CFS 2025

CONSERVATION LAWS

based on chapter 9 "CFS" by Finster, Kindermann, Treude

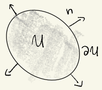
A CFS describes spacetime as a whole, but in daily life, we encounter objects that "live in space".

In this lecture, we'll see how to formulate them in terms of Surface Layer Integrals. In order to concretely see what SLI are, we begin by considering the question of how conservation law can be formulated

1) CONSERVATION LAWS

physical observable $Q = \int_{\mathcal{U}\subset\Sigma} \rho(x)\, d\mathrm{vol}_\Sigma$.
density

rate of change $\frac{d}{dt} Q = \int_{\mathcal{U}} \frac{\partial}{\partial t}\rho \, d\mathrm{vol}_\Sigma = -\int_{\partial \mathcal{U}} \vec{J}\cdot\vec{n}\, d\mathrm{Sup}_{\partial\Sigma}$
Typical current $\vec{v}_\rho$



$\Updownarrow$

conserved $\int_{\partial\mathcal{U}} \vec{J}\cdot\vec{n}\, d\mathrm{Sup}_{\partial\Sigma} = 0$

EXAMPLE Dirac current

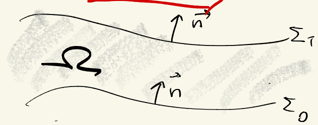
$J \cdot X := \langle \psi | \gamma(X) \psi \rangle$

$\forall X$

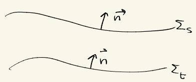
Green's identity

$$0 = \int_\Omega \underbrace{\left(\langle \psi | D\varphi\rangle - \langle D^* \psi | \varphi \rangle\right)}_{\mathrm{div}\, J} d\mathrm{vol}_M .$$

$$= \int_{\partial\Omega} \langle \psi | \gamma(n)\varphi \rangle \, d\mathrm{Sup}_{\partial\Omega}$$



THE CURRENT CONSERVATION on a spacetime $M = \mathbb{R}\times\Sigma$ for symmetric local operator.



$$\int_{\Sigma_t} \vec{J}\cdot\vec{n}\, d\Sigma_t = \int_{\Sigma_s} \vec{J}\cdot\vec{n}\, d\Sigma_s$$

PROBLEM: this law holds when the dynamic is local. For nonlocal theory it cannot be expected. In CFS there is an integration measure for the whole spacetime but none for Σ_t.

2) NOETHER'S THEOREM

continuous symmetry of a Lagrangian system $\Rightarrow$ Noether's current $\Rightarrow$ CURRENT CONSERVATION

smooth family of $\{\psi_s\}_s$ with $\psi_{s=0} = \psi$
spacetime points $\{x_s\}_s$ with $x_{s=0} = x$

for $s \in (-\tau_M, \tau_M)$

Symmetry of the action for any compact $\Omega \subset M$

$$\int_\Omega \mathcal{L}(\psi, \partial_\mu\psi, x)\, d\mathrm{vol}_M = \int_{\Omega_s = \{x_s \mid x\in\Omega\}} \mathcal{L}(\psi_s, \partial_\mu\psi_s, x_s)\, d\mathrm{vol}_M$$

$\Rightarrow$

$$J^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\delta\psi + \mathcal{L}\,\delta x^\mu - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\partial_j\psi\,\delta x^j$$

$$\delta\psi := \frac{d}{ds}\psi_s\Big|_{s=0} \qquad \delta x := \frac{d}{ds} x_s\Big|_{s=0}$$

If ψ satisfies Euler-Lagrangian equation $\Rightarrow \partial_\mu J^\mu = 0 \overset{\text{Gauß THM}}{\Longleftrightarrow} \int_{\Sigma_s} J\cdot n\, d\Sigma = \int_{\Sigma_t} J\cdot n\, d\Sigma$

EXAMPLE Dirac current symm: $\psi_s = e^{is}\psi$, $x_s = x$ conserved charges = Particles numb.

3) Extension to CFS.

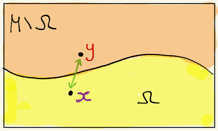
1- WHAT SHOULD BE A CURRENTS? integration measure for the whole spacetime but none for Σ_t

2- WHAT SHOULD REPLACE NOETHER'S THEOREM

the alternative for CURRENTS — SURFACE LAYER INTEGRAL

$\rho :=$ minimazing measure

$\Omega \subset M := \text{supp}\,\rho$ spacetime



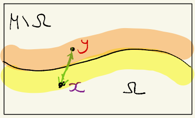
"connect" the points Ω and $M\setminus\Omega$

$$\int_\Omega d\rho(x) \int_{M\setminus\Omega} d\rho(y)\, (\cdots)\, \mathcal{L}(x,y)$$

suitable differential operator, skew-symm. in $x \leftrightarrow y$

FOR SMALL NONLOCALITIES

$\mathcal{L}(x,y) = 0$ for $|x-y|>d$



the alternative for CONSERVATION LAW — NOETHER-LIKE THEOREMS

a symmetry in a CFS $\{\Phi_s\}_{s\in(-\delta,\delta)}$ with $\Phi_s : M := \text{supp}\,\rho \to \mathcal{F} \subset \text{Lin}(\mathcal{H})$

variations

$\Phi_{s=0} = \mathbb{1}$

$\Phi_s \cdot \Phi_t = \Phi_{s+t}$

self-adj. and finite rank

which leaves invariant the Lagrangian $\mathcal{L}(x, \Phi_s(y)) = \mathcal{L}(\Phi_{-s}(x), y)$

CONSERVATION LAW for Ω compact:

$$\frac{d}{ds} \int_\Omega d\rho(x) \int_{M\setminus\Omega} d\rho(y) \Big(\mathcal{L}(\Phi_s(x), y) - \mathcal{L}(x, \Phi_s(y))\Big)\Big|_{s=0} = 0$$

INFINITESIMALLY

generator: $v := \frac{d}{ds}\Phi_s\Big|_{s=0}$ $\quad \mathbf{v} = (0, v)$

symmetry: $(\nabla_{1,\mathbf{v}} + \nabla_{2,\mathbf{v}})\, \mathcal{L}(x,y) = 0$

$\Downarrow$

CONSERVATION LAW

$$\int_\Omega d\rho(x) \int_{M\setminus\Omega} d\rho(y)\, (\nabla_{1,\mathbf{v}} - \nabla_{2,\mathbf{v}})\, \mathcal{L}(x,y) = \mathcal{I}_\rho^\Omega(\mathbf{v})$$

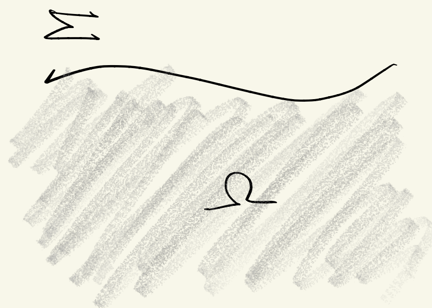
EXAMPLE generalized Dirac current $\Phi(s,x) = U_s \, x \, U_s^{-1}$ $\quad U_s := e^{isA}, \; A \in B(\mathcal{H})$

Last but not least, we can represent the SLI on $\mathcal{H}$.

Consider a finite dim. CFS and define

commutator vector field $\quad \mathcal{C}(x) := i\,[\,|u\rangle\langle u|, x\,]$

commutator jet $\quad \mathfrak{v} = (0, \mathcal{C})$



For Ω be $J^-(\Sigma)$ Let's define

$$\mathcal{C}(u)^{\Omega} = \int_{\Omega} d\rho(x) \int_{M \setminus \Omega} d\rho(y) \left(\nabla_{1,\mathfrak{c}} - \nabla_{2,\mathfrak{c}} \right) \mathcal{L}(x,y)$$

POSITIVE FUNCTIONAL

and using polarization formula COMMUTATOR INNER PRODUCT

$$\langle u | v \rangle_{\rho}^{\Omega} = 2i \left(\int_{\Omega} d\rho(x) \int_{M \setminus \Omega} d\rho(y) - \int_{M \setminus \Omega} d\rho(x) \int_{\Omega} d\rho(y) \right) \langle \psi^{u}(x) \,|\, Q(x,y)\, \psi^{v}(y) \rangle$$

physical wave function