CFS as a Web of Correlations

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Joint work with Patrick Fischer

Causal Fermion Systems 2025, Regensburg

Goals

- Provide an impressionistic rapid fire sketch of a complete physical interpretation of the theory

Causal Fermion Systems

Definition (Causal fermion system)

Let $(\mathcal{H}, \langle .|. \rangle_{\mathcal{H}})$ be Hilbert space

Given parameter $n \in \mathbb{N}$ ("spin dimension")

$$\mathcal{F} := \Big\{ x \in L(\mathcal{H}) \text{ with the properties:}$$

- x is symmetric and has finite rank
- x has at most n positive and at most n negative eigenvalues $\Big\}$

ρ a measure on $\mathcal{F}$

Causal structure

Let $x, y \in \mathcal{F}$. Then

$x \cdot y \in L(H)$ has non-trivial complex eigenvalues $\lambda_1^{xy}, \ldots, \lambda_{2n}^{xy}$

Definition (causal structure)

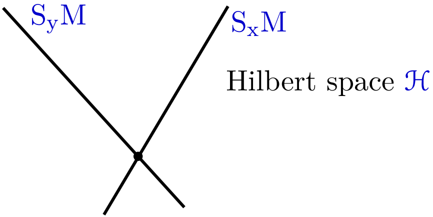
The points $x, y \in \mathcal{F}$ are called

$$\begin{cases} \text{spacelike separated} & \text{if } |\lambda_j^{xy}| = |\lambda_k^{xy}| \text{ for all } j, k = 1, \ldots, 2n \\ \text{timelike separated} & \text{if } \lambda_1^{xy}, \ldots, \lambda_{2n}^{xy} \text{ are all real} \\ & \text{and } |\lambda_j^{xy}| \neq |\lambda_k^{xy}| \text{ for some } j, k \\ \text{lightlike separated} & \text{otherwise} \end{cases}$$

Inherent Structures

- For $x \in M$, the eigenspace of x is the vector space $S_x M$.
  $\rightarrow$ Spinors

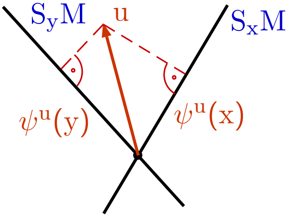
$$S_x M := x(\mathcal{H}) \subset \mathcal{H} \qquad \text{"spin space"}, \; \dim S_x M \leq 2n$$



Inherent Structures

- Physical wave functions:

$$\begin{aligned} \psi^{u}(x) = \pi_x \, u \;\; \text{with } u \in \mathcal{H} &\quad \text{physical wave function} \\ \pi_x : \mathcal{H} \to \mathcal{H} &\quad \text{orthogonal projection on } x(\mathcal{H}) \end{aligned}$$



Subsystems

$\tilde{\mathcal{H}} \supset \mathcal{H}$ an auxiliary Hilbertspace $\Omega = \sum_{i=1}^{N} |u_i\rangle \langle u_i| : \tilde{\mathcal{H}} \to \mathcal{H}$ a projection with $(|u_i\rangle)_i$ orthonormal basis.

$$\mathsf{tr}_{\tilde{\mathcal{H}}}[\Omega] = \mathsf{tr}_{\mathcal{H}}[\mathsf{id}_{\mathcal{H}}] = \dim \mathcal{H} = N$$

Total physical system: $(\mathcal{H}, \Omega)$

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Definition ((Sub-)System)

Let $(\mathcal{H}, \mathcal{F}_n, \rho)$ be a CFS, $\tilde{\mathcal{H}}$ an auxiliary Hilbert space and $\mathcal{H}_A \subseteq \mathcal{H}$ a sub Hilbert space of $\mathcal{H}$. Then a subsystem is the tuple $(\mathcal{H}_A, \omega_A)$, where $\omega_A : \tilde{\mathcal{H}} \to \mathcal{H}_A$ is the projection onto $\mathcal{H}_A$. Further, the particle number of a subsystem $(\mathcal{H}_A, \omega_A)$ is defined as $\mathrm{tr}_{\tilde{\mathcal{H}}}\, \omega_A$. We call $(\mathcal{H}, \Omega)$ the total system.

One-particle subsystem: $(\mathrm{span}\, \{|u\rangle\}\, , |u\rangle \langle u|)$

Pauli Exclusion Principle

Definition (Occupation number)

Let $|\mathrm{u}\rangle \in \tilde{\mathcal{H}}$ be a state and $(\mathcal{H}_\mathrm{A}, \omega_\mathrm{A})$ a (sub-)system, then the occupation number of $|\mathrm{u}\rangle$ in $(\mathcal{H}_\mathrm{A}, \omega_\mathrm{A})$ is defined as $\langle \mathrm{u}|\omega_\mathrm{A} \mathrm{u}\rangle$.

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Proposition (Pauli Exclusion Principle)

Let $|\mathrm{u}\rangle \in \tilde{\mathcal{H}}$ be a state and $(\mathcal{H}_\mathrm{A}, \omega_\mathrm{A})$ a (sub-)system, then the occupation number satisfies

$$0 \leq \langle \mathrm{u}|\omega_\mathrm{A}\mathrm{u}\rangle \leq 1. \tag{1}$$

Fundamental Observables in CFS

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Definition (Position Observables)

Let $(\mathcal{H}, \mathcal{F}_{\mathrm{n}}, \rho)$ be a CFS and $\mathcal{U} \subseteq \mathcal{F}_{\mathrm{n}}$, then the observable of the region $\mathcal{U}$ is the operator

$$\mathcal{O}(\mathcal{U}) := \int_{\mathcal{U}} \pi_{\mathrm{x}} \mathrm{d}\rho(\mathrm{x}). \tag{2}$$

The expectation value of an operator $\mathcal{O} : \mathcal{H} \to \mathcal{H}$ for a (sub-) system $(\mathrm{A}, \omega_{\mathrm{A}})$ is defined as

$$\langle \mathcal{O} \rangle_{\omega_{\mathrm{A}}} := \frac{1}{2\mathrm{n}} \operatorname{tr}_{\mathcal{H}} \left[\omega_{\mathrm{A}} \mathcal{O}(\mathcal{U}) \right]. \tag{3}$$

Properties of the Observables

Proposition

Let $\mathcal{U} \subseteq \mathcal{F}_{\mathrm{n}}$, then the observable of $\mathcal{U}$ satisfies

1. $\langle \mathcal{O}(\mathcal{U}) \rangle_{\omega_{\mathrm{A}}} \geq 0$ for every subsystem $(\mathcal{H}_{\mathrm{A}}, \omega_{\mathrm{A}})$,
2. $\mathcal{O}(\mathcal{U}) = 0 \Leftrightarrow \mathrm{M} \cap \mathcal{U} = 0$.

Properties of the Observables

Proposition

For a subset of the regular correlation operators $\mathcal{U} \subset {\mathcal{F}_\mathrm{n}}^{\mathrm{reg}}$, the expectation value of the position observable $\mathcal{O}(\mathcal{U})$ for the total system $(\mathcal{H}, \Omega)$ is the volume of the spacetime region $\mathcal{U} \cap \mathrm{M}$

$$\langle \mathcal{O}(\mathcal{U}) \rangle_\Omega = \rho(\mathcal{U}).$$

Spacetime Superposition

Definition

Let $(\mathcal{H}_\mathrm{A}, \omega_\mathrm{A})$ be a (sub-) system, then the measure assigned to this (sub-) system is defined by

$$\rho_{\omega_\mathrm{A}}(\mathcal{U}) := \langle \mathcal{O}(\mathcal{U}) \rangle_{\omega_\mathrm{A}} \text{ for measureable } \mathcal{U} \subseteq \mathcal{F}$$

and the corresponding spacetime is defined by

$$\mathrm{M}_{\omega_\mathrm{A}} := \mathrm{supp}\ \rho_{\omega_\mathrm{A}}.$$

Spacetime Superposition

Proposition

Let $(|\mathrm{u_i}\rangle)_\mathrm{i}$ be a basis of $\mathcal{H}$, then the following statements hold

1. $\displaystyle\sum_{\mathrm{i}=1}^{\mathrm{N}} \rho_\mathrm{i}(\mathcal{U}) = \rho(\mathcal{U})$ for every measurable subset $\mathcal{U} \subseteq \mathcal{F}^{\mathrm{reg}}$,
2. $\displaystyle\bigcup_{\mathrm{i}=1}^{\mathrm{N}} \mathrm{M_i} = \mathrm{M}.$

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Corollary

Let M_1 and M_2, $M_1 \cap M_2 \neq \emptyset$, be two one-particle spacetimes. The causal relation between $x, y \subset M_1 \cap M_2$ is independent of whether we consider them as points in M_1 or as points in M_2.

(De)localization of States

Definition (Localization of States)

A state u_i is localized in a spacetime region $\mathcal{U} \subset M$ if

$$\{x \in M_i | \exists y \in \mathcal{U} \text{ such that x and y are causally separated } \} = M_i \, .$$

Definition (Delocalized States)

A state u_i is said to be delocalized if

$$M_i = M$$

The Fermionic Projector

Working with the fermionic projector is more convenient

$$\mathrm{P(x,y)} := \pi_\mathrm{x}\mathrm{y}|_{\mathrm{S_y}} = \sum_{\mathrm{i=1}}^{\mathrm{N}} |\psi^{\mathrm{u_i}}(\mathrm{x}) \succ \prec \psi^{\mathrm{u_i}}(\mathrm{y})| : \mathrm{S_y} \to \mathrm{S_x}\,.$$

The closed chain

$$\mathrm{A_{xy}} := \mathrm{P(x,y)\,P(y,x)} = \pi_\mathrm{x}\,\mathrm{y}\,\pi_\mathrm{y}\,\mathrm{x} : \mathrm{S_x} \to \mathrm{S_x}$$

has the same Eigenvalues as the operator product xy.

Causal Correlations

Definition (Causal Correlation Operator)

Let $x, y \in M$, A_{xy} the closed chain between x and y as defined above, then the causal correlation operator is defined as

$$\tilde{A}_{xy} : S_x \to S_x = \begin{cases} 0 & \text{if x and y spacelike separated} \\ A_{xy} & \text{otherwise.} \end{cases} \tag{4}$$

Correlation Strength

Definition (One-Particle-Two-Point-Correlation-Strength)

Let $|u\rangle \in \mathcal{H}$ and $x, y \in M$, then the one-particle-two-point-correlation-strength $b_u(x, y)$ is defined as

$$b_u(x, y) := \frac{1}{2n} \operatorname{tr}_{S_x} \left[|u\rangle \langle u| \tilde{A}_{xy} \right]. \tag{5}$$

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Definition (Two-Point-Correlation-Strength)

Let $(|u_i\rangle)_i$ be a basis of $\mathcal{H}$, $x, y \in M$, then the two-point-correlation-strength is defined as

$$b(x,y) := \sum_{i=1}^{N} b_{u_i}(x,y) = \left\langle \tilde{A}_{xy} \pi_x \right\rangle_\Omega = \frac{1}{2n} \operatorname{tr}_{S_x} \tilde{A}_{xy}. \tag{6}$$

The Principle of Minimal Fluctuations

Proposition

Let $\mathrm{P}(\mathrm{x},\mathrm{y})$ be the kernel of the fermionic projector of the scalar-vector-form, then the Lagrangian is given by

$$\mathcal{L}(\mathrm{x},\mathrm{y}) = 4\,\mathsf{Var}_\Omega\left[\tilde{\mathrm{A}}_{\mathrm{xy}}\right] := 4\left\langle\left(\tilde{\mathrm{A}}_{\mathrm{xy}} - \left\langle\tilde{\mathrm{A}}_{\mathrm{xy}}\right\rangle_\Omega\right)^2\right\rangle_\Omega \tag{7}$$

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If we consider the correlation strength to be a proxy for causal distance then this amounts to a principle of minimal flucuation for the causal structure of the one-particle spacetimes.

Description of Experiments

1. How do you test a system that comprises the entire universe from within itself?
2. How does a purely relational theory give rise to our conventional experience of time and space?

The Probe-Background Split

Split the total physical system $(\mathcal{H}, \Omega)$ into a probe $(\mathcal{H}_\mathrm{p}, \omega_\mathrm{p})$ and the background $(\mathcal{H}_\mathrm{b}, \omega_\mathrm{b})$ such that

$$\mathcal{H} = \mathcal{H}_\mathrm{p} \oplus \mathcal{H}_\mathrm{b} \qquad \text{and} \qquad \Omega = \omega_\mathrm{p} + \omega_\mathrm{b} \tag{8}$$

with $\mathsf{dim}(\mathcal{H}_\mathrm{p}) \ll \mathsf{dim}(\mathcal{H}_\mathrm{b})$.

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Find a macroscopic continuum limit description of $(\mathcal{H}_\mathrm{b}, \omega_\mathrm{b})$ in terms of emergent variables $\mathrm{g}_{\mu\nu}, \mathrm{A}_\mu, \dots$ via the local correlation map $\mathrm{F}[\mathrm{g}_{\mu\nu}, \mathrm{A}_\mu, \dots] : \mathcal{M} \to \mathcal{F}_\mathrm{n}$.

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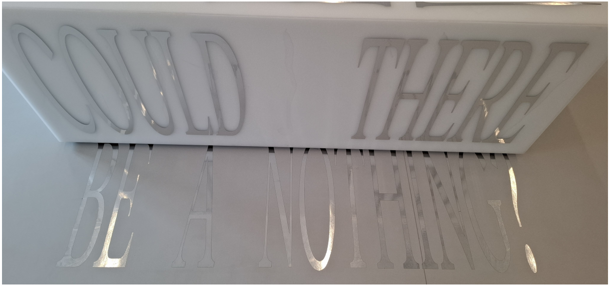
In experiments, we study the evolution of a probe with respect to the web of correlations spanned by the background.

The Probe-Background Split

Two caveats:

1. The web of correlation spanned by the background subsystem $(\mathcal{H}_\mathrm{b}, \omega_\mathrm{b})$ has to admit an approximate effective description in terms of a continuum limit.
2. To be able to study the evolution of a system in a laboratory, we have to be able to meaningfully localize the probe $(\mathcal{H}_\mathrm{p}, \omega_\mathrm{p})$ on the scale of the experiment.

Summary and Conclusion



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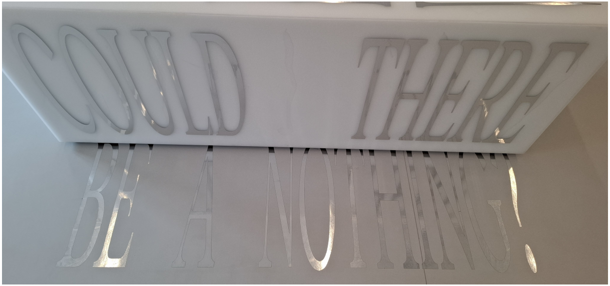

There is nowhere where there is nothing!

But there are many places where nothing can be localized.