Motivation in Examples

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Goals

- Provide you with a solid motivation for the structures of the theory.

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- Introduce you to the basic ideas of correlation geometry.

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In a purely relational theory we aim instead to formulate the entire physical system in terms of the (cor)relations between different elementary objects.

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- Phase has no significance: $\psi \to e^{i\Lambda}\psi$,
  instead of ψ consider ray generated by ψ.
- Only probability density $|\psi(t, \vec{x})|^2$ is of physical significance

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- General question: Suppose we know $|\psi(t,\vec{x})|^2$ for all the wave functions of the system, what can we say about the spacetime structures (causality, metric, fields, ...).
- Try to "probe" spacetime by looking at $|\psi(t,\vec{x})|^2$.

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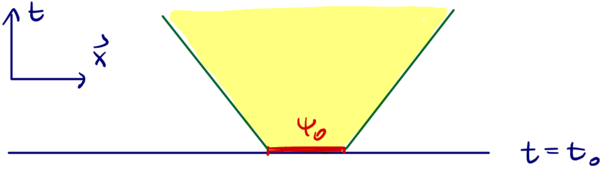
- Allows for detecting aspects of the causal structure of spacetime:



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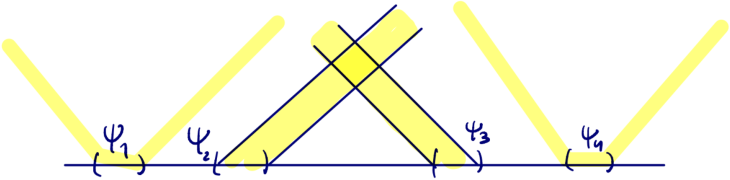
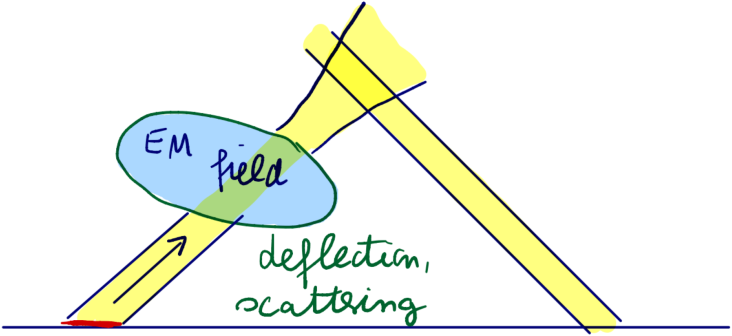


- More wave functions ⇒ More information about spacetime (spacetime resolution)
  E.g. collection of all compactly supported solutions
  → complete causal structure.
  Hawking-King-McCarthy and Malament
  → metric up to volume form.

Which spacetime structures are fundamental?

Possible to detect an electromagnetic field:



Two Point Correlation

It is known that the two point correlation function $G_F(x, y)$ of the free massless scalar field encodes the metric [1]

$$g_{\mu\nu}(x) = -\frac{1}{2}\left(\frac{\Gamma\left(\frac{D}{2}-1\right)}{4\pi^{\frac{D}{2}}}\right)^{\frac{2}{D-2}} \lim_{x\to y} \frac{\partial}{\partial x}\frac{\partial}{\partial y}\left(G_F(x,y)^{\frac{2}{2-D}}\right) \quad (1)$$

$\Rightarrow$ use correlations between wave functions to encode a physical system

[1] M. Saravani, S. Aslanbeigi, and A.Kempf. "Spacetime curvature in terms of scalar field propagators." Physical Review D 93.4 (2016): 045026.

Some Physical Musings

Principle of equivalence states that a freely falling observer feels no force. In mathematical terms this translates to the fact that freely falling observers follow geodesics γ

$$\nabla_{\dot{\gamma}}\dot{\gamma} = 0\,. \tag{2}$$

(Generalization of "straight lines" to curved manifolds.)

What structures does Spacetime provide

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$\Rightarrow$ To make sense of the equivalence principle we need a notion of parallel transport.

Vector Bundles

In the following we want to consider vector bundles

$$\mathrm{F}\mathcal{M} := \bigcup_{\mathrm{x}\in\mathcal{M}} \mathrm{F}_{\mathrm{x}}\mathcal{M} \tag{3}$$

Here $\mathrm{F}_{\mathrm{x}}\mathcal{M}$ is the fiber space at a point. For vector bundles $\mathrm{F}_{\mathrm{x}}\mathcal{M}$ is a vector space.

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Parallel transport tells you how the vector spaces at nearby points are related.

Sections

A section is a map that assigns to every point in $\mathcal{M}$ a unique element in the vector space $\mathrm{F_x}\mathcal{M}$.

$$\begin{array}{lcccc} \xi : & \mathcal{M} & \longrightarrow & \mathrm{F}\mathcal{M} & \\ & \mathrm{x} & \longrightarrow & \xi(\mathrm{x}) & \in \mathrm{F_x}\mathcal{M} \end{array}$$

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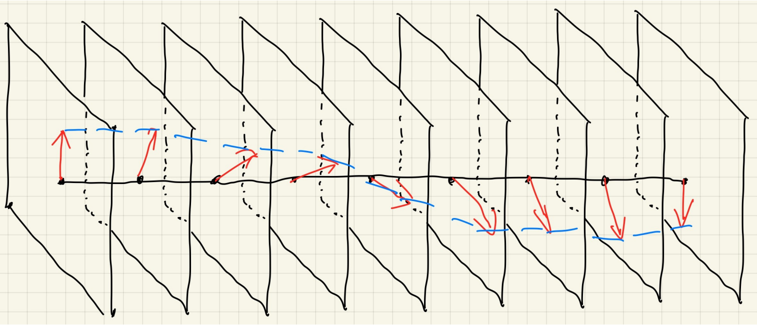
We denote with $\Gamma(\mathrm{F}\mathcal{M})$ the set of smooth sections over the fiber bundles.

How to Replace Parallel Transport

Key idea: Use preferred sections in your fiber bundle to link local structures in the fiber at points in your manifold through sections in your fiber bundle.



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Local structure:

- $\mathrm{b_x} : \mathrm{F_x}\mathcal{M} \times \mathrm{F_x}\mathcal{M} \longrightarrow \mathbb{C} \quad$ (or $\mathbb{R}$) non-degenerate hermitian form on $\mathrm{F_x}\mathcal{M} \times \mathrm{F_x}\mathcal{M}$.
  For example $\mathrm{b_x} = \langle \cdot | \cdot \rangle_\mathrm{x}$ the fiber-inner product at x.

The Principles of Correlation Geometry

Encode the local hermitian form in the global Hilbert space structure.

$$(\psi|\tilde{\mathbb{F}}(\mathrm{x})\phi) := \mathrm{b}_{\mathrm{x}}(\psi(\mathrm{x}), \phi(\mathrm{x})) \qquad \forall \psi, \phi \in \mathrm{V}$$

(using Riesz representation theorem)

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- Number of positive and negative eigenvalues of $\mathbb{F}(\mathrm{x})$ correspond to the signature (p, q) of the hermitian form.
- Local correlation map $\mathrm{F} : \mathcal{M} \to \mathcal{F}^{\mathrm{p},\mathrm{q}} \subset \mathrm{BL}(\mathcal{H})$ with $\mathcal{F}^{\mathrm{p},\mathrm{q}}$ a manifold.

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The local correlation map $\mathrm{G} : \mathcal{M} \to \mathcal{F}^{\mathrm{N},0} \subset \mathrm{BL}(\mathcal{H})$ identifies the metric $\mathrm{g}(\mathrm{x})_{\mu\nu}$ with a self-adjoint operator $\mathbb{G}(\mathrm{x})$.

The Missing Piece

Compare the scalar product and the local hermitian form:

- $(V_1|V_2) := \int_{\mathcal{M}} g(x)_{\mu\nu} V_1^{\mu}(x) V_2^{\nu}(x) \sqrt{|g|} d^N x$
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The missing piece in the local correlation operators that allows us to encode the geometry (M, g) in the operator manifold is the measure.

Let $\Omega \subset \mathcal{F}^{\mathrm{N},0}$ then the pushforward measure of the metric volume form of the manifold under the local correlation map is given by

$$\rho(\Omega) := \mathrm{G}_*[\mu_{\mathcal{M}}](\Omega) = \mu_{\mathcal{M}}(\mathrm{G}^{-1}(\Omega)) = \int_{\mathrm{G}^{-1}(\Omega)} \sqrt{|\mathrm{g}|}\mathrm{d}^{\mathrm{N}}\mathrm{x}$$

Example continued

Let (M, g) be a compact Riemannian manifold that is parallelizable. Let $\mathrm{V} \subset \Gamma(\mathrm{TM})$ be the span of N everywhere orthonormal vector fields $\mathrm{V}_1, \ldots, \mathrm{V}_\mathrm{N}$.

Example continued

Let (M, g) be a compact Riemannian manifold that is parallelizable. Let $V \subset \Gamma(TM)$ be the span of N everywhere orthonormal vector fields $V_1, \ldots, V_N$.
In this case we have

$$g(x)_{\mu\nu} V_i^{\mu}(x) V_j^{\nu}(x) = \delta_{ij} \qquad \forall\, x \in \mathcal{M}$$

and hence $\mathbb{G}(x) = 1$ for all $x \in \mathcal{M}$.

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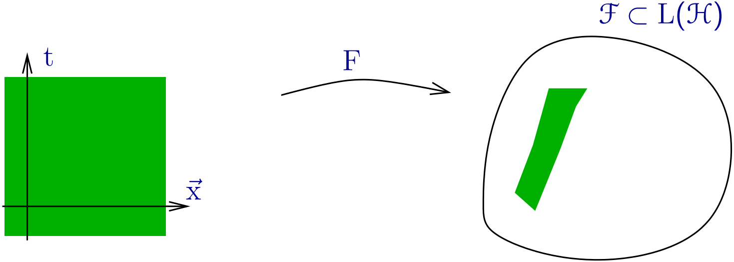
As a consequence we find that

$$\rho(1) = \int_{\mathbb{G}^{-1}(\Omega)=\mathcal{M}} \sqrt{|\mathrm{g}|}\mathrm{d}^\mathrm{N}\mathrm{x} = \mathrm{Vol}(\mathcal{M})$$

Some Observations

We see from the example above, that in order to retain substantial information about our original geometry $F\mathcal{M}$ we need to choose $\mathsf{dim}(V) >> \mathsf{dim}(F_x\mathcal{M})$



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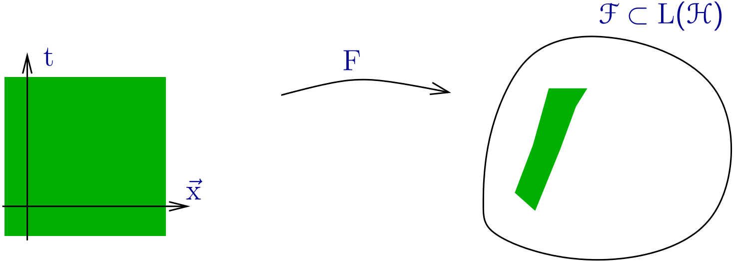

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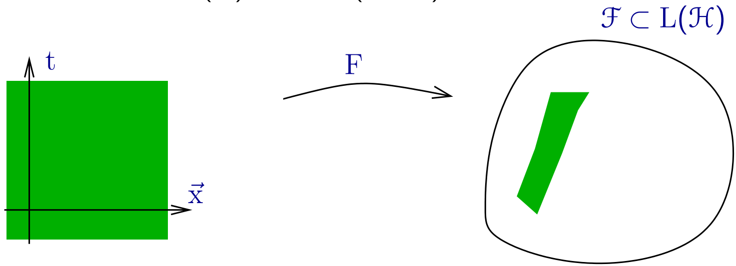

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- If V is chosen as a subspace of the solution space of a particular equation, the right hand side also encodes information about that equation.

More Observations

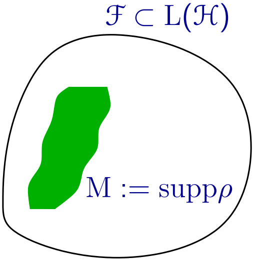

Image of F recovered as the support of the measure,

$$\mathrm{M} := \mathrm{supp}\ \rho = \Big\{ \mathrm{F} \in \mathcal{F} \mid \rho(\Omega) \neq 0 \text{ for every open neighborhood } \Omega \text{ of } \mathrm{F} \Big\}.$$

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- In particular we generically expect that $\mathrm{M}_1 := \operatorname{supp} \rho_1 \neq \operatorname{supp} \rho_2 =: \mathrm{M}_2$.
- Geometries with entirely different topologies can be encoded in the same $\mathcal{F}^{\mathrm{N},0}$

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Then F_1 and F_2 give rise to measures ρ_1 and ρ_2 on $\mathcal{F}^{\mathrm{N},0}$.

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- In particular we generically expect that $\mathrm{M}_1 := \operatorname{supp} \rho_1 \neq \operatorname{supp} \rho_2 =: \mathrm{M}_2$.
- Geometries with entirely different topologies can be encoded in the same $\mathcal{F}^{\mathrm{N},0}$

The same is also true when $(\mathcal{M}_1, \mathrm{g}_1) = (\mathcal{M}_2, \mathrm{g}_2)$ but $\mathrm{V}_1 \neq \mathrm{V}_2$ such that $\mathcal{H}_1 \neq \mathcal{H}_2$.

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We then consider the map

$$\mathrm{P}(\mathrm{x},\mathrm{y}) := \sum_{\mathrm{i}} |\psi(\mathrm{x})\rangle_{\mathrm{x}}\langle\psi(\mathrm{y})|_{\mathrm{y}} \qquad \in \mathrm{F}_{\mathrm{x}}\mathcal{M} \otimes (\mathrm{F}_{\mathrm{y}}\mathcal{M})^* \tag{4}$$

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If you know the value of an element u in $\mathcal{H}$ at y, then $P(x,y)$ tells you what its value at x is.

Correlation Geometry for Causal Fermion Systems

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- Let $\mathrm{F}\mathcal{M} = \mathrm{S}\mathcal{M}$ the spinor bundle over the manifold.
- Let $\mathrm{V} \subset \Gamma(\mathrm{S}\mathcal{M})$ be a suitable subset of the solution space to the Dirac equation.
- The hermitian form is given by $\mathrm{b_x}(\psi, \phi) = \psi^\dagger \phi$ is the indefinite inner product on $\mathbb{C}^{2\mathrm{n}}$ with signature (n, n).

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General strategy:

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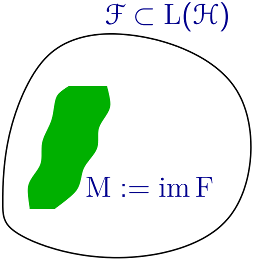
  (spacetime, matter fields, ...)
- Formulate a theory entirely with the objects on the right.



Causal Fermion Systems

Definition (Causal fermion system)

Let $(\mathcal{H}, \langle .|. \rangle_{\mathcal{H}})$ be Hilbert space

Given parameter $n \in \mathbb{N}$ ("spin dimension")

$\mathcal{F} := \Big\{ x \in \mathrm{L}(\mathcal{H})$ with the properties:

- x is symmetric and has finite rank
- x has at most n positive and at most n negative eigenvalues $\Big\}$

ρ a measure on $\mathcal{F}$