Mathematical Preliminaries

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Motivation and Outline

The definition of a causal fermion system $(\mathcal{H}, \mathcal{F}, \rho)$ of spin dimension n contains

- a **Hilbert space** $\mathcal{H}$
- the space $\mathcal{F} \subset L(\mathcal{H})$ of **symmetric operators of finite rank** on $\mathcal{H}$ with at most n positive and n negative eigenvalues
- a **positive Borel measure** ρ on $\mathcal{F}$

The measure ρ should be a minimizer of the **causal action**:

$$S(\rho) = \iint_{\mathcal{F}\times\mathcal{F}} \mathcal{L}(x,y)\,\mathrm{d}\rho(x)\,\mathrm{d}\rho(y)\,.$$

Motivation and Outline

1. Metric spaces
2. Measures and integration
3. Hilbert spaces and bounded linear operators

For references:

Finster, Kindermann, T. *Causal Fermion Systems: An Introduction to Fundamental Structures, Methods and Applications*, https://arxiv.org/abs/2411.06450, to appear in Cambrige Mongraphs on Mathematical Physics, Cambridge University Press, 2025

1. Metric spaces

Metric spaces

Definition

A function $d : X \times X \to \mathbb{R}$ is called **metric on** X ("distance") if

(i) $d(x,y) \geq 0$ and $d(x,y) = 0 \Leftrightarrow x = y$ for all $x, y \in X$

(ii) $d(x,y) = d(y,x)$ for all $x, y \in X$

(iii) $d(x,y) \leq d(x,z) + d(z,y)$ for all $x, y, z \in X$

- **Open ball** $B_r(x) := \{y \in X : d(x,y) < r\}$
- **Open set** $U \subset X$: $x \in U \implies B_r(x) \subset U$ for some $r > 0$

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Open sets $\mathcal{O} \subset \mathcal{P}(X)$ satisfy properties of a **topology** on X.

$\rightsquigarrow$ Framework for continuity, convergence, compactness, ...

Completeness

Definition

- $(x_n)_{n\in\mathbb{N}}$ **converges** to x if

$$\forall \varepsilon > 0 : \exists N \in \mathbb{N} : \forall n \geq N : d(x_n, x) < \varepsilon$$

- $(x_n)_{n\in\mathbb{N}}$ is called **Cauchy sequence** if

$$\forall \varepsilon > 0 : \exists N \in \mathbb{N} : \forall n, m \geq N : d(x_n, x_m) < \varepsilon$$

- (X, d) is called **complete** if: "Cauchy $\Longrightarrow$ convergent"
  The reverse implication "Convergent $\Longrightarrow$ Cauchy" is always true.

- Convenient mathematical property since it gives *existence*.
- Any metric space can be embedded into a (larger) complete metric space ("completion")

2. Measure and integration

Basic idea of a measure

- A measure μ on $\mathcal{F}$ measures the **"volume"** of subsets of $\mathcal{F}$:

$$\begin{aligned} \text{certain subsets of } \mathcal{F} &\longrightarrow [0, \infty] \\ A &\longmapsto \mu(A) \end{aligned}$$

- Measures are closely related to **integration**:

$$\int_{\mathcal{F}} f(x) \, \mathrm{d}\mu(x) \approx \sum_i (\text{"value" of } f \text{ on } A_i) \times \mu(A_i)$$

Integration is accumulating / averaging values with respect to a notion of size.
Measures provide that notion of size in a very general way.

Definition of a measure

Definition

A **measure space** $(\mathcal{F}, \mathfrak{M}, \mu)$ consists of a set $\mathcal{F}$ together with

- a σ**-algebra** $\mathfrak{M}$, i.e. a collection of subsets of $\mathcal{F}$ such that
  (i) $\emptyset \in \mathfrak{M}$
  (ii) $A \in \mathfrak{M} \implies \mathcal{F} \setminus A \in \mathfrak{M}$
  (iii) $(A_n)_{n\in\mathbb{N}} \subset \mathfrak{M} \implies \bigcup_{n\in\mathbb{N}} A_n \in \mathfrak{M}$

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  (iii) $(A_n)_{n\in\mathbb{N}} \subset \mathfrak{M} \implies \bigcup_{n\in\mathbb{N}} A_n \in \mathfrak{M}$
- a **measure** μ, i.e. a mapping $\mu : \mathfrak{M} \to [0, \infty]$ satisfying

$$(A_n)_{n\in\mathbb{N}} \subset \mathfrak{M} \text{ pairwise disjoint} \implies \mu\Big(\bigcup_{n\in\mathbb{N}} A_n\Big) = \sum_{n\in\mathbb{N}} \mu(A_n)$$

Integration with respect to a measure (Lebesgue integral)

Suppose $(\mathcal{F}, \mathfrak{M}, \mu)$ is a measure space.

- For simple functions

$$f = \sum_{i=1}^{n} c_i \cdot \chi_{A_i} \quad \Longrightarrow \quad \int_{\mathcal{F}} f(x)\, \mathrm{d}\mu(x) := \sum_{i=1}^{n} c_i \cdot \mu(A_i)$$

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- Nonnegative functions: approximate by simple functions.
- $\mathbb{R}$-valued functions: split into positive and negative part.
- $\mathbb{C}$-valued functions: split into real and imaginary part.

Integrable and square integrable functions

For $f : \mathcal{F} \to \mathbb{C}$ write

$$f \in L^1(\mathcal{F}, \mathrm{d}\mu) \quad \text{if} \quad \|f\|_{L^1} := \int_{\mathcal{F}} |f(x)| \, \mathrm{d}\mu(x) < \infty$$

$$f \in L^2(\mathcal{F}, \mathrm{d}\mu) \quad \text{if} \quad \|f\|_{L^2} := \left(\int_{\mathcal{F}} |f(x)|^2 \, \mathrm{d}\mu(x) \right)^{\frac{1}{2}} < \infty$$

The spaces $L^1(\mathcal{F}, \mathrm{d}\mu)$ and $L^2(\mathcal{F}, \mathrm{d}\mu)$ with these norms are Banach spaces, i.e. complete normed spaces.

More precisely: Elements of these spaces are not functions but equivalence classes of functions which differ only on sets of measure zero.

The support of a measure

Suppose $\mu : \mathcal{B}(\mathcal{F}) \to [0, \infty]$ is a Borel measure on $\mathcal{F}$.

- $V := \bigcup_{\substack{U \text{ open} \\ \mu(U)=0}} U$ is largest open set of measure zero.
- $\operatorname{supp} \mu := \mathcal{F} \setminus V$ is called **support of** μ.

3. Hilbert spaces and bounded lin. operators

Hilbert spaces

$\mathcal{H}$ a complex vector space with **scalar product** $\langle\cdot|\cdot\rangle : \mathcal{H}\times\mathcal{H}\to\mathbb{C}$

$$
\begin{aligned}
&\text{(i)} && \langle u|\alpha v+\beta w\rangle = \alpha\,\langle u|v\rangle + \beta\,\langle u|w\rangle \\
&\text{(ii)} && \overline{\langle u|v\rangle} = \langle v|u\rangle \\
&\text{(iii)} && \langle u|u\rangle > 0 \text{ for } u\neq 0
\end{aligned}
$$

- $\|u\| := \sqrt{\langle u|u\rangle}$ defines a norm,
- $d(u,v) := \|u-v\|$ a metric ("distance") on $\mathcal{H}$

$(\mathcal{H},\langle\cdot|\cdot\rangle)$ is called **Hilbert space** if the metric d is complete.

Orthogonality

- $u, v \in \mathcal{H}$ are **orthogonal** if $\langle u|v\rangle = 0$.
- $(e_i)_{i\in I} \subset \mathcal{H}$ is **orthonormal Hilbert basis** if
  - pairwise orthogonal and of unit length: $\langle e_i|e_j\rangle = \delta_{ij}$
  - completeness: for any $u \in \mathcal{H}$ have $u = \sum_{i\in I} \langle e_i|u\rangle\, e_i$

    $\left(\sum_{i\in I} |e_i\rangle\langle e_i| = \mathbf{1}\right)$

  Any (separable) Hilbert space has a (countable) orthonormal Hilbert basis.
- For a linear subspace $U \subset \mathcal{H}$
  - $U^\perp := \{v \in \mathcal{H} : \langle u|v\rangle = 0 \text{ for all } u \in U\}$ orth. complement
  - Then $\mathcal{H} = \overline{U} \oplus U^\perp$, i.e. for any $x \in \mathcal{H}$ have decomposition

$$x = u^{\parallel} + u^{\perp} \qquad \text{with unique } u^{\parallel} \in \overline{U}, u^{\perp} \in U^{\perp}\,.$$

Bounded linear operators

Definition

A **bounded linear operator on** $\mathcal{H}$ is a map $A : \mathcal{H} \to \mathcal{H}$ with

(i) A is linear, i.e. $A(\alpha u + \beta v) = \alpha A(u) + \beta A(v)$.

(ii) There exists $C > 0$ such that $\|A(u)\| \leq C \|u\|$ for all $u \in \mathcal{H}$.

$L(\mathcal{H})$ denotes set of all bounded linear operators.

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- $L(\mathcal{H})$ is a vector space via pointwise linear combinations
  This means: $(\alpha A + \beta B)(u) := \alpha A(u) + \beta B(u)$
- On $L(\mathcal{H})$ a (complete) norm is given by the *operator norm*

$$\|A\| := \sup \left\{ \|A(u)\| : \|u\| = 1 \right\},$$

$\rightsquigarrow$ $L(\mathcal{H})$ becomes topological space

Special classes of operators

Let $A \in L(\mathcal{H})$.

- A is **symmetric** if $\langle u|Av\rangle = \langle Au|v\rangle$ for all $u, v \in \mathcal{H}$.
  - eigenvalues are real (more generally the spectrum)
  - eigenvectors for different eigenvalues are orthogonal

- A has **finite rank** if its image has finite dimensions, $\dim A(\mathcal{H}) < \infty$

Diagonalizing a symmetric operator of finite rank

Suppose $A \in L(\mathcal{H})$ is of finite rank and symmetric.

- $I := A(\mathcal{H}) \subset \mathcal{H}$ is a finite-dimensional subspace
- $\mathcal{H} = I \oplus I^\perp$
- $A|_{I^\perp} = 0$, since A is symmetric
  For $u \in U^\perp$ have $\|Au\|^2 = \langle Au, Au \rangle = \left\langle A^2 u, u \right\rangle = 0$.

Diagonalizing a symmetric operator of finite rank

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$$\implies \quad A = \begin{pmatrix} A|_I & 0 \\ 0 & 0 \end{pmatrix} : I \oplus I^\perp \longrightarrow I \oplus I^\perp$$

$A|_I : I \to I$ is a linear map on a *finite-dimensional* vector space

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$\rightsquigarrow$ Linear algebra: ONB $e_1, \ldots, e_n \in I$ with $Ae_i = \lambda_i e_i$

$$A = \sum_{i=1}^{n} \lambda_i \langle e_i | \cdot \rangle \, e_i = \sum_{i=1}^{n} \lambda_i |e_i\rangle\langle e_i| \, .$$