An Invitation to Operational Quantum Reference Frames

Leon Loveridge
Mathematical Physics Seminar, CFS Group, 2026

University of South-Eastern Norway

January 30, 2026

Why Quantum Reference Frames?

If one accepts the following three statements, QRFs emerge naturally:

1. Quantum mechanics (QFT) is empirically precise and accurate
2. Quantum mechanics (QFT) is universal
3. Observable quantities are invariant under XXX

E.g., fundamental particles are often associated with irreducible representations of the Poincare group. Tension with 3.

If you don't like these principles, I have others. E.g. Physical situations related by a symmetry, need a (Q)RF, Einstein's *operational* clocks and rods, in a quantum setting.

The reference system/frame is described by the same physical theory as the systems under consideration

Some recent uses (operational framework)[4]

- Helps us to understand better what QM 'is about'
- Frame changes - observables and states relative to frame, can move consistently between descriptions relative to different frames[1]
- Uncertainty relations - QRFs in phase space, frame-relative uncertainties[2]
- 'Type reduction' in AQFT - incorporation of frame 'regularises' the observable algebra[3]

---

[1] *Operational Quantum Reference Frame Transformations* T. Carette, J. Glowacki, LL, Quantum 2025

[2] *Uncertainty Relations Relative to Phase Space Quantum Reference Frames*, M. Jorquera Riera, L.L, PRA Letters 2025

[3] *Quantum Reference Frames, Measurement Schemes and the Type of Local Algebras in Quantum Field Theory*, C. J. Fewster, D. W. Janssen, L.L, K. Rejzner, James Waldron CMP 2025

[4] Other frameworks are available!! E.g., *Perspective-neutral*, P. Hoehn et al, *Extra particle*, Castro-Ruiz, Oreshkov...

Operational Theories I

- Most basic experiment we can do: prepare (T - '*state*') and measure (E - '*observable*)' a system the same way many times (n)
- For each possible outcome $\omega_i \in \Omega$, write n_i for the frequency
- Assign probabilities

$$p_T^{\mathsf{E}}(\omega_i) = \lim_{n \to \infty} \frac{n_i}{n} \tag{1}$$

- Call $E_i : T \mapsto p_T^{\mathsf{E}}(\omega_i)$ *effect*
- Mixing: convex structure on states, effects are (in the) affine maps
- Linear (plus convex) structure on effects - states as affine functionals on convex set of effects through $\hat{T}(E_i) = E_i(T)$
- State space is convex subset of a vector space, total convexity gives Banach structure on dual (base norm space, order unit Banach space)
- Observable $E : \omega_i \mapsto E_i$ 'effect valued measure'
- Pure states: extremal elements of convex state space

Operational Theories II - Classical probability theory

- States are probability measures on Ω, pure states delta measures
- State space is a *simplex* (unique pure state decomposition)

Various ways to think of observables:

- Effect-valued measure, extremal effects are idempotent (projections = characteristic functions)
- Extremal effects give real-valued functions $C(\Omega)$ (Gelfand spectrum $\Sigma(C(\Omega)) \cong \Omega$)
- Algebraic point of view recovered from operational or probabilistic ideas

Operational theories III - Quantum Theory

Quantum Theory:

- (Normal) state space $\mathcal{S}(\mathcal{H}) \subset \mathcal{T}(\mathcal{H}) \cong B(\mathcal{H})_*$, pure states rank-1 projections
- Effects $\mathcal{E}(\mathcal{H})$ are operators in $[\mathbb{0}, \mathbb{1}] \subset B(\mathcal{H})$
- Suppose now outcome space Ω is topological space,
- Observables are POVMs $\mathsf{E} : \mathcal{F}(\Omega) \to \mathcal{E}(\mathcal{H})$ ($\mathsf{E}(X) \geq 0$, additive on disjoint sets, $\mathsf{E}(\Omega) = \mathbb{1}$)
- Sharp observables on $\mathbb{R}$ are PVMs = s.a. ops through spectral theorem, C^* algebras etc
- Born rule

$$p_\omega^{\mathsf{E}}(X) = \mathrm{tr}[\omega \mathsf{E}(X)] \tag{2}$$

- POVMs exhaust the probabilistic structure of Hilbert space: any set function $\mathsf{E} : \mathcal{F}(\Omega) \to B(\mathcal{H}^+)$ is a POVM iff $X \mapsto \langle x | \mathsf{E}(X) x \rangle$ is a measure

POVMs in QM: overview

- Natural fit with probabilistic/operational structure
- Arise naturally in measurement (cf. Busch, Fewster/Verch)
- Can be used to model 'classical noise' (unsharp version of sharp observable)
- Can be used to model 'intrinsic unsharpness' (time, phase)
- Allow for operational definition of incompatibility: POVMs A on Ω_A and B on Ω_B if there is an M on $\Omega_A \times \Omega_B$ such that $\mathrm{A}(X) = \mathrm{M}(X \times \Omega_B)$ and $\mathrm{B}(Y) = \mathrm{M}(\Omega_A \times Y)$ (need not commute)

Our main interest will be in *Covariant POVMs* ('transform well under a representation')

Some POVMs I

- Pauli (1926): Given self-adjoint operator A with spectrum bounded below, there does not exist (s.a.) B for which $[A, B] = i\mathbb{1}$.
- (Nearly!) equivalently, no PVM E^B on $\mathcal{B}(\mathbb{R})$ for which $e^{itA}\mathsf{E}^B(X)e^{-itA} = \mathsf{E}^B(X + t)$

Some POVMs satisfy the covariance, e.g.,

- Ex.1: Standard number observable $N = \sum_{n\geq 0} nP_n$, 'canonical phase'

$$\mathsf{E}^{\text{phase}}(X) = \sum_{n,m=0}^{\infty} c_{n,m} \int_X e^{i(n-m)\theta} |n\rangle\langle m| d\theta \ , \ X \in \mathcal{B}((0, 2\pi]) \quad (3)$$

is covariant (addition modulo 2π) under conjugation.

Some POVMs II

Ex.2: 'Hamiltonian' given by q_+ in $L^2(\mathbb{R}_+)$

- Can construct covariant time observable $E^T : \mathcal{B}(\mathbb{R}) \to B(L^2(\mathbb{R}_+))$:

$$e^{itq_+} E^T(X) e^{-itq_+} = E^T(X+t). \tag{4}$$

- Construction follows Naimark: embed $L^2(\mathbb{R}_+)$ in $L^2(\mathbb{R})$ where the sharp momentum conjugate to q lives, project back down.
- Example of a covariant Naimark dilation (always exists)

Some POVMs III

Ex. 3: Covariant smearings of (spectral measures of) Q, P give joint observable as covariant phase space POVM (covariant under phase space translations)

$$\mathsf{G}^T(Z) := \frac{1}{2\pi}\int_Z W(q,p)TW(q,p)^* dqdp \in B(L^2(\mathbb{R}))$$

$$\mathsf{G}^T(X\times\mathbb{R}) = (\mu_T \star \mathsf{Q})(X) = \mathsf{Q}^{\mu_T}(X); \tag{5a}$$
$$\mathsf{G}^T(\mathbb{R}\times Y) = (\nu_T \star \mathsf{P})(Y) = \mathsf{P}^{\nu_T}(Y), \tag{5b}$$

where

$$\mu_T(X) = \mathrm{tr}[\Pi T \Pi\, \mathsf{Q}(X)]; \tag{6a}$$
$$\nu_T(Y) = \mathrm{tr}[\Pi T \Pi\, \mathsf{P}(Y)]. \tag{6b}$$

'Measurement Uncertainty Relation' $\Delta(\mathsf{Q}^{\mu_T},\rho)\Delta(\mathsf{P}^{\nu_T},\rho) \geq 1$

Even More POVMs

- Von Neumann algebra $\mathcal{M}$ comes with intrinsic dynamics (1-parameter group of automorphisms, φ faithful normal state)

$$\sigma_t^{\varphi}(A) = \Delta^{-it} A \Delta^{it} \in \mathcal{M} \text{ for all } t \in \mathbb{R} \tag{7}$$

- Δ modular operator (depends on state, but not strongly)
- Generalises to faithful normal semifinite weight
- σ_t is the unique 1-parameter automorphism group for which
  - $\varphi \circ \sigma_t = \varphi$ for all t
  - φ satisfies the KMS condition (roughly existence of holo. F on strip s.t. $F(t) = \varphi(\sigma_t(a)b)$ and $F(t+i) = \varphi(b)\sigma_t(a)$ for a, b satisfying some finiteness conditions)
- Some work on when thermal time is unsharp (covariant POVM) [5]

Never ending POVMs

POVMs arise through measurement:

- Apparatus $\mathcal{H}_\mathcal{A}$, pointer observable P^Z, initial state ϕ, unitary coupling U on $\mathcal{H}_\mathcal{S} \otimes \mathcal{H}_\mathcal{A}$ such that for all $\varphi \in \mathcal{H}_\mathcal{S}$:

$$\left\langle U\varphi \otimes \phi \middle| \mathbb{1} \otimes \mathsf{P}^Z(X)(U\varphi \otimes \phi)\right\rangle = \langle \varphi | \mathsf{E}(X)\varphi\rangle \tag{8}$$

  or $\mathsf{E}(X) = \Gamma_\phi(U^* \mathbb{1} \otimes \mathsf{P}^Z(X)U)$ - probability reproducibility
- Two readings: (i) LHS is some Naimark dilation of RHS (ii) LHS is fixed and dictates measured POVM E
- $\langle \mathcal{H}_\mathcal{A}, U, \mathsf{P}^Z, \phi\rangle$ is called a *measurement scheme* for E, recently extended to AQFT by Fewster and Verch [4]

(Operational) Quantum reference frames: motivation

- Start from a general principle - ‘true observables’ are (gauge-)invariant (differs from other contemporary approaches)
- l.c.s.c. G, unitary rep U in $\mathcal{H}$
- Nothing left if U irreducible (cannot observe single particles?)
- We know invariance and ‘relativity’ intimately connected: introduce frame, stipulate invariance for system-plus-frame (c.f. relative position, etc)
- Within invariants $B(\mathcal{H})^G$, some special ‘relative’ (relational) observables, which we will get at next
- NB: may be no invariant states, but can identify states which agree on $B(\mathcal{H})^G$, write $S(\mathcal{H})_G$

(Operational) Quantum Reference Frames: definition, taxonomy

A quantum reference frame based $\Sigma = G/H$ is a *System of Covariance* $(U_R, \mathsf{E}_R, \mathcal{H}_\mathcal{R})$.

- Sharp if E is a PVM (and unsharp otherwise)
- Principal if Σ is a principal homogeneous space
- Ideal if it is both principal and sharp
- Compactly stabilised if $\Sigma \cong G/H$ with $H \subset G$ compact
- Localisable if E satisfies the norm-1 property
- Complete if there is no (non-trivial) subgroup $H_0 \subset G$ acting trivially on all the effects of E
- Coherent state systems, informationally complete, etc
- Covariant POVMs are related to 'standard' covariant PVMs (Imprimitivity) through covariant dilation/compression (complete characterisation due to Mackey, Cattaneo, others)

Relative states and observables for principal frames

We can *relativise* observables with $\yen^R : B(\mathcal{H}_\mathcal{S}) \to B(\mathcal{H}_\mathcal{S} \otimes \mathcal{H}_\mathcal{R})^G$:

$$\yen^R(A) = \int_G U_S(g) A U_S(g)^* \otimes \mathsf{E}^\mathcal{R}(dg) \tag{9}$$

- Linear completely positive normal unital contraction, *-preserving, injective homomorphism iff E is PVM (also works for compactly stabilised)
- Relative observables $B(\mathcal{H}_\mathcal{S})^R := \mathrm{Im}(\yen^R)$ (u.w. closure)
- Identify trace class ops indistinguishable on $B(\mathcal{H}_\mathcal{S})^R$:

$$\mathcal{T}(\mathcal{H}_\mathcal{S})^R := \mathcal{T}(\mathcal{H}_\mathcal{S} \otimes \mathcal{H}_\mathcal{R})/\sim_\mathcal{R}; \quad \mathcal{T}(\mathcal{H}_\mathcal{S})^R \cong B(\mathcal{H}_\mathcal{S})^R_* \tag{10}$$

- Relative states:

$$\mathcal{S}(\mathcal{H}_\mathcal{S})^R := \mathcal{S}(\mathcal{H}_\mathcal{S} \otimes \mathcal{H}_\mathcal{R})/\sim_R \cong \yen^R_*(\mathcal{S}(\mathcal{H}_\mathcal{R} \otimes \mathcal{H}_\mathcal{S})); \tag{11}$$

Restriction and conditioning

Fix frame state ω, define $\Gamma_\omega : B(\mathcal{H}_\mathcal{S} \otimes \mathcal{H}_\mathcal{R}) \to B(\mathcal{H}_\mathcal{S})$ by $\Gamma_\omega(A \otimes B) = A\,\mathrm{tr}[\omega B]$ (extended by linearity, continuity). CP normal conditional expectation.

- Frame-conditioned relative observables $\mathrm{Im}(\yen^R_\omega) := \Gamma_\omega \circ \yen^R$
- If A is invariant, $\yen^\mathcal{R}_\omega(A) = A$ for all frame states ω (frame-independent)
- Identify states, quotient, $[B(\mathcal{H}_\mathcal{S})^R_\omega]_* \cong \mathcal{T}(\mathcal{H}_\mathcal{S})^R_\omega$
- $\rho^{(\omega)} = \yen^R_*(\rho \otimes \omega)$; $\rho^{(g.\omega)} = (g^{-1}.\rho)^{(\omega)}$ (active/passive)
- E.g. ρ invariant, then $\rho^{(\omega)} = \rho$ for any ω

Theory is most tractable when frame is localizable (norm-1 property): for any X s.t. $\mathrm{E}(X) \neq 0$, $\exists$ a sequence (ω_n) of states s.t. $\lim \omega_n(\mathrm{E}(X)) = 1$.

Localization

'Fundamental theorem of OQRFs'?

Theorem

Let $\mathcal{R} = (U_\mathcal{R}, \mathrm{E}_\mathcal{R}, \mathcal{H}_\mathcal{R})$ be a localizable principal frame and (ω_n) a localizing sequence centered at $e \in G$. Then for any $A_\mathcal{S} \in B(\mathcal{H}_\mathcal{S})$ we have

$$\lim_{n\to\infty} (\Gamma_{\omega_n} \circ \yen^\mathcal{R})(A_\mathcal{S}) = A_\mathcal{S}, \tag{12}$$

- Ordinary QM captures relation between quantum system and 'suitably classical' reference
- Not possible for bad frames (lower bound)
- Dual result: $\lim_{n\to\infty} \yen_*(\rho \otimes \omega_n) = \rho$; (conditioned) relative states are dense (and converse - bad agreement for very unsharp frames)

Applications II: Frame-relative Uncertainty Relations

Can take covariant phase space observable as frame, relativise and restrict

- $¥^T_\omega \equiv \Gamma_\omega \circ ¥^T$ breaks the incompatibility of position and momentum for any T, ω of the frame
- Novel frame-dependent uncertainty relation (several of these)

$$\Delta(¥^T_\omega \circ \mathsf{Q}_\mathcal{S}, \rho)\Delta(¥^T_\omega \circ \mathsf{P}_\mathcal{S}, \rho) \geq 3/2. \tag{13}$$

- cf $\Delta(\mathsf{Q}, \rho)\Delta(\mathsf{P}, \rho) \geq 1/2$ and $\Delta(\mathsf{Q}^{\mu_T}, \rho)\Delta(\mathsf{P}^{\nu_T}, \rho) \geq 1$
- Which is right? Experiment?
- Classical limit (very dodgy): set ω, T localised at $(0, 0)$ (Landsman framework to make rigorous?)

$$\Delta(¥^{T_0'}_{\omega_0} \circ \mathsf{Q}_\mathcal{S}, \rho)\Delta(¥^{T_0'}_{\omega_0} \circ \mathsf{P}_\mathcal{S}, \rho) \geq 1/2 \tag{14}$$

Applications III: 'type reduction'

- Algebraic Quantum Field Theory: Manifold M, regions $U \subset M$, von Neumann algebras $m\mathcal{M}(U)$, axioms (isotony, Einstein Causality etc)
- Local algebras in AQFT are typically Type III von Neumann algebras (no good trace, no entropies) (hyperfinite factor of type III_1!)

Types

Choice of algebra comes from physics

- I_n 'qubits' etc ($\mathcal{H} = \mathbb{C}^n$, $\mathcal{M} = M_n(\mathbb{C})$)
- I_∞ 'wave mechanics' ($\mathcal{M} = B(L^2(\mathbb{R}))$)
- II_1 maths: i.c.c. groups, physics: infinite spin chain (with conditions!)
- II_∞ e.g. $II_1 \otimes I_\infty$
- III quantum field theory (mathematical example: $\mathrm{II}_\infty \rtimes \mathbb{R}$ for suitable action of $\mathbb{R}$)

Can be characterised by **traces**:

- $\tau : \mathcal{M}_+ \to [0, \infty]$ with $\tau(u^* a u) = \tau(a)$ (plus additive, homogeneous)
- Called finite if never ∞
- Semifinite if finite trace elements are dense
- If finite, extends to $\mathcal{M}$ with cyclicity $\tau(ab) = \tau(ba)$

Characterisation

Full characterisation (factors) comes from value of trace on $\mathcal{P}(\mathcal{M})$; here we simplify:

- I_n **Finite trace**
- I_∞ **Semifinite trace** (semi-finite normal trace is the usual one)
- II_1 **Finite trace**
- II_∞ **Semifinite trace**
- III **No finite trace**

This is a problem. Want to understand, e.g., black hole or cosmological entropies. E.g., Bekenstein, Hawking (black holes), Hawking + Gibbons - areas of cosmological horizons are entropies. How should we understand this as part of (A)QFT proper? Need traces.

Chandrasekaran, Longo, Penington, and Witten

- CLPW [1]: 'observer' on a worldline in de Sitter 'static patch' (causally accessible region)
- Time translation-invariant algebra (plus details) gives II_1 - *type reduction*
- 'FJLRW': view the observer as a **quantum reference frame**, use and develop existing machinery to make more precise, operational, general

Big picture

- Theory based on local measurements appears to break diffeomorphism/symmetry invariance. Suppose weak gravity, fixed background, interested in isometries, true observables isometry-invariant (system + frame)
- We will analyse the type of a QFT on a background with isometry group $\mathbb{R} \times H$ (H compact) 'relative to' a QRF on G/K (K compact).
- We do this by describing the algebra of invariants of system + frame together (see also [2])
- The techniques are based on crossed products and modular theory, combined with systems of covariance (QRFs) and induced representations.

QRFs

Mackey 1949, Cattaneo 1979: our QRFs are subrepresentations of induced representations

- QRF $(U, E, \mathcal{H}_\mathcal{R})$ on $\Sigma = G/H$ (assume H compact)
- $W : \mathcal{H}_R \to \mathcal{K} \otimes L^2(G)$, new QRF given by

$$(1_\mathcal{K} \otimes \lambda, 1_\mathcal{K} \otimes P, \mathcal{K} \otimes L^2(G));$$

$$E(X) = W^*(1_\mathcal{K} \otimes P(X))W \tag{15}$$

- Any QRF is of the form

$$(1_\mathcal{K} \otimes \lambda|_{\mathcal{H}_\mathcal{R}}, p(1_\mathcal{K} \otimes P)|_{\mathcal{H}_\mathcal{R}}, p(\mathcal{K} \otimes L^2(G))); \tag{16}$$

obtained by setting $p = WW^*$ Naimark projection.

Invariants and Crossed Products: General

Set-up: QFT Von Neumann algebra $\mathcal{M}_S$ in $\mathcal{H}_S$ with action α_S of G, compactly stabilised QRF $(U_R, E, \mathcal{H}_\mathcal{R})$. Want $(\mathcal{M}_S \otimes B(\mathcal{H}_\mathcal{R}))^{\alpha_S \otimes \mathrm{Ad} U_R}$.

- Well known: $(\mathcal{M} \otimes B(L^2(G))^{\alpha \otimes \mathrm{Ad}\lambda} = \mathcal{M} \rtimes_\alpha G$
- For us: There is a Hilbert space $\mathcal{K}$ and projection $p \in \lambda(G)' \otimes B(\mathcal{K})$ such that

$$(\mathcal{M}_S \otimes B(\mathcal{H}_\mathcal{R}))^{\alpha_S \otimes \mathrm{Ad} U_R} \cong (1_{\mathcal{H}_S} \otimes p)((\mathcal{M} \rtimes_{\alpha_S} G) \otimes B(\mathcal{K}))(1_{\mathcal{H}_S} \otimes p) \tag{17}$$

- Note that the action on $B(\mathcal{K})$ is trivial and the iso above is spatial through $\mathcal{H}_\mathcal{R} \cong p(L^2(G) \otimes \mathcal{K})$
- α_S is unitarily implementable - some simplifications can be made

Invariants and Crossed Products: $\mathbb{R}\times$compact

Theory is tractable for $G = \mathbb{R} \times H$ (H compact) plus compactly stabilised QRF

- $\mathcal{H}_{\mathcal{R}} \cong p(L^2(\mathbb{R}) \otimes L^2(H) \otimes \mathcal{K})$
- Assume that $\mathcal{M}_S$ has a faithful normal KMS state ω (inverse temp. β) and the corresponding modular action is unitarily implemented, then

$$(\mathcal{M}_S \otimes B(\mathcal{H}_R))^{\mathrm{Ad}\, U_S\restriction_{\mathbb{R}} \otimes U_R\restriction_{\mathbb{R}}}$$
$$\cong (1_{\mathcal{H}_S} \otimes p)((\mathcal{M}_S \rtimes_{\mathrm{Ad}\, U_S\restriction_{\mathbb{R}}} \mathbb{R}) \otimes B(L^2(H) \otimes \mathcal{K}))(1_{\mathcal{H}_S} \otimes p).$$

- We will now analyse the structure of this algebra

Type reduction I

- Well known that $(\mathcal{M}_S \rtimes_{\mathrm{Ad}\, U_S\restriction_{\mathbb{R}}} \mathbb{R})$ is semifinite
- Can show that $(\mathcal{M}_S \otimes B(\mathcal{H}_R))^{\mathrm{Ad}\, U_S\otimes U_R}$ is semifinite
- Proof comes from constructing a semi-finite trace τ
- **Finite** on $(1_{\mathcal{H}_S} \otimes p)((\mathcal{M}_S \rtimes_{\mathrm{Ad}\, U_S\restriction_{\mathbb{R}}} \mathbb{R}) \otimes B(L^2(H) \otimes \mathcal{K}))(1_{\mathcal{H}_S} \otimes p)$ if and only if
$$\tau(1_{\mathcal{H}_S} \otimes p) < \infty \tag{18}$$
- Finiteness of $(\mathcal{M}_S \otimes B(\mathcal{H}_R))^{\mathrm{Ad}\, U_S\otimes U_R}$ follows
- Can be shown to hold if there is a KMS weight on frame at inverse temperature β ($=$ inverse temp of KMS state on $\mathcal{M}_S$)
- If H is trivial and $\mathcal{M}_S$ is a III_1 *factor*, then $(\mathcal{M}_S \otimes B(\mathcal{H}_R))^{\mathrm{Ad}(U_S\otimes U_R)}$ is a type II_1 factor

As a Theorem

(See [3], Thm. 5.5)
Let $\mathcal{M}_S$ describe a QFT, and suppose there is a (faithful, normal, $G-$invariant) KMS state ω, and $\mathcal{M}_S$ is acted on unitarily by $G = \mathbb{R} \times H$ (H compact). When coupled to a compactly stabilised QRF, the algebra

$$(\mathcal{M}_S \otimes B(\mathcal{H}_R))^{\mathrm{Ad}U_S \otimes U_R} \tag{19}$$

is semifinite. If moreover the frame admits a normal KMS weight (with finite values on a dense subalgebra of $B(\mathcal{H}_R)$) at the same inverse temperature as ω, resulting algebra is II_1 (finite!). (Missing here - explicit constraint on frame properties that gives sufficient condition for finiteness - see paper!)

CLPW revisited: $G = \mathbb{R} \times SO(n-1)$

Consider the $SO(n-1)$ part to act trivially on system and frame, CLPW [1] Hamiltonian q_+.

- Hilbert space of frame is $\mathcal{H}_\mathcal{R} = p_+ L^2(\mathbb{R})$
- Time observable is $E^T(X) = p_+ \mathcal{F} T_{\chi_X} \mathcal{F}^*$
- Dilation is canonical PVM T (self-adjoint momentum)
- (Time translation) invariant algebra is

$$(\mathcal{M}_S \otimes B(\mathcal{H}_R))^{\mathrm{Ad}\, U_S|_\mathbb{R} \otimes U_R|_\mathbb{R}} = (1 \otimes p_+)(\mathcal{M}_S \rtimes_{\mathrm{Ad}\, U_S|_\mathbb{R}} \mathbb{R})(1 \otimes p_+) \quad (20)$$

- Crossed product takes us from III to II_∞, then the projection p_+ takes us from II_∞ to II_1
- This final step turns the sharp time into an unsharp one

Comments

- We have shown that the inclusion of a QRF with given properties yields a II_1 algebra of observables
- Type reduction takes place in 'stages': modular crossed product plus projection
- II_1 follows from QRF admitting a normal KMS weight at inverse temp β
- Non-triviality of *p* means QRF is non-ideal (unsharp time observable) - unsharpness regularises QFT
- Which unsharp time observables yield type reduction? Inevitable for QFT QRFs?
- New theory is 'quantum stat mech' rather than QFT. Meaning?
- Gravity?!

Summing up

- QRFs are indispensable for describing QM and QFT when there is symmetry
- Lots to do: QRF is a QFT, relational AQFT 'from scratch',...

Some more references

[1] V. Chandrasekaran, R. Longo, G. Penington, and E. Witten. An algebra of observables for de sitter space. *Journal of High Energy Physics*, 2023(2):1–56, 2023.

[2] J. De Vuyst, S. Eccles, P. A. Höhn, and J. Kirklin. Crossed products and quantum reference frames: on the observer-dependence of gravitational entropy. *Journal of High Energy Physics*, 2025(7):1–118, 2025.

[3] C. J. Fewster, D. W. Janssen, L. D. Loveridge, K. Rejzner, and J. Waldron. Quantum reference frames, measurement schemes and the type of local algebras in quantum field theory. *Communications in Mathematical Physics*, 406(1):1–81, 2025.

[4] C. J. Fewster and R. Verch. Quantum fields and local measurements. *Communications in Mathematical physics*, 378:851–889, 2020.

[5] J. van Neerven and P. Portal. Thermal time as an unsharp observable. *Journal of Mathematical Physics*, 65(3), 2024.

Types

von Neumann algebras are built from *factors*, and each factor has a *type*, characterised by properties of projections, or equivalently traces:

- $B(\mathcal{H})$ type I_∞ (faithful normal semifinite trace with values on projections in $\mathbb{N}_0 \cup \{\infty\}$)
- $\mathcal{R}(G) =$ type II_1 (faithful normal tracial state with values on projections in $[0, 1]$)
- $\mathcal{A}$ type II_∞ (faithful normal semifinite trace with values on projections in $[0, \infty]$)
- $\mathcal{A} \rtimes \mathbb{R}$ type III for some given actions of $\mathbb{R}$ (no good trace)

Semidirect products

Suppose we have a group K acting on another group H by automorphisms, i.e., $\varphi : K \to \mathrm{Aut} H$ is a homomorphism. Question: Can we construct a group in a 'natural' way that 'encodes' the action of K on H?

Yes! The ('outer') semidirect product $H \rtimes_\varphi K$. As a set, this is $H \times K$, but the group law is

$$(h, k).(h', k') = (h.\varphi_k(h'), k.k') \tag{21}$$

Now suppose that we have a group G acting by automorphisms on a von Neumann algebra $\mathcal{M}$. Question: Can we construct a von Neumann algebra in a 'natural' way that 'encodes' the action of G on $\mathcal{M}$? Yes...

Crossed product

Let $\mathcal{M}$ be a von Neumann algebra, G a (l.c.s.c.H) group, $\alpha : G \to \mathrm{Aut}\mathcal{M}$ (covariant system $(\mathcal{M}, G, \alpha)$). Construct $L^2(G, \mathcal{H})$ and fix a representation of $\mathcal{M}$ as

$$(\pi_\alpha(x)f)(g) = \alpha_g^{-1}(x)f(g) \tag{22}$$

and of G as

$$(\lambda(g)f)(h) = f(g^{-1}h). \tag{23}$$

Then $\mathcal{M} \rtimes_\alpha G = \{\pi_\alpha(M), \lambda(G)\}''$.

[Connection to previous example: $\mathcal{G}(H \rtimes K) \cong \mathcal{G}(H) \rtimes K$]