Cone structures and their applications to fundamental theories and discretization

Miguel Sánchez (U. Granada)

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Aims

1. **Background** on Cone structures
   - Finsler metrics
   - Cone structures
   - Lorentz-Finsler metrics
2. **Prospects** on Fundamental Physics and discretization
3. **Technical ideas** on Fermat principle and Snell law

1. Cone structures

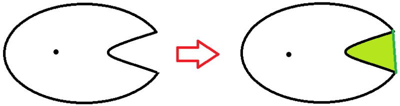
Common Knowledge

- **Finsler**: MA Javaloyes, MS, Annali Pisa (2014) arxiv: 1111.5066
- **Lorentz-Finsler**: MA Javaloyes, MS, RACSAM (2020) arxiv: 0903.3501

1.1 Finsler metrics

Anisotropic propagation in classic setting

Space of velocities $\Rightarrow$ Convex **effective** space

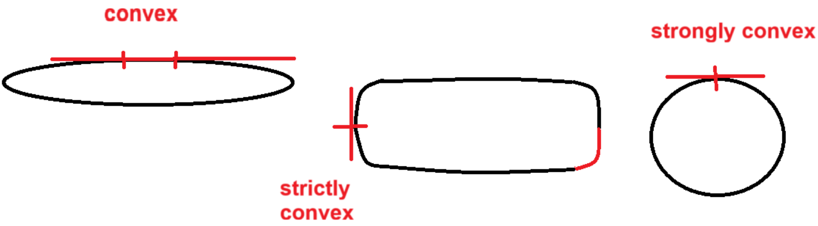


(possibly non-symmetric around 0)

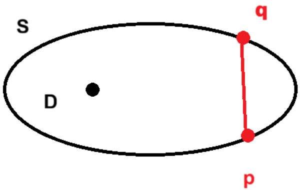
Types of convexity

For a smooth hyp. S in an n-affine space, each $p \in S$:

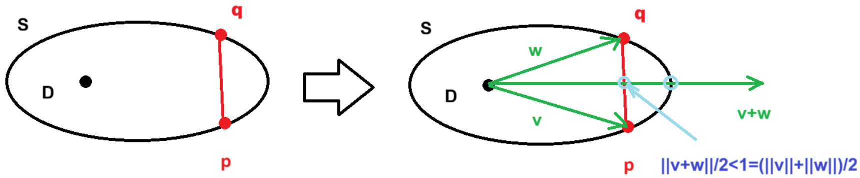
1. Convex: when infinitesimal (equivalent to local), $\sigma_p^+ \geq 0$
2. Strictly convex: close to p, $T_pS \cap S = \{p\}$
3. Strongly convex: when $\sigma_p > 0$ (positive definite)



- When $S = \partial D$ and $\bar{D}$ compact:
  - S convex: each $p, q \in \bar{D}$ joined by minimizing γ in $\bar{D}$
  - S strictly convex: γ will not touch $S = \partial D$ but at p or q.



Convex $\Sigma \longleftrightarrow$ norm indicatrix (unit sphere)



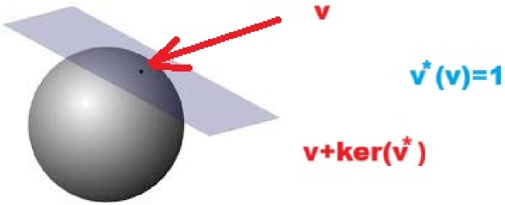
(Non-symmetric) norm on $V = V^n(\mathbb{R})$:

- Positiveness: $\| v \| \geq 0$, equality only at $v = 0$
- Positive homogeneity: $\| \lambda v \| = \lambda \| v \|$, for $\lambda > 0$
- **(Strict) triangle inequality**: $\| v + w \| \leq \| v \| + \| w \|$

$\Longleftrightarrow$

(Strictly) convex indicatrix $\Sigma := \{\| v \| = 1\}$ (0 inside Σ)

$\|\cdot\|^2$ When smooth (away 0!):
continuous (non-linear) flat, Legendre $V \setminus \{0\} \to V^* \setminus \{0\}$



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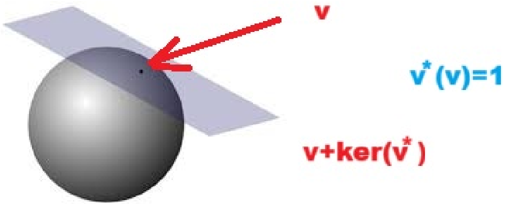

- Σ strictly convex $\Longleftrightarrow$ bijective flat map

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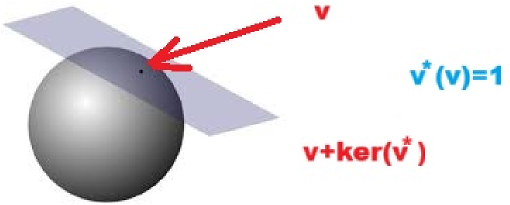

- Σ strictly convex $\Longleftrightarrow$ bijective flat map
- Σ strongly convex $\Longleftrightarrow$ diffeomorphism $\rightsquigarrow$ Minkowski norm

Our standard regularity choice: Minkowski norm

Minkowski norm $\| \cdot \|: V \to \mathbb{R}$

$\Longleftrightarrow$

1. Positive
2. Positive homogeneous
3. Smooth outside 0 with

fundamental tensor $\quad g_v := \dfrac{1}{2} \mathrm{Hess}_v \| \cdot \|^2 > 0$ for $v \neq 0$

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$$\textbf{fundamental tensor} \qquad g_v := \frac{1}{2}\text{Hess}_{\,v}\, \|\cdot\|^2 > 0 \;\; \text{for } v \neq 0$$

- g_v affine second fundamental form of Σ respect to $-v$
- Depends on oriented direction: $g_{\lambda v} = g_v$ for $\lambda > 0$

Finsler metric

Finsler metric on M: Minkowski norm varying smoothly with p
Notation: $L = F^2$ (2-homogeneous) used when convenient

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Definition

$L : TM \setminus \mathbf{0} \to \mathbb{R}$ is a *Finsler metric* if it satisfies:

1. Positive: $L(v) > 0$
2. Positive 2-homogeneus: $L(\lambda v) = \lambda^2 L(v)$ for $\lambda > 0$
3. Smooth with positive definite **fundamental tensor** g_v

$$g_{v_p}(u_p, w_p) = \frac{1}{2}(\mathrm{Hess}_{v_p} L)(u_p, w_p) = \frac{1}{2}\,\frac{\partial^2}{\partial r \partial s} L(v_p + r u_p + s w_p)\Big|_{r,s=0}$$

Some Finsler notions extending Riemannian

- Length of **oriented** curves:

$$\ell(\gamma) = \int F(\gamma'(s))ds$$

$\rightsquigarrow$ invariant under monotonically **increasing** parametrization

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$\rightsquigarrow$ Euler Lagrange equation $\ddot{x}^k + \Gamma_{ij}^k(x, \dot{x})\dot{x}^i\dot{x}^k = 0$
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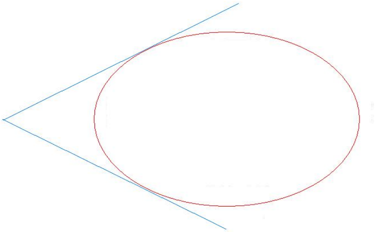
  - $\Gamma^k_{ij}(v_p)$ depend on p **and the (oriented) direction** $\mathbb{R}^+ \cdot v_p$
  - Locally minimize (oriented) distance.
- Geodesic spray $\rightsquigarrow$ canonic nonlinear connection

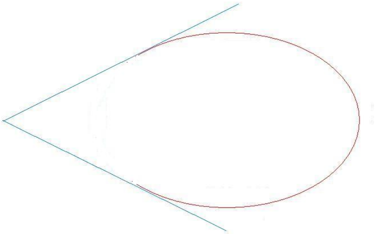
A further question

What about if 0 not inside Σ_0?:

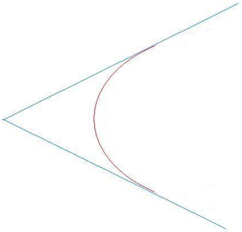


Conic region with

- Convex piece of Σ_0 triang. ineq.



- **Concave** piece of Σ_0 with **reversed** triang ineq
  straight lines (geodesics) in the conic region locally maximize
  $\rightsquigarrow$ "Lorentz norm" g_v $(+,-,\ldots,-)$,



Wind Finsler

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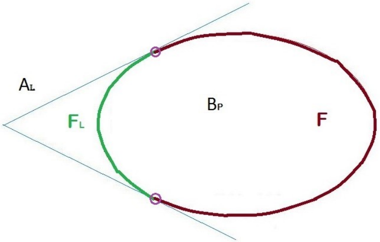

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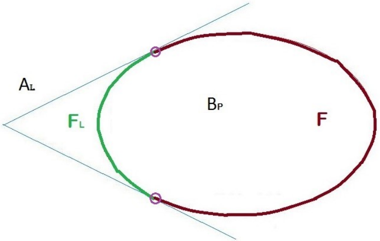
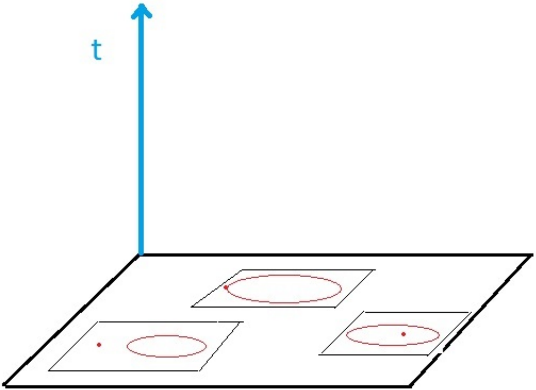


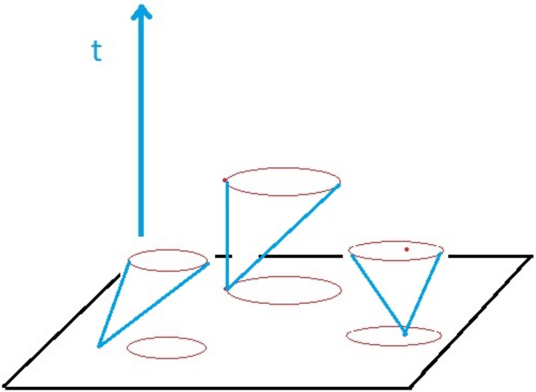
- **Wind Finsler metric** on M:
  hypersurface $\Sigma \subset TM$ s.t.:
  - $\Sigma_p = T_pM \cap \Sigma$ is a wind norm
  - technicality: Σ transverse to each T_pM

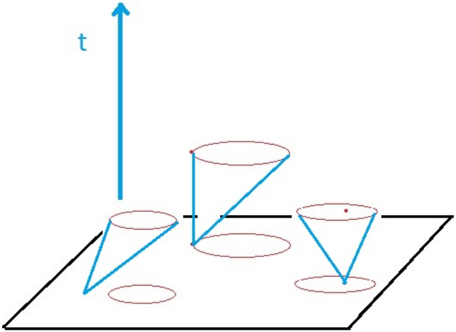
1.2 Cone structures

Time evolution of Wind F. $\rightsquigarrow$ **cones**



Introducing time





1. Desingularization of wind Finsler structures
2. Posssible time-dependence of velocities

Intrinsic definition

Cone in V: smooth hyp. $\mathcal{C}_0$ embedded in $V \setminus \{0\}$ satisfying

1. Conic: $v \in \mathcal{C}_0 \Rightarrow \lambda v \in \mathcal{C}_0, \forall \lambda > 0$
2. Salient: if $v \in \mathcal{C}_0$, then $-v \notin \mathcal{C}_0$.
3. Convex interior: $\mathcal{C}_0 = \partial A_0$,
   with $A_0 \subset V \setminus \{0\}$ open *convex* subset (*interior*)
4. **(Non-radial) strong convexity**: 2^{nd} fundamental form σ of $\mathcal{C}_0$ with radial spanned by $\{\lambda v : \lambda > 0\}$

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... always constructable as in the picture

Cone structure $\mathcal{C}$ on M: smoothly varying cone $\mathcal{C}_p$ at each T_pM

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Formally: $\mathcal{C}$ (smooth) embedded hypersurface of TM s.t.:

1. $\mathcal{C}_p := T_pM \cap \mathcal{C}$ is a cone
2. Technicality: $\mathcal{C}$ is tranverse to each T_pM

Some notions related to $\mathcal{C}$

$\mathcal{C}$ **naturally extend future cones in GR**:

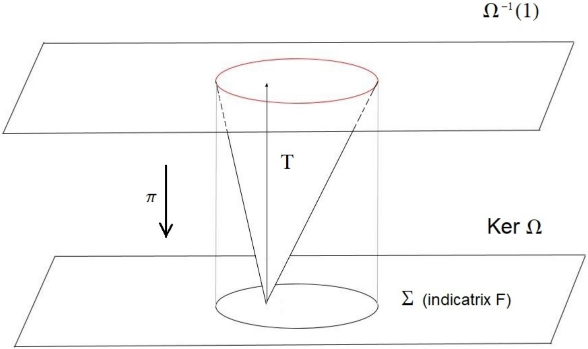
- Cone domain: $A = \cup_{p \in M} A_p$
  (where each A_p is the interior of $\mathcal{C}_p$)
- $v \in TM$ is
  - Lightlike when $v \in \mathcal{C}$
  - Timelike when $v \in A$
  - Causal when timelike or lightlike
- Causality:
  - Future/past: $I^{\pm}(p), J^{\pm}(q), E^{\pm}(p) := J^{\pm}(q) \setminus I^{\pm}(q)$...
  - Global hyperbolicity, hierarchy of spacetimes
- Cone (lightlike) geodesics: locally horismotic curves

Description of cone structures

- Any $\mathcal{C}$ admits a (highly non unique) **cone triple** (Ω, T, F)
  1. timelike 1-form Ω: $\Omega(v) > 0$ for all v causal.
  2. timelike vector field T with $\Omega(T) \equiv 1$
  3. and, then, a (unique) Finsler metric F on $\ker(\Omega)$

  s.t. if $\pi : TM \to \ker(\Omega)$ is the natural T-projection:

  $$v_p \in \mathcal{C} \iff v_p = F(\pi(v_p))T_p + \pi(v_p) \qquad \forall v_p \in TM \setminus \mathbf{0}$$



- Conversely: *cone triple* (Ω, T, F) as above $\Longrightarrow \mathcal{C}$.

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Especially useful when

- $\exists$ time t: $\Omega = dt$
- $\exists$ field of observers T
  (non-necessarily the barycentric one from Ω)

1.3 Lorentz-Finsler metrics

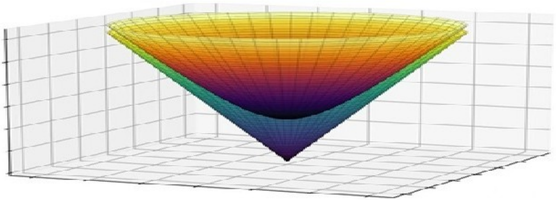
Definition

Lorentz-Finsler L: smooth pointwise Lorentz-Minkowski norm L_p

$\rightsquigarrow$ $\mathcal{C}$

$\rightsquigarrow$ concave indicatrix $\Sigma_p := L_p^{-1}(1)$ asymptotic to each $\mathcal{C}_p$

$\rightsquigarrow$ Lorentz sign $(+, -, \ldots, -)$ of g_{v_p} at each $v_p \in \Sigma_p \cup \mathcal{C}_p$



Definition

$L : \bar{A} \to \mathbb{R}$ smooth is a **Lorentz-Finsler metric** (and (M, L) a **Finsler spacetime**) if:

1. $\bar{A} = A \cup \mathcal{C} \subset TM \setminus \mathbf{0}$ is a connected $2n$-manifold with boundary $\mathcal{C}$ and interior A
2. $L \geq 0$ with $= 0$ on $\mathcal{C}$
3. L is 2-homogeneous: $L(\lambda v) = \lambda^2 L(v)$ (and $\lambda A = A$) for $\lambda > 0$
4. $g_{v_p} := \mathrm{Hess}_{v_p} L$ with Lorentz sign for $v_p \in \bar{A}$

In this case,

- $\mathcal{C}$ cone structure, A interior domain.
- $\Sigma_p = L_p^{-1}(1)$ concave asymptotic to $\mathcal{C}$

Consistency

Proposition (focus on $\bar{A} = A \cup \mathcal{C}$)
Let $L : TM \setminus \{\mathbf{0}\} \to \mathbb{R}$ smooth 2-homog.:

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When $\exists T$ timelike v.f $(L(T) > 0)$
$\Longrightarrow \exists! A \subset TM$:

- T lies in A, $L|_{\bar{A}}$ Lorentz-Finsler

Consistency

Prop. (cone geodesics) Let $L : \bar{A}(= A \cup \mathcal{C}) \to \mathbb{R}$ Lorentz-F.:

- { lightlike pregeod. of L } = { cone geod. of $\mathcal{C}$}

Lorentz-Finsler m. with same $\mathcal{C} \Longrightarrow$ same lightlike pregeod.

Example: L associated with (Ω, T, F)

$$L := \Omega^2 - F^2$$

$$L(tT_p + w_p) = t^2 - F(w_p)^2, \qquad \forall t \in \mathbb{R},\ \forall w_p \in \ker(\Omega_p),\ \forall p \in M$$

- L-lightlike vectors $\longleftrightarrow$ cone structure $\mathcal{C}$

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2. Fundamental Physics and discretization

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4. Foundations for particle Physics?

Some ideas

- **Fundamental particles**: emerging semi-classical propagation

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1. Cones should work for an effective theory
2. **Naively**: elements for a more fundamental modelization?

Some ideas

Causal sets[1]: points + (causal) relations

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[1] S. Surya, Living Reviews (2019) arxiv:1903.11544

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3. **Extreme (quantum) discretization**

$\rightsquigarrow$ In an effective or less strict limit:

- spacetimes modelled by cells of points endowed with cones

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Non-speculative

Discretization:

- Numerical Relativity
- ... and in "real world" space+time

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Standard: grid

- cells c_k centered at p_k
- metric $g(p_k)$

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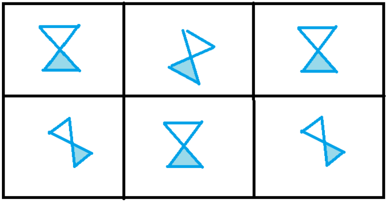

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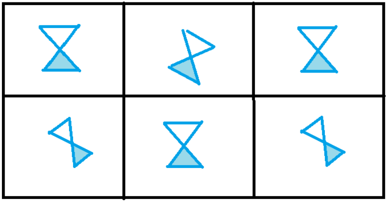

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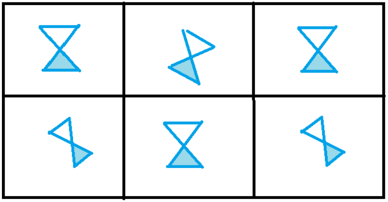

1. For **Numerics**: feasible
2. For **Fundamental Physics**: applicable?

A foundational problem

How do **discrete** *lightlike geodesics look like?*

A foundational problem

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Are they still locally horimotic?

A foundational problem

How do **discrete** *lightlike geodesics look like?*

Are they still locally horimotic?

Can they help to determine causal futures?

3. Fermat principle and Snell law

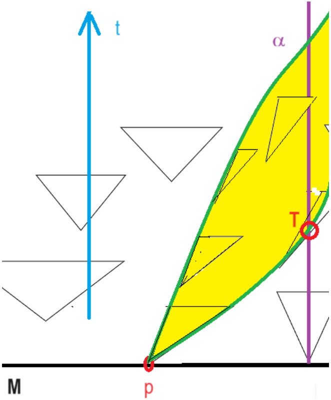
References

- I Kovner, Astrophys. J. (1990).
- V Perlick, CQG (1990), GRG (2006)
- V Perlick, P Piccione, GRG (1998)
- S Markvorsen, E Pendás-Recondo, IJGMMP (2023)
- **M. A. Javaloyes, S. Markvorsen, E. Pendás-Recondo**, M. Sánchez, arxiv: 2509.26492

Fermat problem for a cone structure $\mathcal{C}$

Given $p \in M$ and a timelike curve α parametrized by a time

- $\{\gamma : [0,1] \to M | \gamma$ lightlike $p = \gamma(0), \gamma(1) \in \alpha\}$
- Arrival time functional $\mathcal{T}$: $\alpha(\mathcal{T}[\gamma]) = \gamma(1)$



Theorem

$\{$ *Critical points of* $\mathcal{T}$ $\} = \{$ *cone geod. from* p *to* α $\}$
$\rightsquigarrow$ *first-arriving causal curves are cone geod.*

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Under **very general hypotheses**:

- $\mathcal{C}$ possibly anisotropic
- Extensible when departing from a spacelike submanifold P (or just P non tangent to $\gamma \rightsquigarrow$ orthogonality at departure)
- α non necessarily timelike (nor a time is needed) (transversality of γ and α suffices)
- Extensible to arrival at timelike submanifolds with a time

Existence results and further interpretations:
wind Finsler structures, stationary and SSTK spacetimes...

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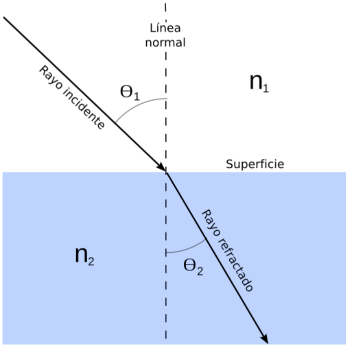
Let us focus on discretization

Discretization → discontinuous medium → Snell law

$$n_1 \sin(\theta_1) = n_2 \sin(\theta_2) \qquad n = c/v$$



Fuente: Wikipedia "Ley de Snell"

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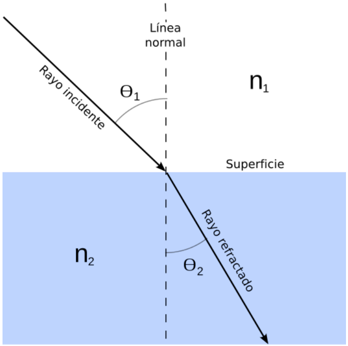
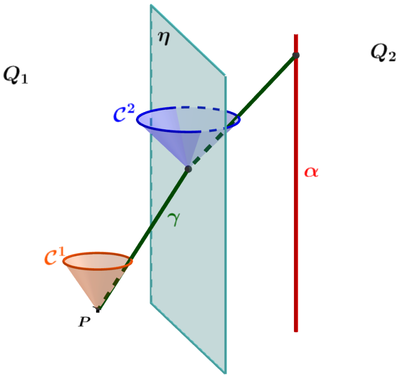


Fuente: Wikipedia "Ley de Snell"

Fermat problem for discontinuos media...

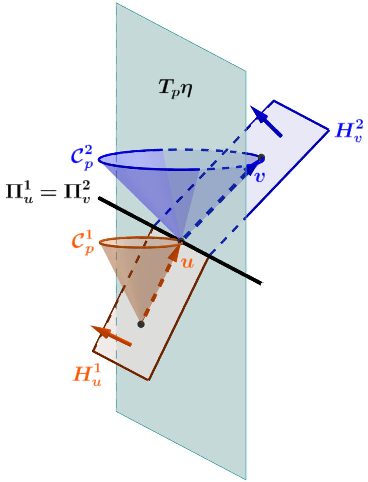
...rheonomic formulation



- P source: medium Q_1, cone $\mathcal{C}_1$
- α observer: medium Q_2 cono $\mathcal{C}_2$ (wider $\Rightarrow v_2 > v_1$)
- η interface: γ broken geodesic

Essential Snell law for (anisotropic) cones:

$$\Pi_u^1 (:= H_u^1 \cap T_p\eta) = \Pi_v^2 (:= H_v^2 \cap T_p\eta)$$



Remarks on $\Pi^1_u(:= H^1_u \cap T_p\eta) = \Pi^2_v(:= H^2_v \cap T_p\eta)$

- Characterizes critical points, not necessarily minimizers ...

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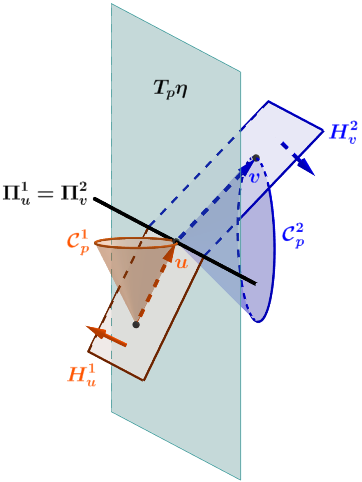
- Characterizes critical points, not necessarily minimizers ...
- **relevance of orientations!**

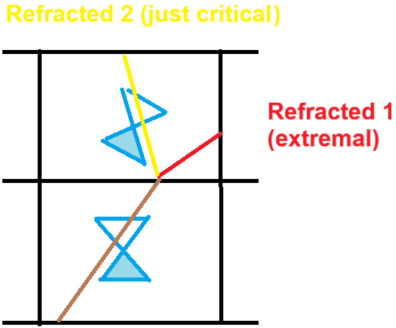


Classical issues

- Non transverse incident cone: $H^1_u = T_p\eta$
- Total reflection (and just reflection)

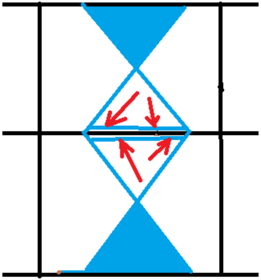
Relativistic issues

- Even two refracted trajectories (only one minimizing)...



Relativistic issues

- Even two refracted trajectories (only one minimizing)...
- ...or none!



But all this can be handled!

Conclusion

In discretized spacetimes, lightlike geodesics:

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In discretized spacetimes, lightlike geodesics:

- consistently defined
- horismotic ones tidely determined
- they determine causality
  [proof much subtler in the anisotropic case]

Conclusion

Applications:

1. General Relativity: Numerics
2. Classical Mechanics: propagation of waves (e.g., sismic waves on discontinuous layers)
3. Fundamental Physics ? : prospective
   - effective quantum propagation
   - quantum modelization

Thank you
for your attention