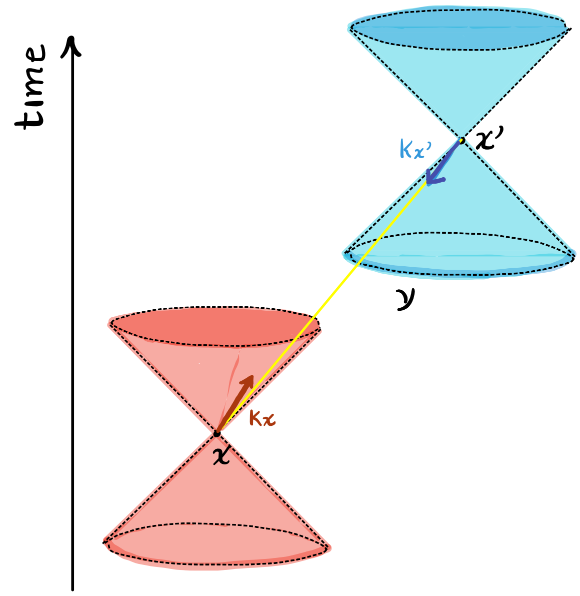

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$$\sigma_\varepsilon(x,x') := \sigma(x,x') + i\varepsilon\,(t(x)-t(x')) + \varepsilon^2,\qquad \delta_d = \begin{cases}1 & d \text{ even}\\ 0 & d \text{ odd}\end{cases}$$

Radzikowski theorem ('96):

global Hadamard form $\Longleftrightarrow$ local Hadamard form

A brief digression: No boundary (III)

Hadamard recursion relations:

$$U = \sum_{j=0}^{\infty} u_j(x,x') \left(\frac{\sigma}{\lambda}\right)^j \qquad V = \sum_{j=0}^{\infty} v_j(x,x') \left(\frac{\sigma}{\lambda}\right)^j$$

Set $u_{-1} \equiv 0$, $v_{-1} \equiv 0$

$$\begin{cases} Pu_j + (2j+4-d)\,\sigma^{\mu}\partial_{\mu} u_{j+1} + (j+1)\left(\sigma^{\mu}{}_{\mu} + 2j + 4 - 2d\right) u_{j+1} + \frac{(2-d)}{2}\left(\sigma^{\mu}{}_{\mu} - d\right) u_{j+1} = 0 \\ [u_0] = 1\,, \quad [u_{j+1}] = -\dfrac{[Pu_j]}{(j+1)(2j+4-d)} \\ \\ Pv_j + (2j+1)\,\sigma^{\mu}\partial_{\mu} v_{j+1} + (j+1)\left(\sigma^{\mu}{}_{\mu} + 2j\right) v_{j+1} = 0 \qquad \text{only in even } d \\ [v_0] = -\dfrac{[P u_{\frac{d-2}{2}}]}{(d-2)}\,, \quad [v_{j+1}] = -\dfrac{[Pv_j]}{(j+1)(d+2j)} \end{cases}$$

Why Hadamard Recursions ?

- Wick Polynomials:

$$:\Phi_\omega^2:(f) := (\omega_2 - H)(f\delta_2), \quad \forall f \in \mathcal{D}(\mathcal{M})$$

- Stress-Energy tensor[1] (regularized)

$$:(T_{\mu\nu})_\omega(f): = \underbrace{D_{\mu\nu}}_{\text{differential operator}}(\omega_2 - H)(f\delta_2)$$

Trace anomaly on $(\mathbb{R}^4, \eta)$

$$:T_\omega(x): = m^2 :\Phi_\omega^2(x): + \frac{[v_1]}{4\pi}$$

(1) B.C, C.Dappiaggi, M.Goi, Accepted for publication in Annales Henri Poincaré, (2025)

Formulation of the Problem

Advanced (+)\ Retarded (−) Robin propagators

$$G^{\pm}_{\kappa} \in \mathcal{D}'(\mathbb{H}^d \times \mathbb{H}^d) \quad \text{s.t.}$$

1. dynamics:

$$(P \otimes \mathbb{1})\, G^{\pm}_{\kappa}\Big|_{\mathring{\mathbb{H}}^d \times \mathring{\mathbb{H}}^d} = (\mathbb{1} \otimes P)\, G^{\pm}_{\kappa}\Big|_{\mathring{\mathbb{H}}^d \times \mathring{\mathbb{H}}^d} = \delta\Big|_{\mathring{\mathbb{H}}^d \times \mathring{\mathbb{H}}^d}$$

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2. boundary conditions:

$$\underbrace{(\partial_z \otimes \mathbb{1})\, G^{\pm}_{\kappa}\big|_{\partial\mathbb{H}^d}}_{} = \underbrace{(-\kappa \otimes \mathbb{1})\, G^{\pm}_{\kappa}\big|_{\partial\mathbb{H}^d}}_{}, \quad \kappa \leq 0$$

pull back to $z=0$

Formulation of the Problem

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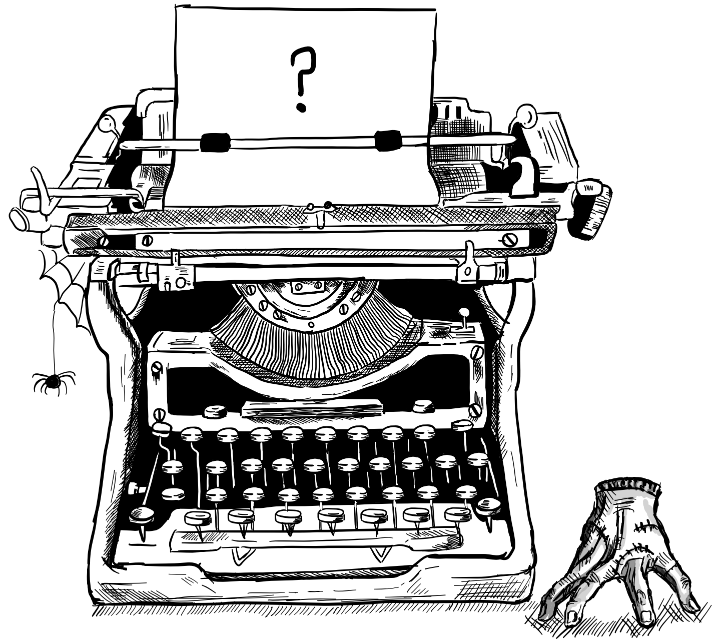
2. boundary conditions:

$$(\partial_z \otimes \mathbb{1})\, G^{\pm}_{\kappa}\big|_{\partial\mathbb{H}^d} = (-\kappa \otimes \mathbb{1})\, G^{\pm}_{\kappa}\big|_{\partial\mathbb{H}^d}, \quad \kappa \leq 0$$

3. support property:

$$\operatorname{supp}(\mathcal{G}^{\pm}_{\kappa}(f)) \subseteq J^{\mp}(\operatorname{supp}(f)), \quad \forall f \in \mathcal{D}(\mathring{\mathbb{H}}^d)$$

Open Questions



* existence of $G_\kappa^{\pm}$ [1] ✓

* causal support property ?

    ✓ only for $\kappa = 0$ and $\kappa \to -\infty$

* local expression of $G_\kappa^{\pm}(x, x')$?

* Hadamard states ?

(1) C. Dappiaggi, N. Drago, H.R.C. Ferreira, Lett. Math. Phys. **109** 10 (2019)

Dirichlet & Neumann fundamental solutions

P.G. Lejeune Dirichlet

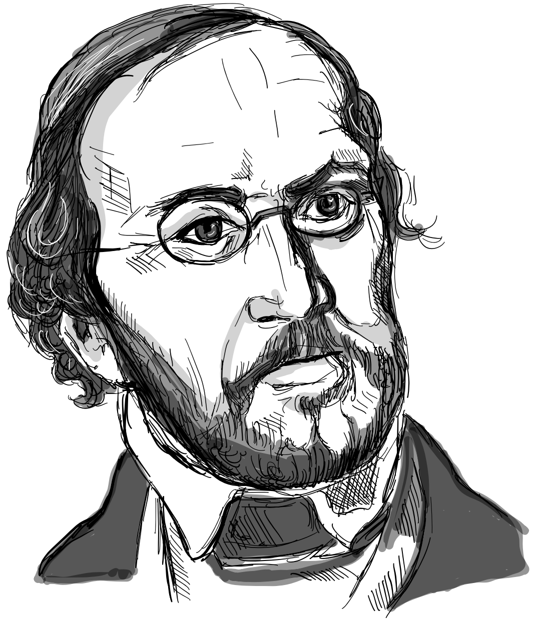
Dirichlet causal propagator $G_D(\underline{x}, z, \underline{x}', z')$

$$= G_{\mathbb{R}^d}(\underline{x}-\underline{x}', z-z') - \underbrace{G_{\mathbb{R}^d}(\underline{x}-\underline{x}', z+z')}$$

integral kernel of $(i_z^* \otimes \mathrm{id}|_{\mathbb{R}^d})\, G_{\mathbb{R}^d}$

$$i_z : \mathbb{R}^d \longrightarrow \mathbb{R}^d$$
$$(\underline{x}, z) \longmapsto (\underline{x}, -z)$$

Dirichlet & Neumann fundamental solutions



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Dirichlet causal propagator $G_D(\underline{x}, z, \underline{x}'; z')$

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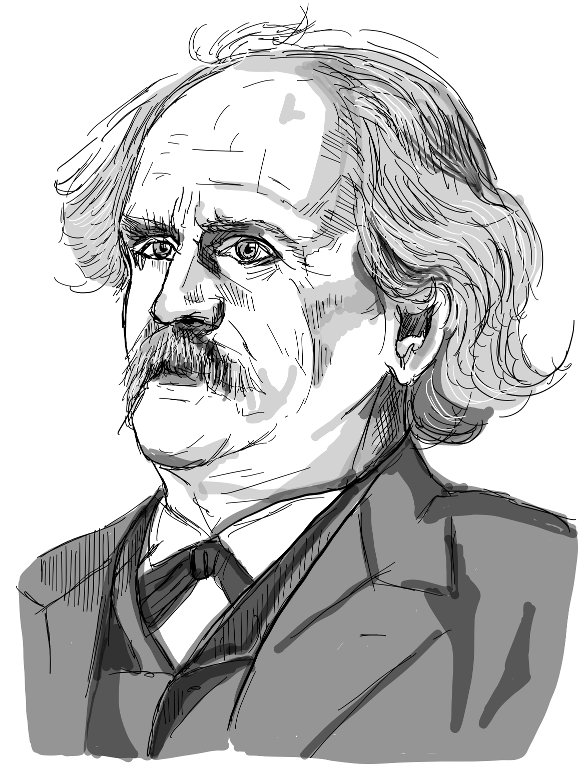
$$\begin{aligned} i_z :\ \mathbb{R}^d &\longrightarrow \mathbb{R}^d \\ (\underline{x}, z) &\longmapsto (\underline{x}, -z) \end{aligned}$$

Neumann causal propagator $G_N(\underline{x}, z, \underline{x}'; z')$

$$= G_{\mathbb{R}^d}(\underline{x}-\underline{x}', z-z') + G_{\mathbb{R}^d}(\underline{x}-\underline{x}', z+z')$$



Carl Neumann

Robin-to-Dirichlet map (I)

Dirichlet functions

- $C_D^\infty(\mathbb{H}^d) := \{ f \in C^\infty(\mathbb{H}^d) \mid f|_{z=0} = 0 \}$

Robin functions

- $C_\kappa^\infty(\mathbb{H}^d) := \{ f \in C^\infty(\mathbb{H}^d) \mid \partial_z f|_{z=0} = -\kappa f|_{z=0} \}$

Robin-to-Dirichlet map [1]

$$T_\kappa : C_\kappa^\infty(\mathbb{H}^d) \longrightarrow C_D^\infty(\mathbb{H}^d)$$

$$f \longmapsto T_\kappa(f) := \left(\mathbb{1}_{\underline{x}} \otimes \left(\partial_z + \kappa \mathbb{1}_z \right) \right) f$$

T_κ can be read as a linear operator on $C_\kappa^\infty(\mathbb{R}^d)$

[1] J.D. Bondurant, S.A. Fulling, J. Phys. A **38** (7) (2005)

Robin-to-Dirichlet map (II)

The map :

$$\tilde{T}_\kappa : \frac{C^\infty_\kappa(\mathbb{R}^d)}{\mathrm{Ker}(T_\kappa)} \longrightarrow \mathrm{Ran}(T_\kappa) \subseteq C^\infty_D(\mathbb{R}^d)$$

$$f \longmapsto \tilde{T}_\kappa[f] := T_\kappa(f)$$

is bijective , hence $\exists\, \mathcal{L}_\kappa := \tilde{T}_\kappa^{-1}$

$$\mathcal{L}_\kappa \in \mathcal{D}'(\mathbb{R}^d) \quad \text{s.t.} \quad \tilde{T}_\kappa \mathcal{L}_\kappa = \delta \qquad \Rightarrow \qquad \boxed{\mathcal{L}_\kappa(z) = -\Theta(-z)\, e^{-\kappa z}}$$

$\kappa < 0 \;\Rightarrow\; \mathcal{L}_\kappa \in \mathcal{S}'(\mathbb{R}^d)$ ✓

$\Rightarrow \boxed{\kappa < 0}$

$\kappa > 0 \;\Rightarrow\;$ bounded state modes ✗

Spaces of distributions

- $\mathcal{D}'_-(\mathbb{R}^d) := \{ u \in \mathcal{D}'(\mathbb{R}^d) \mid \exists\, v \in \mathcal{D}'(\mathbb{R}^d),\ u = v - i_z^* v \}$

  $\mathcal{S}'_-(\mathbb{R}^d) := \mathcal{D}'_-(\mathbb{R}^d) \cap \mathcal{S}'(\mathbb{R}^d)$

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  $\mathcal{S}'_-(\mathbb{R}^d) := \mathcal{D}'_-(\mathbb{R}^d) \cap \mathcal{S}'(\mathbb{R}^d)$

- Dirichlet (tempered) distributions:

  $\mathcal{D}'_0(\mathbb{R}^d) := \{ u \in \mathcal{D}'(\mathbb{R}^d) \mid j_0^* u = 0 \}$, where

  $$j_0 : \mathbb{R}^{d-1} \longrightarrow \mathbb{R}^d, \quad \underline{x} \longmapsto j_0(\underline{x}) := (\underline{x}, 0)$$

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- Robin tempered distributions:

  $$\mathcal{S}'_\kappa(\mathbb{R}^d) := \mathcal{L}_\kappa \star \mathcal{S}'_0(\mathbb{R}^d)$$

Advanced & Retarded Robin propagators (I)

Thm. 4.12 [CDJ+25]

Dirichlet propagators

$$G^{\pm}_{\kappa} := (\mathcal{L}_{\kappa} \otimes \delta) \star_1 G^{\pm}_{D} \circ (\mathbb{1} \otimes \widetilde{T}_{\kappa}) \qquad (1)$$

are the advanced (+)\retarded (−) Robin propagators

Advanced & Retarded Robin propagators (I)

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are the advanced (+)\retarded (−) Robin propagators

Proof: Eq. (1) is well-defined & $G^{\pm}_{\kappa}(f) \in \mathcal{S}'(\mathbb{R}^d)$, $\forall f \in \mathcal{D}(\mathbb{R}^d)$

Since $[P, \widetilde{T}_{\kappa}] = 0$,

$$(P \otimes \mathbb{1})\, G^{\pm}_{\kappa} = (\mathcal{L}_{\kappa} \otimes \delta) \star_1 (P \otimes \mathbb{1})\, G^{\pm}_{D} \circ (\mathbb{1} \otimes \widetilde{T}_{\kappa}) =$$

$$= (\mathcal{L}_{\kappa} \otimes \delta) \star_1 \delta|_{\mathrm{Diag}(\mathbb{R}^d \times \mathbb{R}^d)} \circ (\mathbb{1} \otimes \widetilde{T}_{\kappa}) =$$

$$= \delta|_{\mathrm{Diag}(\mathbb{R}^d \times \mathbb{R}^d)} \qquad \square$$

Advanced & Retarded Robin propagators (II)

Prop. 4.15 [CDJ+25]

Neumann causal propagator

$$G_{\kappa} := G^-_{\kappa} - G^+_{\kappa} = G_N - 2\kappa(\mathcal{L}_{\kappa} \otimes \delta) \star_1 G_r \quad , \quad (2)$$

where $G_r(\underline{x} - \underline{x}', z + z') = (i^*_Z \otimes \mathrm{id}|_{\mathbb{R}^d})\, G_{\mathbb{R}^d}(\underline{x} - \underline{x}', z - z')$

Advanced & Retarded Robin propagators (II)

Prop. 4.15 [CDJ+25]

$$G_\kappa := G_\kappa^- - G_\kappa^+ = \overbrace{G_N}^{\text{Neumann causal propagator}} - 2\kappa(\mathcal{L}_\kappa \otimes \delta) \star_1 G_r \,, \qquad (2)$$

where $G_r(\underline{x}-\underline{x}', z+z') = (i_z^* \otimes \mathrm{id}|_{\mathbb{R}^d})\, G_{\mathbb{R}^d}(\underline{x}-\underline{x}', z-z')$

Proof. At the level of integral kernels, $\forall f' \in \mathcal{D}(\mathbb{R}^d)$

$$\big((\mathcal{L}_\kappa \otimes \delta) \star_1 G_D(\widetilde{T}_\kappa f')\big)(z) =$$

$$= \big((\mathcal{L}_\kappa \otimes \delta) \star_1 G_{\mathbb{R}^d}(\widetilde{T}_\kappa f')\big)(z) - \big((\mathcal{L}_\kappa \otimes \delta) \star_1 G_r(\widetilde{T}_\kappa f')\big)(z)$$

$$= G_{\mathbb{R}^d}(f')(z) - \Big(-G_r(f')(z) + 2\kappa\big((\mathcal{L}_\kappa \otimes \delta) \star_1 G_r(f')\big)(z)\Big)$$

$$= G_N(f')(z) - 2\kappa\,(\mathcal{L}_\kappa \otimes \delta) \star_1 G_r(f')(z) \qquad \square$$

Advanced & Retarded Robin propagators (III)

Lem. 4.16 & 4.17 [GDJ+25] (Support properties of $G^{\pm}_{\kappa}$)

$$\mathrm{supp}\left((\mathcal{L}_\kappa \otimes \delta) *_1 \delta(2\partial_-)\right) \cap (\mathbb{H}^d \times \mathbb{H}^d) \subset \mathrm{supp}\left(\Theta(2\partial_-)\right) \cap (\mathbb{H}^d \times \mathbb{H}^d)$$

and the same holds true also for $(\mathcal{L}_\kappa \otimes \delta) *_1 \Theta(2\partial_-)$

Advanced & Retarded Robin propagators (III)

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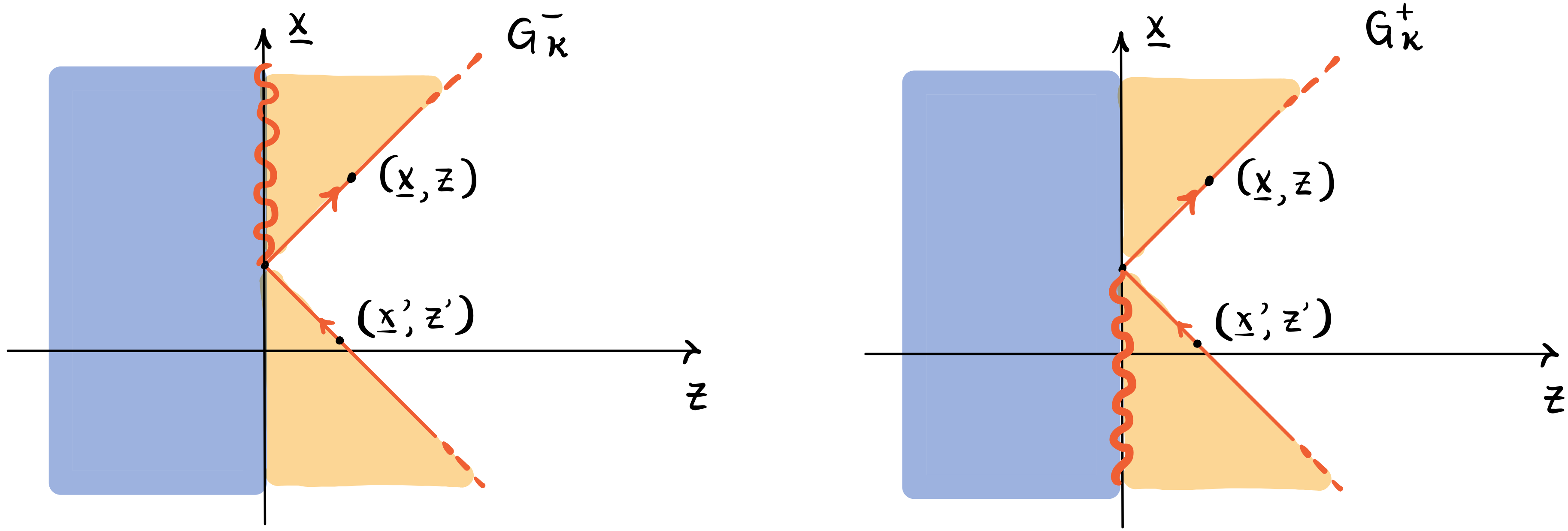
and the same holds true also for $(\mathcal{L}_\kappa \otimes \delta) *_1 \Theta(2\sigma_-)$

$\Rightarrow$ $G^{\pm}_{\kappa}$ abide by the causal support property ✓



Advanced & Retarded Robin propagators (IV)

Prop. 4.20 [GDJ+25]

$G^{\pm}_{\varkappa} \in \mathcal{D}'(\mathbb{H}^d \times \mathbb{H}^d)$ as in Eq. (2) are the advanced (+) and retarded (−) Robin propagators in the sense of [1]

[1] C. Dappiaggi, N. Drago, H.R.C. Ferreira, Lett. Math. Phys. **109** 10 (2019)

Advanced & Retarded Robin propagators (IV)

Prop. 4.20 [GDJ+25]

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Proof. (1) In Fourier modes, denoting by $\omega^2 := |k_\perp|^2 + k_z^2 + m^2$,

$$G_\varkappa(x,x') = \int_{\mathbb{R}^{d-2}} \frac{dk_\perp}{(2\pi)^{d-2}} e^{ik_\perp(x_\perp - x'_\perp)} \int_0^\infty dk_z \, \Psi_k(z) \, \overline{\Psi_k(z')} \, \frac{\sin(\omega(t-t'))}{\omega}$$

where $\Psi_k(z) = \frac{1}{\sqrt{2}} \left(e^{-ik_z z} - \frac{\kappa + ik_z}{\kappa - ik_z} e^{ik_z z} \right)$.

$$G_\kappa(f,f') = \int_{\mathbb{R}^2} dt \, dt' \left(f, A_\varkappa^{-1/2} \sin\left(A_\varkappa^{1/2}(t-t')\right) f' \right), \quad \forall f, f' \in \mathcal{D}(\mathbb{H}^d)$$

□

(1) C. Dappiaggi, N. Drago, H.R.C. Ferreira, *Lett. Math. Phys.* **109** 10 (2019)

Advanced & Retarded Robin propagators (IV)

Prop. 4.21 [CDJ+25]

Exact sequence:

$$0 \longrightarrow C^\infty_{tc,\kappa}(\mathbb{H}^d) \xrightarrow{P} C^\infty_{tc}(\mathbb{H}^d) \xrightarrow{G_\kappa} C^\infty_\kappa(\mathbb{H}^d) \xrightarrow{P} C^\infty(\mathbb{H}^d) \longrightarrow 0$$

In addition:

$$\frac{C^\infty_{tc}(\mathbb{H}^d)}{P[C^\infty_{tc,\kappa}(\mathbb{H}^d)]} \simeq \ker(P)\Big|_{C^\infty_\kappa(\mathbb{H}^d)}$$

Solution space

Advanced & Retarded Robin propagators (IV)

Prop. 4.22 [GDJ+25]

$\mathcal{G}^{\pm}_{\varkappa}$ are the unique advanced $(+)$ and retarded $(-)$ Robin Green's operators.

Advanced & Retarded Robin propagators (IV)

Prop. 4.22 [CDJ+25]

$\mathcal{G}^{\pm}_{\kappa}$ are the unique advanced $(+)$ and retarded $(-)$ Robin Green's operators.

Proof. Assume $\exists\, \mathcal{G}^{-}_{\kappa}, \tilde{\mathcal{G}}^{-}_{\kappa}$ and consider $\Delta\mathcal{G}^{-}_{\kappa} := \mathcal{G}^{-}_{\kappa} - \tilde{\mathcal{G}}^{-}_{\kappa}$ s.t.

$$P\Delta\mathcal{G}^{-}_{\kappa} = 0 \quad \text{and} \quad (\partial_z + \kappa)\Delta\mathcal{G}^{-}_{\kappa}(f) = 0, \ \forall f \in C^{\infty}_{0}(\mathbb{H}^d)$$

Suppose $\exists \tilde{f} \in C^{\infty}_{0}(\mathbb{H}^d)$ s.t. $\Delta\mathcal{G}^{-}_{\kappa}(\tilde{f}) := \Phi_{\tilde{f}} \neq 0$.

Then, $\Phi_{\tilde{f}} \in C^{\infty}_{pc,\kappa}(\mathbb{H}^d)$ and

$$\Phi_{\tilde{f}} = \mathcal{G}^{-}_{\kappa} P \Phi_{\tilde{f}} = 0.$$

$\square$

Example : $(\mathbb{H}^4, \eta)$, $m = 0$

Let $|\tau| := +\sqrt{(t-t')^2 - (x-x')^2 - (y-y')^2}$

$$G_\kappa(\tau, z, z') = \frac{\operatorname{sgn}(t-t')}{2\pi}\Big(\underbrace{\delta(\sigma) + \delta(\sigma_-)}_{G_N} - \underbrace{\frac{2\kappa}{|\tau|}\, \Theta(\sigma_-)\, e^{-\kappa(-|\tau|+z+z')}}_{\text{correction term } \propto \kappa} \Big)$$

Recalling that $G_\kappa^-(x, x') = \Theta(t-t')\, G_\kappa(x, x')$

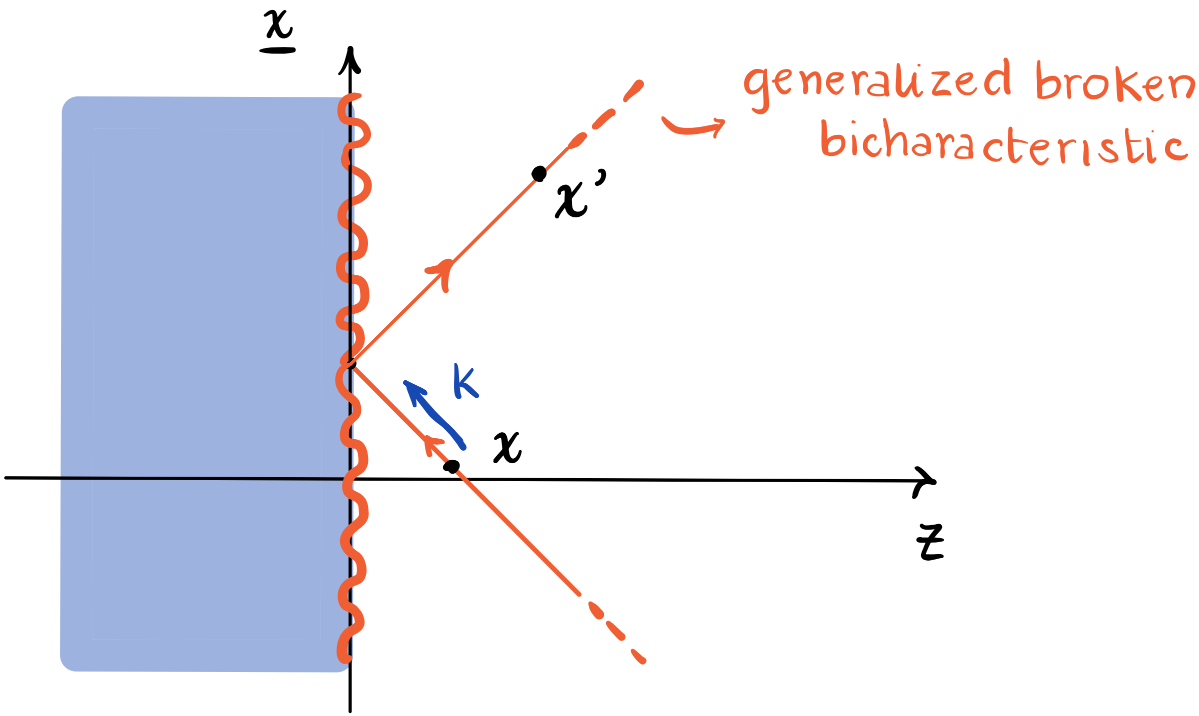
$$G_\kappa^-(t, \underline{x}, t', \underline{x}') = \frac{\Theta(t-t')}{2\pi}\Big(\delta(\sigma) + \delta(\sigma_-) + \Theta(\sigma_-) \underbrace{\sum_{j=0}^{\infty} d_j(z+z', \kappa)\, \sigma_-^{\,j}}_{\text{power series in } \sigma_-} \Big)$$

where d_j are suitable functions.

Robin Hadamard states on $\mathbb{H}^d(I)$

$\omega_{2,\kappa} \in \mathcal{D}'(\mathbb{H}^d \times \mathbb{H}^d)$ is of **global Hadamard form** if

$$WF(\omega_{2,\kappa}) = \{ (x, k, x', -k') \in T^*(\mathbb{H}^d \times \mathbb{H}^d) \setminus \{0\} \mid (x,k) \,\dot{\sim}\, (x', k'),\ k \triangleright 0 \}$$



Jacques Hadamard

Robin Hadamard states on $\mathbb{H}^d$ (I)

$\omega_{2,\kappa}$ is a **Robin Hadamard two-point function** if:

1. $WF(\omega_{2,\kappa}) = \{(x,k,x',-k') \in T^*(\mathbb{H}^d \times \mathbb{H}^d) \setminus \{0\} \mid (x,k) \dot{\sim} (x',k'),\ k \triangleright 0\}$

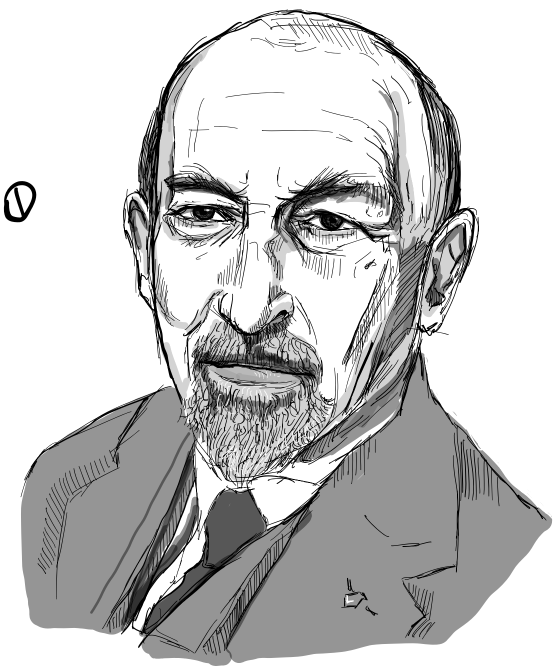
2. [Dynamics] $(P \otimes \mathbb{1})\,\omega_{2,\kappa} = (\mathbb{1} \otimes P)\,\omega_{2,\kappa} = 0$

   [Robin BCs] $\left(\partial_z \omega_{2,\kappa} + \kappa\,\omega_{2,\kappa}\right)\Big|_{\partial\mathbb{H}^d} = 0,\ \kappa \leq 0$

   $\forall f, f' \in \mathcal{D}(\mathbb{H}^d)$

3. [Positivity] $\omega_{2,\kappa}(f,f) \geq 0$

4. [CCR] $\omega_{2,\kappa}(f,f') - \omega_{2,\kappa}(f',f) = i\,G_\kappa(f,f')$



Jacques Hadamard

Robin Hadamard states on $\mathbb{H}^d$ (II)

Prop. 5.6 [GDJ+25] ($d > 2$, $m \neq 0$)

$\omega_{2,\kappa} \in \mathcal{D}'(\mathbb{H}^d \times \mathbb{H}^d)$ s.t.:

$$\omega_{2,\kappa} := (\mathcal{L}_\kappa \otimes \delta) \star_1 \omega_{2,D} \circ (\mathbb{I} \otimes \widetilde{T}_\kappa), \quad \kappa \leq 0 \qquad (3)$$

is a Robin Hadamard two-point function

Robin Hadamard states on $\mathbb{H}^d(\mathbb{I})$

Prop. 5.6 [GDJ+25] ($d>2$, $m\neq 0$)

$\omega_{2,\kappa} \in \mathcal{D}'(\mathbb{H}^d \times \mathbb{H}^d)$ s.t.

$$\omega_{2,\kappa} := (\mathcal{L}_\kappa \otimes \delta) \star_1 \omega_{2,D} \circ (\mathbb{1} \otimes \widetilde{T}_\kappa), \quad \kappa \leq 0 \qquad (3)$$

is a Robin Hadamard two-point function

Proof. Eq. (3) is equivalent to

$$\omega_{2,\kappa} = \omega_{2,N} + 2\kappa (\mathcal{L}_\kappa \otimes \delta) \star_1 \omega_{2,r} \qquad (4)$$

$$WF(\omega_{2,\kappa}) \subseteq WF(\omega_{2,N}) \cup WF((\mathcal{L}_\kappa \otimes \delta) \star_1 \omega_{2,r})$$

$$WF(\omega_{2,N}) \overset{(1)}{=} WF(\omega_{2,r}) \cup WF(\omega_2)$$

$$WF((\mathcal{L}_\kappa \otimes \delta) \star_1 \omega_{2,r}) \subseteq WF(\omega_{2,r})$$

Hadamard

$\square$

(1) O. Gannot, M. Wrochna, J. Ins. Math. Jussieu **21** 67(1) (2022)

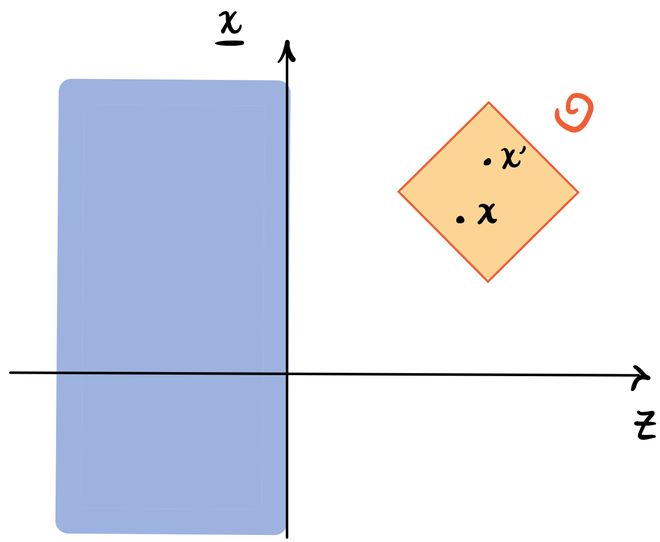
Local Hadamard form

$\omega_{2,\kappa} \in \mathcal{D}'(\mathbb{H}^d \times \mathbb{H}^d)$ is of local Hadamard form if

* if $\mathcal{O} \cap \partial\mathbb{H}^d = \phi$,

$\exists\exists$ $U, V, W \in C^\infty(\mathcal{O} \times \mathcal{O})$ s.t.

on $\mathcal{O} \times \mathcal{O}$



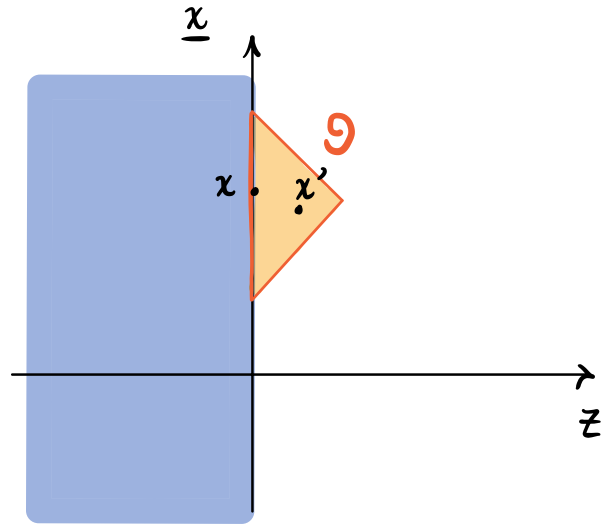
$$\omega_{2,\kappa}(x,x') = \lim_{\varepsilon \to 0^+} \frac{U(x,x')}{\sigma_\varepsilon^{\frac{d-2}{2}}} + \delta_d V(x,x') \ln\left(\frac{\sigma_\varepsilon}{\lambda^2}\right) \quad (4a)$$

$$\mod C^\infty$$

Local Hadamard form

$\omega_{2,\kappa} \in \mathcal{D}'(\mathbb{H}^d \times \mathbb{H}^d)$ is of **local Hadamard form** if

* if $\mathcal{O} \cap \partial\mathbb{H}^d \neq \emptyset$,



$\exists\, U, V, U', V', W \in C^\infty(\mathcal{O} \times \mathcal{O})$ s.t.

on $\mathcal{O} \times \mathcal{O}$

$$\omega_{2,\kappa}(x,x') = \widetilde{H}_\kappa(x,x') + W(x,x') =$$

$$= \lim_{\varepsilon \to 0^+} \frac{U(x,x')}{\sigma_\varepsilon^{\frac{d-2}{2}}} + \delta_d V(x,x') \ln\left(\frac{\sigma_\varepsilon}{\lambda^2}\right) + \frac{U'(x,x')}{\sigma_{-,\varepsilon}^{\frac{d-2}{2}}} + \delta_d V'(x,x') \ln\left(\frac{\sigma_{-,\varepsilon}}{\lambda^2}\right)$$

$$\text{mod } C^\infty \qquad (4b)$$

Hadamard recursion relations (I)

d even:

For $U, V \longrightarrow$ same as on $(\mathbb{R}^d, \eta)$

For U', V': leading singularity same as $\omega_{2,N}$

$$
\begin{cases}
(2-d)\,\sigma_-^{\mu}\partial_\mu u_0' = 0 \\
u_0'\big|_{z=0} = u_0\big|_{z=0} \\
P u_j' + (2j+4-d)\,\sigma_-^{\mu}\partial_\mu u_{j+1}' + (j+1)(2j+4-d)\,u_{j+1}' = 0 \\
(\partial_z + \kappa)(u_j + u_j')\big|_{z=0} + \frac{1}{2}(2j+4-d)\,\partial_z\sigma\,(u_{j+1} - u_{j+1}')\big|_{z=0} = 0, \quad 0 \leq j \leq \frac{d}{2} - 3 \\
P u_{\frac{d}{2}-2}' + 2\sigma_-^{\mu}\partial_\mu v_0' + (d-2)\,v_0' = 0 \\
\partial_z\left(u_{\frac{d}{2}-2} + u_{\frac{d}{2}-2}'\right)\big|_{z=0} + \partial_z\sigma\,(v - v_0')\big|_{z=0} = -\kappa\left(u_{\frac{d}{2}-2} + u_{\frac{d}{2}-2}'\right)\big|_{z=0} \\
P v_j' + (2j+1)\,\sigma_-^{\mu}\partial_\mu v_{j+1}' + (j+1)(d+2j)\,v_{j+1}' = 0 \\
(\partial_z + \kappa)(v_j + v_j')\big|_{z=0} + (j+1)\,\partial_z\sigma\,(v_{j+1} - v_{j+1}')\big|_{z=0} = 0
\end{cases}
$$

Hadamard recursion relations (II)

d odd:

For $U \longrightarrow$ same as on $(\mathbb{R}^d, \eta)$

For U':

$$\begin{cases} (2-d)\,\sigma_-^{\mu}\,\partial_\mu u_0' = 0 \\ u_0'\big|_{z=0} = u_0\big|_{z=0} \\ P u_j' + (2j+4-d)\,\sigma_-^{\mu}\partial_\mu u_{j+1}' + (j+1)(2j+4-d)\,u_{j+1}' = 0 \\ (\partial_z + k)(u_j + u_j')\big|_{z=0} + \frac{1}{2}(2j+4-d)\,\partial_z\sigma\,(u_{j+1} - u_{j+1}')\big|_{z=0} = 0, \end{cases}$$

leading singularity same as $\omega_{2,N}$

Radzikowski Theorem[1],[2]

Thm. 5.16 [CDJ+25] (Global ≡ local)

The following are equivalent:

1. $\omega_{2,\kappa}$ is of local Hadamard form, see Eqs. (4a) & (4b)
2. $\omega_{2,\kappa}$ satisfies Robin BC & [GGR] mod C^∞ &

$$WF(\omega_{2,\kappa}) = \{(x,k,x',-k') \in T^*(\mathbb{H}^d \times \mathbb{H}^d)\setminus\{0\} \mid (x,k) \dot{\sim} (x',k') \text{ and } k' \triangleright 0\}$$

(1) M. J. Radzikowski, Comm. Math. Phys. **179**, 529 (1996)

(2) M. J. Radzikowski, Comm. Math. Phys. **180**, 1 (1996)

Proof. ($1 \Rightarrow 2$) On $\mathcal{O} \subseteq \mathbb{R}^d$, define:

$$b_1(x,x') := \lim_{\varepsilon\to 0^+} \frac{1}{\sigma_\varepsilon^{\frac{d-2}{2}}(x,x')} \quad , \quad b_2(x,x') := \lim_{\varepsilon\to 0^+} \ln\left(\sigma_\varepsilon(x,x')\right)$$

$$\overset{(1)}{\Longrightarrow} \quad WF(b_i) = \{(x,k,x',-k') \in T^*(\mathcal{O}\times\mathcal{O})\setminus\{0\} \mid (x,k)\sim(x',k'), k \rhd 0\}$$

Let $b_{i,-} := (i_z^* \otimes id)\, b_i$, $i=1,2$. Since i_z is a diffeomorphism,

$$WF(b_{i,-}) = WF((i_z^* \otimes id)\, b_i) = \{(i_z(x), i_z^*(k), x', -k') \in T^*(\mathcal{O}\times\mathcal{O})\setminus\{0\} \mid (x,k) \dot{\sim} (x',k'), k \rhd 0\}$$

Note that: $\omega_{2,\kappa}(x,x') = \Theta(z)\,\Theta(z')\,\tilde{\omega}_{2,\kappa}(x,x')$, $WF(\omega_{2,\kappa}) = WF(\tilde{\omega}_{2,\kappa})$.

(1) M. J. Radzikowski, Comm. Math. Phys. **179**, 529 (1996)

Proof. $(1 \Rightarrow 2)$ On $\mathcal{O} \subseteq \mathbb{R}^d$, define:

$$b_1(x,x') := \lim_{\varepsilon \to 0^+} \frac{1}{\sigma_\varepsilon^{\frac{d-2}{2}}(x,x')} , \qquad b_2(x,x') := \lim_{\varepsilon \to 0^+} \ln\left(\sigma_\varepsilon(x,x')\right)$$

$$\overset{(1)}{\Longrightarrow} \quad WF(b_i) = \{(x,k,x',-k') \in T^*(\mathcal{O} \times \mathcal{O}) \setminus \{0\} \mid (x,k) \sim (x',k'), k \triangleright 0\}$$

Let $b_{i,-} := (i_z^* \otimes id) b_i$, $i = 1,2$. Since i_z is a diffeomorphism,

$$WF(b_{i,-}) = WF((i_z^* \otimes id) b_i) = \{(i_z(x), i_z^*(k), x', -k') \in T^*(\mathcal{O} \times \mathcal{O}) \setminus \{0\} \mid (x,k) \dot{\sim} (x',k'), k \triangleright 0\}$$

Note that: $\omega_{2,\kappa}(x,x') = \Theta(z)\Theta(z')\tilde{\omega}_{2,\kappa}(x,x')$, $WF(\omega_{2,\kappa}) = WF(\tilde{\omega}_{2,\kappa})$.

$(2 \Rightarrow 1)$ Consider $\omega'_{2,\kappa} \neq \omega_{2,\kappa}$ as per 2. On $\mathcal{O} \subseteq \mathbb{H}^d$, let

$$\Delta\omega_{2,\kappa} = \omega'_{2,\kappa} - \omega_{2,\kappa}$$

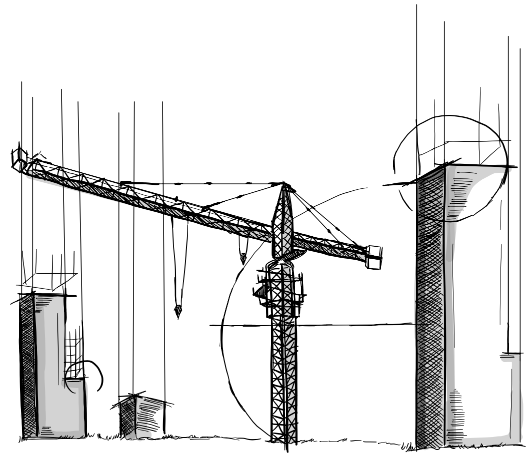
$$\Rightarrow \Delta\omega_{2,\kappa}(x,x') = \Delta\omega_{2,\kappa}(x',x) \;\&\; WF(\Delta\omega_{2,\kappa}) \subset WF(\omega_{2,\kappa})$$

$$\Rightarrow WF(\Delta\omega_{2,\kappa}) = \emptyset$$

□

(1) M. J. Radzikowski, Comm. Math. Phys. **179**, 529 (1996)

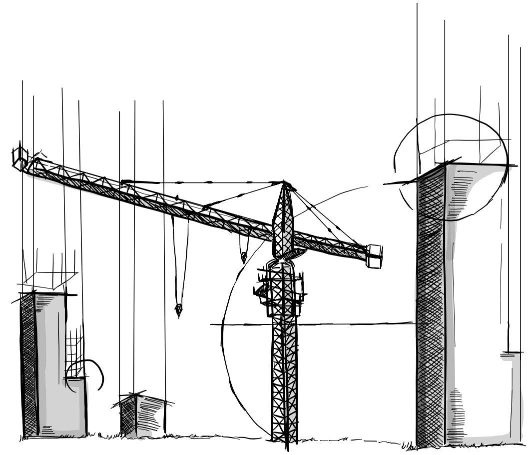
On-going Projects and Outlooks



- Generalization to curved spacetime[(1)]

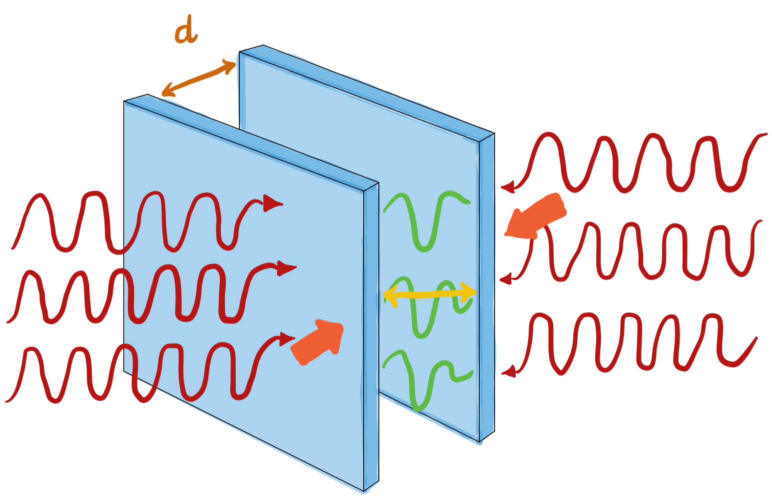
(1) B.C., C. Dappiaggi, B.A. Juárez-Aubry, *In preparation*

On-going Projects and Outlooks



- Generalization to curved spacetime[1]

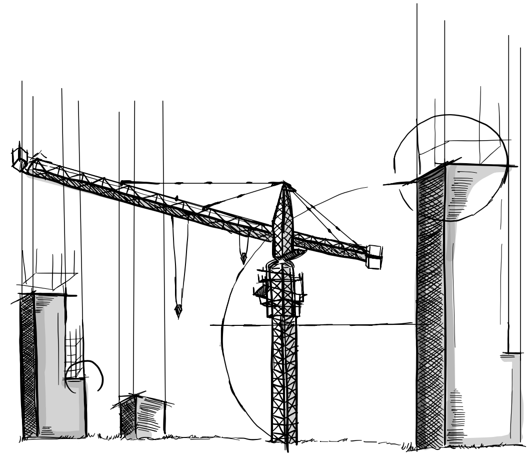
•• Dynamical Casimir Effect[2] in Random media



(1) B.C., C. Dappiaggi, B.A. Juárez-Aubry, *In preparation*

(2) C. Dappiaggi, A. Marta, MPAG, **24** (3) (2021)

On-going Projects and Outlooks

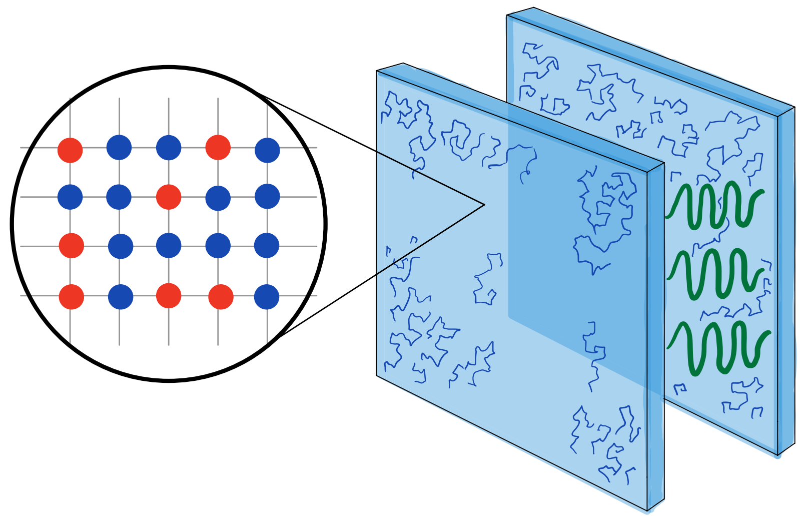


- Generalization to curved spacetime[1]

•• Dynamical Casimir Effect[2] in Random media

••• Anderson model[3]

$$H_L f = -\Delta f + f\xi \quad \text{on} \quad \left(-\frac{L}{2}, \frac{L}{2}\right)^d$$



(1) B.C., C. Dappiaggi, B.A. Juárez-Aubry, *In preparation*

(2) C. Dappiaggi, A. Marta, MPAG, **24** (3) (2021)

(3) C. Labbé, J. Func. Anal., **227** (9) (2019)