Quantum field theory: Constructive versus perturbative

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Introduction

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- But first, let's start with a detour. I want to talk to you about graphs, trees and forests.

Graphs, trees and forests

- A graph G is a set of vertices and of edges (in physics edges might be often called lines or propagators) together with a relation: to any edge is associated two vertices called its ends. A graph is said to be connected if one can join any of its vertices to any other one through a chain of edges.

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- A forest of G is an acyclic subgraph of G, hence a subgraph without any cycle.
  (a cycle is a set of distinct edges $e_1 = (v_1, v_2), e_2 = (v_2, v_3) \cdots e_n = (v_n, v_1)$ with its vertices $v_1, \cdots v_n$ all distinct).

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- A tree is a forest of G which is connected.
- Trees and forests are essential combinatorial objects which appear in a myriad of different problems; in particular they index algorithms in computer science and they index solutions of algebraic or differential equations.

The BKAR Forest Formula

Let f be a smooth function of $n(n-1)/2$ edge variables $x_\ell \in [0,1]$, $\ell = (i,j)$, $1 \le i < j \le n$. The BKAR forest formula states

$$f(1, \cdots, 1) = \sum_{\mathcal{F}} \Big\{ \prod_{\ell \in \mathcal{F}} \Big[\int_0^1 dw_\ell \Big] \Big\} \Big\{ \prod_{\ell \in \mathcal{F}} \frac{\partial}{\partial x_\ell} f \Big\} [X^{\mathcal{F}}(\mathbf{w}_{\mathcal{F}})],$$

where the sum over $\mathcal{F}$ is over all forests over n vertices.

We explain this formula in the next slide.

The BKAR Forest Formula

This formula was invented by David Brydges and Tom Kennedy and perfected in its combinatorial aspects by Malek Abdesselam and me (R), hence it is sometimes called the BKAR formula.

- It's important to realize that $X^{\mathcal{F}}(\mathbf{w}_{\mathcal{F}})$ is a real symmetric n x n matrix whose coefficients are $X^{\mathcal{F}}_{ij}(\mathbf{w}_{\mathcal{F}}) = X^{\mathcal{F}}_{ji}(\mathbf{w}_{\mathcal{F}})$;
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- the "weakening parameter" $X_{ij}^{\mathcal{F}}(\mathbf{w}_{\mathcal{F}})$ is 1 on the diagonal $i = j$, and 0 if i and j don't belong to the same connected component of $\mathcal{F}$, i.e. in trivial cases;

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- in non-trivial cases, it is the minimum of the $w_{\ell'}$ for ℓ' running over the unique path from i to j in $\mathcal{F}$;
- furthermore the matrix $X^{\mathcal{F}}(\mathbf{w}_{\mathcal{F}})$ is positive.

Constructive Quantum Field Theory, I

- Cluster expansion = Taylor-Lagrange expansion of the functional integral:

$$F = 1 + H, \quad H = -\lambda \int_0^1 dt \int_{-\infty}^{+\infty} x^4 e^{-\lambda t x^4 - x^2/2} \frac{dx}{\sqrt{2\pi}}$$

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- Mayer expansion: define $H_i = -\lambda \int_0^1 dt \int_{-\infty}^{+\infty} x^4 e^{-\lambda t x^4 - x^2/2} \frac{dx}{\sqrt{2\pi}} = H \ \forall i$, $\epsilon_{ij} = 0 \ \forall i, j$ and write

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- To prove this, let us define $\eta_{ij} = -1$, $\epsilon_{ij} = 1 + \eta_{ij} = 1 + x_{ij}\eta_{ij}|_{x_{ij}=1}$ and apply the BKAR forest formula.

Constructive Quantum Field Theory, II

- $$F = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\mathcal{F}} \prod_{i=1}^{n} H_i(\lambda) \Big\{ \prod_{\ell \in \mathcal{F}} [\int_0^1 dw_\ell] \eta_\ell \Big\} \prod_{\ell \notin \mathcal{F}} [1 + \eta_\ell x_\ell^{\mathcal{F}}(w)]$$

  The logarithm of the forest formula is simply a tree formula. Then defining $G = \log F$,

  $$G = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\mathcal{T}} \prod_{i=1}^{n} H_i(\lambda) \Big\{ \prod_{\ell \in \mathcal{T}} [\int_0^1 dw_\ell] \eta_\ell \Big\} \prod_{\ell \notin \mathcal{T}} [1 + \eta_\ell x_\ell^{\mathcal{F}}(w)]$$

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- The convergence is easy because each H_i contains a different "copy" $\int dx_i$ of functional integration, and $|1 + \eta_\ell x_\ell^{\mathcal{F}}(w)| \leq 1$.
- It generalizes well to the case of lattice statistical mechanics ($d > 0$).
- However the link with Feynman graphs is somewhat lost, and furthermore il may be not optimal for curved or random space-time geometries.

My present personal motivation

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- I am currently working on several papers combining the LVE or LVR for cumulants.
- However, I am eager to learn more about *perturbative quantum field theory*, at this or other meetings, as some of my closest friends work in this field.

Loop Vertex Expansion, I

Intermediate field representation

$$
\begin{aligned}
F &= \int_{-\infty}^{+\infty} e^{-\lambda x^4 - x^2/2} \frac{dx}{\sqrt{2\pi}} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-i\sqrt{2\lambda}\sigma x^2 - x^2/2 - \sigma^2/2} \frac{dx}{\sqrt{2\pi}}\frac{d\sigma}{\sqrt{2\pi}} \\
&= \int_{-\infty}^{+\infty} e^{-\frac{1}{2}\log[1+2i\sqrt{2\lambda}\sigma] - \sigma^2/2} \frac{d\sigma}{\sqrt{2\pi}} = \int_{-\infty}^{+\infty} \sum_{n=0}^{\infty} \frac{V^n}{n!} d\mu(\sigma)
\end{aligned}
$$

Let us apply the forest formula, but using "replicas" of the intermediate field:

$$
V^n(\sigma) \to \prod_{i=1}^{n} V_i(\sigma_i), \quad d\mu(\sigma) \to d\mu_C(\{\sigma_i\}),
$$

$C_{ij} = \mathbf{1}_n = x_{ij}|_{x_{ij}=1}$, where $\mathbf{1}_n$ is the $n \times n$ matrix with entries one everywhere.

Loop Vertex Expansion, II

$$F = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\mathcal{F}} \Big\{ \prod_{\ell \in \mathcal{F}} [\int_0^1 dw_\ell] \Big\} \int \Big\{ \prod_{\ell \in \mathcal{F}} \frac{\partial}{\partial \sigma_{i(\ell)}} \frac{\partial}{\partial \sigma_{j(\ell)}} \prod_{i=1}^{n} V(\sigma_i) \Big\} d\mu_{C^{\mathcal{F}}}$$

where $C_{ij}^{\mathcal{F}} = x_\ell^{\mathcal{F}}(\{w\})$ if $i < j$, $C_{ii}^{\mathcal{F}} = 1$.

$$G = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\mathcal{T}} \Big\{ \prod_{\ell \in \mathcal{T}} [\int_0^1 dw_\ell] \Big\} \int \Big\{ \prod_{\ell \in \mathcal{T}} \frac{\partial}{\partial \sigma_{i(\ell)}} \frac{\partial}{\partial \sigma_{j(\ell)}} \prod_{i=1}^{n} V(\sigma_i) \Big\} d\mu_{C^{\mathcal{T}}}$$

where the second sum runs over trees !

Link with Feynman graphs can be recovered, and the conclusion is that the LVE should be better adapted for general background geometries, such as curved geometries, random geometries... $\to$ in short, to quantum gravity.

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The work of [Gurau-Krajewski,2014]

Gurau and Krajewski combines the LVE with cumulants. These cumulants, in physics, are usually referred to as "connected Dyson-Schwinger functions".

- They *define ordinary cumulants* which are based on ordinary Feynman graphs and amplitudes. They then define *scalar cumulants* for the topological expansion in the genus of combinatorial maps. A significant portion of their article is dedicated to Weingarten calculus [Collins,2003].

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- They *prove* that any ordinary cumulant is an analytic function inside a cardioid domain in the complex plane. They also demonstrate that all cumulants are Borel summable at the origin.

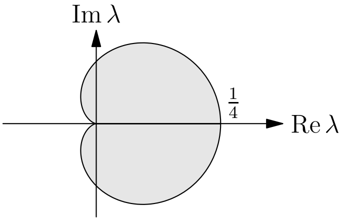

Figure: Analyticity domain in the complex λ plane

The work of [Gurau-Krajewski,2014]

For scalar cumulants regarded as functions of λ with N considered as a parameter, the domain of analyticity is reduced by a factor $1/4$:

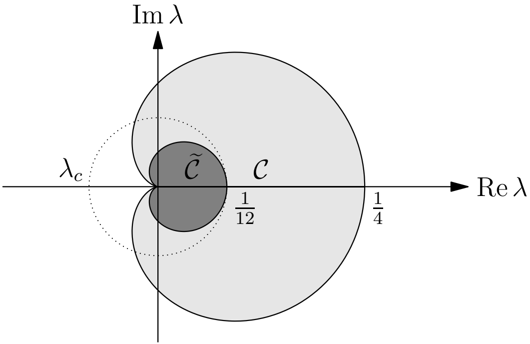

Figure: Analyticity domain of the topological expansion

Their work is an essential piece for anyone seriously interested in cumulants, even to this day. I am currently following in their footsteps.

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- I find also that Fuss-Catalan functions are shown rather easily to have bounded derivatives of all orders; that is the second feature which allows the LVR to work.
- Fuss-Catalan functions of order p also govern the leading term in the $N \to \infty$ limit of random tensor models of rank p [Bonzom et al., 2011].

The work of [Krajewski-R.-Sazonov,2019]

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- We apply the idea of reparametrization invariance to monomial interactions of arbitrarily high even order.
- The key notion is to use the Fuss-Catalan function T_p defined to be the solution analytic at the origin of the algebraic equation

$$zT_p^p(z) - T_p(z) + 1 = 0\,.$$

For any square matrix X we define the matrix-valued function

$$A(\lambda, X) \quad := \quad XT_p(-\lambda X^{p-1})$$

and also an N_l by N_l square matrix X_l and an N_r by N_r square matrix X_r through

$$X_l \quad := \quad MM^\dagger, \quad X_r := M^\dagger M.$$

A lemma of [Krajewski-R.-Sazonov,2019]

Lemma

In the sense of formal power series in λ

$$Z(\lambda, N_l, N_r) \quad = \quad \int dM dM^\dagger \ \exp\{-N_r \mathrm{Tr}_l X_l + \mathcal{S}(X_l, X_r)\}$$

where $\mathcal{S}$, the loop vertex action is

$$\mathcal{S}(X_l, X_r) \quad = \quad -\mathrm{Tr}_{lr} \log \Big[\mathbf{1}_{lr} + \lambda \sum_{k=0}^{p-1} A^k(X_l) \otimes_{lr} A^{p-1-k}(X_r) \Big].$$

The N_l by N_l matrix $A^k(X_l)$ acts on the left index of $\mathcal{H}_{lr}$ and the N_r by N_r matrix $A^{p-1-k}(X_r)$ acts on the right index of $\mathcal{H}_{lr}$, where $\mathcal{H}_l$ is the Hilbert space with $\dim \mathcal{H}_l = N_l$ and $\mathcal{H}_r$ is the Hilbert space with $\dim \mathcal{H}_r = N_r$.

This Lemma is proved by a change of variables $M \to P$ and we define $P(M)$ (up to unitary conjugation) through the implicit function formal power series equation:

$$Y_l := PP^\dagger, \quad Y_r := P^\dagger P, \quad X_l := A(Y_l), \quad X_r := A(Y_r).$$

The key result of [Krajewski-R.-Sazonov,2019]

Theorem

For any $\epsilon > 0$ there exists η small enough such that the expansion defined by $F(\lambda, N) = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\mathcal{T} \in \mathfrak{T}_n} A_{\mathcal{T}}$ is absolutely convergent and defines an analytic function of λ, uniformly bounded in N, in the "pacman domain"

$$P(\epsilon, \eta) := \{0 < |\lambda| < \eta, |\arg \lambda| < \pi - \epsilon\},$$

a domain which is uniform in N.

Here absolutely convergent and uniformly bounded in N means that for fixed ϵ and η as above there exists a constant K independent of N such that for $\lambda \in P(\epsilon, \eta)$

$$\sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\mathcal{T} \in \mathfrak{T}_n} |A_{\mathcal{T}}| \leq K < \infty.$$

where $\mathcal{T}$ is a LVR tree, $A_{\mathcal{T}}$ is the corresponding amplitude and $\mathfrak{T}_n$ is the set of LVR trees with n vertices. But it applies only to the simplest matrix with interaction $\lambda(\bar{M}M)^p$. We thought that to deduce the case of Hermitian or symmetric matrices would be relatively easy. In fact, it took us two years to understand and write the corresponding article!

The work of [Krajewski-R.-Sazonov,2021]

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- In [Krajewski-R.-Sazonov,2021] we extend the LVR to Hermitian matrices and real symmetric matrices.
- Since this paper is a sequel to [Krajewski-R.-Sazonov,2019], we would like to stress that the improved method introduced in this article is both simpler and more powerful. The basic formalism is still the LVR, combined with Cauchy holomorphic matrix calculus. But when Krajewski et al. used contour integral parameters attached to every *vertex* of the loop vertex representation, this paper introduces more contour integrals, one for each loop vertex *corner*. This results in simpler bounds for the norm of the corner operators.

The work of [Krajewski-R.-Sazonov,2021]

- In [Krajewski-R.-Sazonov,2021] we extend the LVR to Hermitian matrices and real symmetric matrices.
- Since this paper is a sequel to [Krajewski-R.-Sazonov,2019], we would like to stress that the improved method introduced in this article is both simpler and more powerful. The basic formalism is still the LVR, combined with Cauchy holomorphic matrix calculus. But when Krajewski et al. used contour integral parameters attached to every *vertex* of the loop vertex representation, this paper introduces more contour integrals, one for each loop vertex *corner*. This results in simpler bounds for the norm of the corner operators.
- In the Hermitian case, it uses a one-to-one change of variables (not singular for real positive values of λ):

$$K := H\sqrt{1 + \lambda^{p-1} H^{2p-2}}, \quad K^2 = H^2 + \lambda^{p-1} H^{2p}.$$

The key result of [Krajewski-R.-Sazonov,2021]

Theorem

For any $\epsilon > 0$ there exists η small enough such that the expansion in LVR trees is absolutely convergent and defines an analytic function of λ, uniformly bounded in N, in the uniform in N in a "pacman domain"

$$P(\epsilon, \eta) := \left\{ 0 < |\lambda| < \eta, |\arg \lambda| < \frac{\pi}{2} + \frac{\pi}{p-1} - \epsilon \right\}.$$

More precisely, for fixed ϵ and η as above there exists a constant $O(1)$, independent of N such that for $\lambda \in P(\epsilon, \eta)$

$$\sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\mathcal{T} \in \mathfrak{T}_n} |A_{\mathcal{T}}| \leq O(1) < \infty \,.$$

where $\mathcal{T}$ is again a LVR tree, $A_{\mathcal{T}}$ is the corresponding amplitude, and $\mathfrak{T}_n$ is the set of LVR trees with n vertices.

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- Secondly, I aim to construct scalar cumulants using Weingarten calculus, obtaining explicit and convergent expansions for these cumulants. This allows me to prove their analyticity and Borel summability in the appropriate norm of $JJ^\dagger$. uniformly as $N \to \infty$.

Variational LVE for Cumulants

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- Applying the idea of selecting the initial approximation based on the coupling constant, Sazonov and I develop an analytic continuation of the cumulants of the quartic matrix model beyond the standard LVE cardioid, taking into account the branch cut and considering arbitrarily large couplings.
- It is a non-trivial extension because of the sources $\bar{J}, J$ caused some difficulties. It may be solved perhaps by using different colors, as in the tensors [Bonzom et al., 2011].

In the more distant future

We want to extend the simple model to Hermitian matrices, the group being $U(N)$, symmetric matrices $O(N)$, symplectic matrices $Sp(N)$, or to the ten different Gaussian random-matrix ensembles of [Altland-Zirnbauer,2001]....

Thanks for your attention!