A baryogenesis mechanism for causal fermion systems

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October 8th 2025,
Causal Fermion Systems 2025 Conference (Regensburg)

Introduction

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- *Baryogenesis*: Matter creation process responsible for the observed asymmetry.
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  (i) **On the dynamics of the spinors:** Spinors evolve according to a slight modification of Dirac dynamics.

  (ii) **On the regularity of spacetime:** On the very small scales, below a minimum length $\varepsilon > 0$, the manifold structure breaks down.
  $\rightarrow$ Consequence: Start from distributional sections of *SM* (e.g., weak solutions to the Dirac equation) and regularize them.

Introduction

(iii) **On the physical input:** The regularization operators

$$R_\varepsilon : W^{1,2}_{\mathrm{loc}}(M, SM) \to C^0(M, SM)$$

have an important physical significance and depend on the spacetime point: for any $p \in M$, $R_\varepsilon(p) : W^{1,2}_{\mathrm{loc}}(M, SM) \to S_p M$.
Moreover, the dynamics of R_ε:

$$M \to \mathcal{L}(W^{1,2}_{\mathrm{loc}}(M, SM), SM) \ , \quad p \mapsto R_\varepsilon(p)$$

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In this talk...

$$\text{Dynamics of } R_\varepsilon \to \text{Spinor dynamics} \to \text{Baryogenesis?}$$

The setup: how to model baryogenesis?

- In $(\mathbb{R}^4, \eta)$ the Dirac Hamiltonian H_η is a selfadjoint operator with absolutely continuous spectrum $\sigma(H_\eta) = (-\infty, -m] \cup [m, \infty)$.
  $\rightarrow$ *Particles* (resp. *antiparticles*) are eigenstates associated to positive (resp. negative) eigenvalues of H_η.

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  $\rightarrow$ *Particles* (resp. *antiparticles*) are eigenstates associated to positive (resp. negative) eigenvalues of H_η.
- **Goal:** In a globally hyperbolic (M, g), describe/quantify baryogenesis as the relative change of spectral subspaces of an operator which *generalizes* H_η.

The setup: Preliminaries

Geometric setup:

- From now on, consider conformally flat spacetimes $(\mathbb{R}^4, g)$, i.e.

$$g = \Omega^2 \eta = \Omega^2(t, r, \theta, \varphi)(dt^2 - dr^2 - r^2 d\theta^2 - r^2 \sin\theta^2 d\varphi^2)$$

  with $\Omega : M \to (0, \infty)$ smooth and the foliation $(N_t)_{t\in\mathbb{R}}$ given by the level sets of the global time function t.

- Moreover, we denote

$$\mathcal{H}_{t,g} := L^2(N_t, SM) \,,$$

  with a suitable spinor scalar product $(\cdot|\cdot)_t$.

- Finally, *the regularizing vector field* $u : M \to TM$ is a timelike and future directed vector field describing the regularization dynamics.

The setup: Main concepts

Definition:

(i) The *symmetrized Hamiltonian* A_t with $D(A_t) = C_0^\infty(N_t, SM)$,

$$A_t := \frac{1}{4}\{u^0, H_g + H_g^*\} + \frac{i}{4}\{u^\mu, \nabla^s_\mu - (\nabla^s_\mu)^*\} \ ,$$

is an essentially self-adjoint operator (cf. [4, Lem. 5.3], proof relies on Chernoff's criteria, see [1]).
$\rightarrow$ From now on, denote with the same symbol the self-adjoint extension $A_t : \mathcal{D} \supset C_0^\infty(N_t, SM) \rightarrow \mathcal{H}_{t,g}$.

Remark:

- Given the selfadjoint operator $A_t : \mathcal{D} \supset C_0^\infty(N_t, SM) \to \mathcal{H}_{t,g}$, $\chi_I(A_t) : \mathcal{D} \to \mathcal{H}_{t,g}$ is a densely defined bounded operator ($I \subset \mathbb{R}$). Hence, there exists a unique extension

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$$\chi_I(A_t) : \mathcal{H}_{t,g} \to \mathcal{H}_{t,g}$$

- If (M, g) has a bounded geometry and I is a bounded interval, $\chi_I(A_t)$ is a regularization operator (cf. [5, Proposition 2.7], proof relies on elliptic regularity):

$$\chi_I(A_t) : \mathcal{H}_{t,g} \to C^\infty(N_t, SM)$$

Its dynamics is determined by the dynamics of $u : M \to TM$.

The setup: Main concepts

(ii) The *locally rigid operator* $V_{t_0}^t : \mathcal{H}_{t_0}^{\varepsilon} \to \mathcal{H}_{t,g}$ is

$$V_{t_0}^t := \lim_{k_{\max}\to\infty} \chi_I(A_t) U_{t-\Delta t}^t \cdots \chi_I(A_{t_0+\Delta t}) U_{t_0}^{t_0+\Delta t} \quad \text{with} \quad \Delta t := \frac{t-t_0}{k_{\max}}.$$

where $\mathcal{H}_{t_0}^{\varepsilon} := \chi_I(A_{t_0})(\mathcal{H}_{t_0,g}) \subset \mathcal{H}_{t_0,g}$, $I := (-1/\varepsilon, -m)$ and for any $t_k < t_{k+1}$, $U_{t_k}^{t_{k+1}} : \mathcal{H}_{t_k,g} \to \mathcal{H}_{t_{k+1},g}$ is the unitary Dirac evolution operator.
Moreover, set $\mathcal{H}_t^{\varepsilon} := V_{t_0}^t(\mathcal{H}_{t_0}^{\varepsilon})$.

The setup: Main concepts

(iii) The *rate of baryogenesis*:

$$B_t := \frac{d}{dt}\mathrm{tr}_{\overline{\mathcal{H}_t^\varepsilon}}(\eta(\tilde{H}_\eta)(\chi_I(A_t) - \chi_I(\tilde{H}_\eta))$$

with $I := (-1/\varepsilon, -m)$ and $\tilde{H}_\eta := \tilde{U}^{-1} H_\eta \tilde{U}$, where $\tilde{U} : \mathcal{H}_{t,g} \to \mathcal{H}_{t,\eta}$ is a unitary operator. Moreover, $\eta \in C_0^\infty((-\Lambda, \Lambda), [0, \infty))$ is a smooth cutoff function with $m \ll \Lambda \ll \frac{1}{\varepsilon}$.

Quantifying the rate of baryogenesis

Goal: Quantify B_t perturbatively for conformally flat spacetimes.

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Assumptions:

(i) $(\mathbb{R}^4, g)$ has a bounded geometry.

(ii) A_t has an absolutely continuous spectrum.

(iii) $A_t = \tilde{H}_\eta + \Delta A$, where $\Delta A : C^\infty(N_t, SM) \to C_0^\infty(N_t, SM)$ has smooth compactly supported coefficients and satisfies that for any $\omega \in \rho(\tilde{H}_\eta)$

$$||R_\omega(\tilde{H}_\eta)\Delta A|| < 1$$

These assumptions allow for a perturbative study of B_t.

Some results (cf. [2], [4], [5]):

(i) No baryogenesis if spinors follow Dirac dynamics nor if $m = 0$ and $u = \partial_t (\implies \Delta A = 0)$.

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(ii) What about Minkowski spacetime? $B_t^\eta = 0$ if $u = \partial_t$.

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(i) No baryogenesis if spinors follow Dirac dynamics nor if $m = 0$ and $u = \partial_t (\implies \Delta A = 0)$.

(ii) What about Minkowski spacetime? $B_t^{\eta} = 0$ if $u = \partial_t$.

(iii) $B_t^{(0)} = B_t^{(1)} = 0$ but

$$B_t^{(2)} = -\int \frac{d^3k}{(2\pi)^3} \int \frac{d^3k'}{(2\pi)^3} \frac{1}{4\omega_k \omega_{k'}} \frac{1}{(\omega_{k'} + \omega_k)^2} G_{\Omega,m,u}(\vec{k}, \vec{k'}) \ ,$$

where $\omega_k := \sqrt{|\vec{k}|^2 + m^2}$, $\omega_{k'} := \sqrt{|\vec{k'}|^2 + m^2}$ and $G_{\Omega,m,u} : (0, \infty) \times (0, \infty) \to \mathbb{R}$ is a function depending on Ω, m and u. For $u = \partial_t$, $B_t \propto m^2$.

Final remark:

(i) How canonical is B_t? Computing the rate of baryogenesis requires choosing a time function t...

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$\rightarrow$ Choose the unique smooth time function t for which the misalignment between ∂_t and u is minimal (work in progress).

Quantifying the rate of baryogenesis

Final remark:

(i) How canonical is B_t? Computing the rate of baryogenesis requires choosing a time function t...
$\rightarrow$ Choose the unique smooth time function t for which the misalignment between ∂_t and u is minimal (work in progress).

Thank you for your attention!

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Extra: The dynamics of u

Definition: Let $u : N_{t_0} \to TM$ be a smooth future directed timelike vector field. We construct a global $u : M \to TM$ as follows

(i) Consider the sets $\mathscr{L}^g$ and $D_p\mathscr{L}^g$, where $p \in M$ is arbitrary,

$$(I, \gamma) \in \mathscr{L}^g :\Longleftrightarrow \begin{cases} \gamma : I \to M \quad \text{is a max. f.d. null geod.} \\ g_{\gamma(s)}(u_{\gamma(s)}, \dot\gamma(s)) = 1 \quad \text{whenever} \quad \gamma(s) \in N_{t_0} \end{cases}$$

$$D_p\mathscr{L}^g := \{\dot\gamma(s) \mid (I,\gamma) \in \mathscr{L}^g \text{ and } \gamma(s) = p\} \subset T_pM\,.$$

(ii) For any $q \in N_{t_0+\Delta t}$ we define the timelike vector fields

$$\xi_q := \frac{1}{\mu_q(D_q\mathscr{L}^g)} \int_{D_q\mathscr{L}^g} \dot\gamma(s)\, d\mu_q(\dot\gamma(s))$$

$$u_q := \frac{1}{|\xi_q|_g^2}\xi_q$$

Extra: The important role of R^ε in CFS

Consider $(M, g) = (\mathbb{R}^{1,3}, \eta)$. We construct a CFS $(\mathcal{H}, \mathcal{F}, \rho)$:

(i) Let $\mathcal{H} \subset W^{1,2}_{loc}(\Omega)$ be the negative energy space of weak solutions to the Dirac equation.

(ii) Let $R^\varepsilon : \mathcal{H} \to \mathcal{H} \cap C^0(M, SM)$ denote a regularization operator. $\to$ Use it to construct $F^\varepsilon : M \to \mathcal{F} \subset L(\mathcal{H})$ as follows

$$\prec\varphi | F^\varepsilon(x)\psi\succ := -\prec(R^\varepsilon\varphi)(x) | (R^\varepsilon\psi)(x)\succ$$

for all $\varphi, \psi \in \mathcal{H}$.

(iii) Finally the pushforward measure, $\rho := (F^\varepsilon)_*\mu_M$ is

$$\rho(\Omega) = \mu_M((F^\varepsilon)^{-1}(\Omega))$$

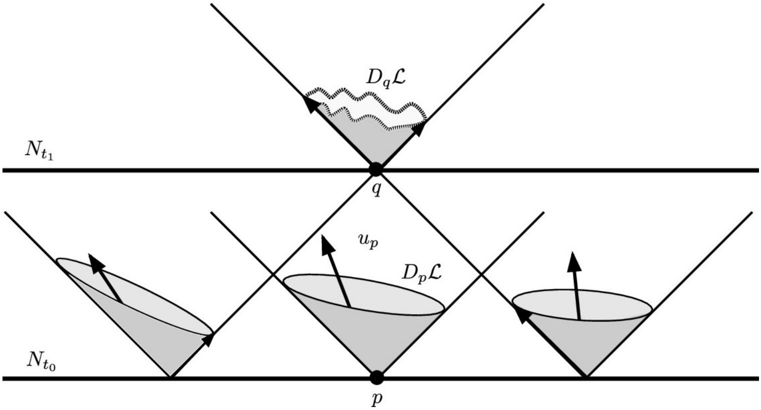
where $\Omega \subset \mathcal{F}$.

(iv) $(\mathcal{H}, \mathcal{F}, \rho)$ minimizes the causal action in the *continuum limit*.

Consider the initial timelike $u : N_{t_0} \to TM$ with

$$u_p = (1 + \lambda f_p)\partial_t + \lambda X_p \quad \text{for} \quad p \in N_{t_0}$$

where $\lambda > 0$, $f \in C^\infty(M)$ and $X_p \in T_pM$ spacelike.



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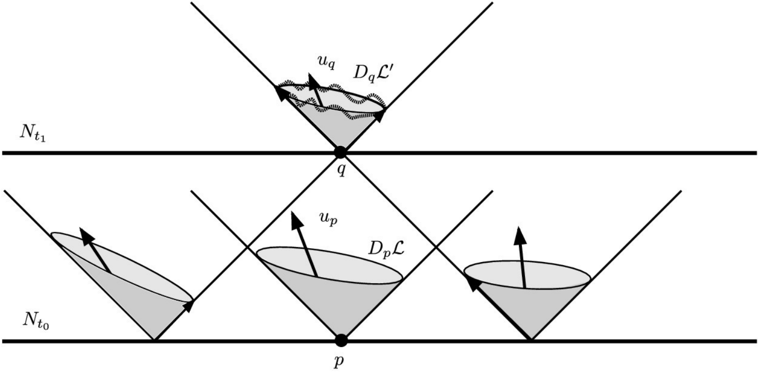

The dynamical equation of $u : M \to TM$ is (see [4, Th. 7.1])

$$\frac{du_p}{dt} = \left[-\operatorname{grad}_\delta(f_p^{-1}) + \frac{1}{3}\operatorname{div}_\delta\left(X_p\right)\nu \right]\lambda + \mathcal{O}(\lambda^2)$$

The dynamics of $u : M \to TM$ is a conformal invariant (cf. [5, Prop. 3.1]).

Extra: The misalignment functional

We introduce the functional $F : \mathcal{T} \subset H^p(\Omega) \to (0, \infty]$ with

$$F(t) = F_h(t) + F_g(t) \; , \tag{1}$$

where $F_h : \mathcal{T} \to [0, \infty)$ is the Riemannian functional

$$F_h(t) = \sum_{i=0}^{p-1} \int_\Omega |(\nabla^h)^i (u - \nabla^h t)|_h^2 d\mu_h = \|u - \nabla^h t\|^2_{H^{p-1}(\Omega, d\mu_h)} \; , \tag{2}$$

with $h := g + 2\frac{u^\flat \otimes u^\flat}{|u|_g^2}$ and $F_g : \mathcal{T} \to (0, \infty]$ the Lorentzian functional

$$F_g(t) := \begin{cases} \int_\Omega \frac{1}{|\nabla^g t|_g^{2q}} d\mu_h & \text{if } \nabla^g t \neq 0 \text{ a.e. in } \Omega \\ +\infty & \text{otherwise .} \end{cases} \tag{3}$$

where $q \in \mathbb{N}$ is arbitrary.

Extra: Schematic derivation of B_t

Step 1: Perturb the spectral projection operator.

$$R_\omega(A_t) = (1 + R_\omega(\tilde{H}_\eta)\Delta A(t))^{-1} R_\omega(\tilde{H}_\eta) = \sum_{p=0}^{\infty} (-R_\omega(\tilde{H}_\eta)\Delta A(t))^p R_\omega(\tilde{H}_\eta)$$

$$=: \sum_{p=0}^{\infty} R_\omega^{(p)}(A_t)$$

By absolute continuity of the spectrum of A_t

$$\chi_I(A_t) = \int_I F_{\omega'}(A_t) d\omega' = \frac{1}{2\pi i} \operatorname*{s-lim}_{\delta \to 0^+} \int_I R_{\omega' + is\delta}(A_t)\Big|_{s=-1}^{s=1} d\omega' \ ,$$

where $F_\omega(A_t) : \mathcal{H}_{t,g} \to \mathcal{H}_{t,g}$ and we used Stone's formula. Analogous for $\tilde{H}_\eta$.

Step 2: Obtain the perturbative power expansion of B_t.
We start with the following operator product

$$\eta_\Lambda(\tilde{H}_\eta)\chi_I(A_t) = \int_{-\infty}^{\infty} d\omega' \eta_\Lambda(\omega') F_{\omega'}(\tilde{H}_\eta) \int_{-\frac{1}{\varepsilon}}^{-m} d\omega F_\omega(A_t)$$

$$= \frac{1}{2\pi i} \operatorname*{s-lim}_{\delta\to 0^+} \int_{-\frac{1}{\varepsilon}}^{-m} d\omega \int_{-\infty}^{\infty} d\omega' \eta_\Lambda(\omega') F_{\omega'}(\tilde{H}_\eta) R_{\omega+is\delta}(A_t)\Big|_{s=-1}^{s=1}$$

Thus, B_t is

$$B_t := \frac{d}{dt}\mathrm{tr}_{\overline{\mathcal{H}_t^\varepsilon}}(\eta(\tilde{H}_\eta)\chi_I(A_t))$$

$$= \frac{d}{dt}\Big(\frac{1}{2\pi i} \operatorname*{s-lim}_{\delta\to 0^+} \int_{-\frac{1}{\varepsilon}}^{-m} d\omega \int_{-\infty}^{\infty} d\omega' \eta_\Lambda(\omega')\, \mathrm{tr}_{\overline{\mathcal{H}_t^\varepsilon}} \left(F_{\omega'}(\tilde{H}_\eta) R^{(p)}_{\omega+is\delta}(A_t)\right)\Big|_{s=-1}^{s=1}\Big)$$

$$=: \sum_{p=0}^{\infty} B_t^{(p)}$$

where the operator product is trace-class ([4, Lemma 7.4]).

Step 3: Determine the strong limit $\delta \to 0^+$ (cf. [4, Lemma 7.3]).

For $p = 2$ we obtain

$$B_t^{(2)} = -\int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} d\omega' \, \partial_\omega \big(\eta_\Lambda(\omega) \, \mathrm{tr}_{\overline{\mathcal{H}_t^\varepsilon}} \Big(\frac{d}{dt} \tilde{Q}(\omega, \omega') \Big) \, \frac{g(\omega') - g(\omega)}{\omega' - \omega}$$

where $\tilde{Q}(\omega, \omega') := \Delta A F_\omega(\tilde{H}_\eta) \Delta A F_{\omega'}(\tilde{H}_\eta) : \mathcal{H}_{t,g} \to C_0^\infty(N_t, SM)$ and g is the characteristic function of $I := (-1/\varepsilon, -m)$. Note that $\tilde{Q}(\omega = -m, \omega' = -m) = 0$.

Extra: Example of B_t

Example: Let $u = \partial_t$ and $m \neq 0$. Then,

$$\Delta A = (\Omega - 1) m \gamma_{\eta 0} \ : \ C^\infty(N_t, SM) \to C_0^\infty(N_t, SM)$$

and the second order rate of baryogenesis is

$$B_t^{(2)} = 2m^2 \int_0^\infty \frac{d\rho}{(2\pi)^4} \left(\hat{\alpha}_1'(\rho) \hat{\alpha}_2(-\rho) + \hat{\alpha}_2'(\rho) \hat{\alpha}_1(-\rho) \right) Q(\rho) \ ,$$

where $\alpha_1 = \frac{d\Omega}{dt}$, $\alpha_2 = (\Omega - 1)$, $\alpha_i' = \Omega^3 \alpha_i$, Q is smooth and $\rho := \frac{1}{2} |\vec{k} + \vec{k'}|$.

Extra: Example of A_t

Let $u = \partial_t$ and $m \neq 0$. Then,

$$A_t = -i\gamma_{\eta 0}\gamma_\eta^\mu \partial_\mu - \frac{3}{2} i \frac{\partial_\mu(\Omega)}{\Omega}\gamma_{\eta 0}\gamma_\eta^\mu + m\gamma_{gt} = \tilde{H}_\eta + (\Omega - 1)m\gamma_{\eta 0} \ ,$$

Let $m = 0$ and assume that initially

$$u_p = \begin{cases} (1+\lambda f_p)\nu + \lambda X_p & \text{for} \quad p \in V \\ \partial_t & \text{for} \quad p \in N_{t_0} \setminus V \ , \end{cases}$$

Then, A_t is

$$\begin{aligned} A_t &= \tilde{H}_\eta + \frac{\lambda}{2}\{f, \tilde{H}_\eta\} \\ &+ i\frac{\lambda}{2}\Big\{X^\mu, \partial_\mu + \frac{\partial_\mu(\sqrt{|\det(g|_{N_t})|})}{2\sqrt{|\det(g|_{N_t})|}} + \frac{1}{4}\frac{\partial_\nu(\Omega)}{\Omega}\eta^{\nu\rho}[\gamma_{\eta\mu}, \gamma_{\eta\rho}]\Big\} \end{aligned}$$

Extra: The Dirac Hamiltonian H_g

The Dirac Hamiltonian $H_g : C^\infty(N_t, SM) \to C^\infty(N_t, SM)$ is

$$H_g = -(\gamma_g^0)^{-1}\left(i\gamma_g^\mu \nabla_\mu^s - m\right) - E_0 - a_0 \ ,$$

where E_j, a_j are linear operators on the spinor space and ∇^s denotes the Levi-Civita spin connection

$$\nabla_j^s = \partial_j - iE_j - ia_j$$

Finally the Dirac operator $D_g : C^\infty(M, SM) \to C^\infty(M, SM)$ is

$$D_g = i\gamma_g^j \nabla_j^s$$

More concretely, in Minkowski spacetime:

$$H_\eta = -i\gamma_{\eta t}\gamma_\eta^\alpha \partial_\alpha + \gamma_{\eta t} m$$

Extra: Some identities

Stone's formula: Given a self-adjoint operator A_t and an interval $(a,b) \subset \mathbb{R}$ it holds that

$$\frac{1}{2}(\chi_{(a,b)}(A_t) + \chi_{[a,b]}(A_t)) = \frac{1}{2\pi i} \underset{\delta \to 0^+}{\text{s-lim}} \int_I R_{\omega' + is\delta}(A_t)\Big|_{s=-1}^{s=1} d\omega'$$

The operators $\chi_I(A_t)$ and $F_\omega(H_\eta)$:

$$\begin{aligned}
&\chi_{(-\frac{1}{\varepsilon},\omega)}(H_\eta)(x,y) \\
&= -\int \frac{d^4k}{(2\pi)^3}(\gamma_{\eta j}k^j + m)\gamma_{\eta 0}\delta(k^2 - m^2)\Theta(-k^0 + \omega)\Theta(1 + \varepsilon k^0)e^{ik\cdot(x-y)} \\
&F_\omega(x,y) = \frac{d}{d\omega}\chi_{(-\frac{1}{\varepsilon},\omega)}(H_\eta) \\
&= -\int \frac{d^3k}{(2\pi)^3}(\gamma_{\eta 0}\omega - \gamma_\eta \cdot k + m)\gamma_{\eta 0}\Theta(1 + \varepsilon\omega)\delta(\omega^2 - \omega_k^2)e^{ik\cdot(x-y)}
\end{aligned}$$

Extra: Some identities

And moreover, for $x = y$ we have that

$$F_\omega(x,x) = -\frac{1}{(2\pi)^2}\Big((\omega + \gamma_{\eta 0} m)\sqrt{\omega^2 - m^2}\Big)\Theta(1+\varepsilon\omega)\,,$$

so $|F_\omega(x,y)| \leq F_{-m}(x,x) = 0 \implies F_{-m}(x,y) = 0.$

Extra: The operator $\tilde{U}$

The operator $\tilde{U} : \mathcal{H}_{t,g} \to \mathcal{H}_{t,\eta}$ with $\tilde{U}\psi = \Omega^{3/2}\psi$ for all $\psi \in \mathcal{H}_{t,g}$ is unitary as the following computation shows:

$$\begin{aligned}
(\psi|\phi)_t &:= \int_{N_t} \prec\psi|\gamma_g(\nu)\phi\succ_{S_pM} d\mu_{N_t} = \int_{\mathbb{R}^3} \prec\psi|\gamma_\eta(\partial_t)\phi\succ_{\mathbb{C}^4} \Omega^3(x) d^3x \\
&= \int_{\mathbb{R}^3} \prec(\Omega^{3/2}\psi)(x)|\gamma_\eta(\partial_t)(\Omega^{3/2}\phi)(x)\succ_{\mathbb{C}^4} d^3x \\
&= (\Omega^{3/2}\psi|\Omega^{3/2}\varphi) =: (\tilde{U}\psi|\tilde{U}\varphi)
\end{aligned}$$