Quantum entropies in Quantum Field Theory

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Outline for this Talk

- Beckenstein Bound
- Generalize to QFT (Casini) $\Rightarrow$ Relative Entropy
- Crash Course Modular Theory
- Relative Entropy defined via Modular operator
- Applications to noncommutative QFT

The Classical Bekenstein Bound

What happens if we throw a hot cup of (filter-)coffee into a black hole?

Black Hole Entropy

For a stationary, non-charged, non-rotating black hole, the entropy is proportional to the event horizon area:

$$S_{BH} = \alpha M^2.$$

Dropping Matter into a Black Hole (Poor man's derivation)

Consider a system outside the black hole with mass $m \ll M$ and entropy S. The total initial entropy is:

$$S^- = S_{BH} + S$$

Dropping m into the black hole (neglecting energy losses) gives:

$$S_{BH+m} = \alpha(M+m)^2 \approx \alpha M^2 + 2\alpha Mm = S_{BH} + 2\alpha Mm$$

Entropy Increase Condition

$$S_{BH+m} - S^- \geq 0$$

Thus,

$$S \leq 2\alpha Mm$$

The Bekenstein Bound

If BH is large enough to swallow the system (radius R), and identifying $E = m$, one finds:

Bekenstein Bound [Beckenstein 1981]

For a system with radius R and energy E:

$$\boxed{S \leq 2\pi k_B R\, E}$$

where k_B is the Boltzmann constant.

- Universal entropy bound on physical systems
- Derived from black hole entropy formula

Quantum Field Theory Formulation of the Beckenstein Bound

Bekenstein Bound: Formulation in QFT

Von Neumann entropy
S diverges for type
III algebras in QFT

Energy E is ill-defined:
Local energy density has
UV divergences in QFT

$$S \leq 2\pi\, k_B\, R\, E$$

Radius R ambiguous

Global vs local: "For global states, E is well-defined, but R is not; for localized states R is well-defined, vacuum entanglement (pair creation) makes both S and E UV-sensitive and ill-defined without careful renormalization."

QFT Challenges: Divergences

Problem [Casini 08]

Naive application of von Neumann entropy in QFT leads to divergences

- Physical origin: UV divergences from vacuum fluctuations

- Solution: Subtract vacuum contributions

Regularized Entropy

$$S_V = S(\rho_V) - S(\rho_V^0) = -\operatorname{tr}(\rho_V \log \rho_V) + \operatorname{tr}(\rho_V^0 \log \rho_V^0)$$

where:

- V: spatial region of interest
- ρ_V: reduced density matrix of the excited state associated with V
- $S(\rho_V^0)$: von Neumann entropy of vacuum state

Local Hamiltonian and Vacuum State

Vacuum Density Matrix

$$\rho_V^0 = \frac{e^{-K}}{\operatorname{tr} e^{-K}}$$

where K is the local Hamiltonian (self-adjoint, dimensionless).

Example

For half-space $V = \{x^1 > 0\}$, K is the boost operator:

$$K = L_{01} = 2\pi \int_{x^1 > 0} x^1 \, T_{00}(0, \mathrm{x}) \, d\mathrm{x}$$

- T_{00}: Hamiltonian density operator, physical units: $[R \cdot E] = [L_{01}]$

Regularized Energy

Problem

Expectation value $\langle K \rangle_{\rho_V}$ diverges due to vacuum fluctuations

Solution

Take for right hand side, difference between excited and vacuum states:

$$K_V = \mathrm{tr}(K\rho_V) - \mathrm{tr}(K\rho_V^0)$$

QFT Bekenstein Bound

$$\boxed{S_V \leq K_V}$$

Relative Entropy

Definition

$$\boxed{S_{rel,V} = \operatorname{tr} \rho_V (\log(\rho_V) - \log(\rho_V^0)) \geq 0}$$

Theorem (Positivity of Relative Entropy)

The relative entropy $S_{rel,V}$ is always non-negative.

Quantum relative entropy $S(\rho \parallel \sigma)$ has two key interpretations:

1. State Distinguishability, where reference state σ
2. Quantifies **unexpected information content** when encountering ρ while expecting σ

Key Result

The positivity of relative entropy implies the Bekenstein bound!

Full Circle

Black hole entropy $\rightarrow$ Bekenstein bound $\rightarrow$ QFT formulation $\rightarrow$ Relative entropy positivity

Mathematical Framework to study relative Entropy

Tomita-Takesaki-Modular theory [1967, 1970]

Tomita-Takesaki Theory & Relative Entropy

Tomita-Takesaki Theory: Foundations

Basic Setup:

- $(\mathcal{M}, \mathscr{H}, |\Omega\rangle)$: von Neumann algebra $\mathcal{M}$ on Hilbert space $\mathscr{H}$
- $|\Omega\rangle$: cyclic and separating vector for $\mathcal{M}$

Tomita Operator S:

- Definition: $S\,A\,|\Omega\rangle = A^*|\Omega\rangle$ for all $A \in \mathcal{M}$
- S is closable and densely defined

Polar Decomposition:

$$S = J\Delta^{1/2}$$

- J: modular conjugation (anti-linear, unitary, $J^2 = 1$)
- $J\mathcal{M}J = \mathcal{M}'$ (commutant relation)
- Δ: modular operator (positive, self-adjoint)

Modular Group and KMS States

Modular Automorphism Group:

- $\sigma_t(A) = \Delta^{it} A \Delta^{-it}$ for $t \in \mathbb{R}$
- One-parameter group of automorphisms: $\sigma_t \circ \sigma_s = \sigma_{t+s}$
- Covariance: $\sigma_t(\mathcal{M}) = \mathcal{M}$

KMS State Definition:

- A state ω is KMS at $\beta > 0$ for $\{\sigma_t\}$ if, for all $A, B \in \mathcal{M}$, the map $t \mapsto \omega\big(A\,\sigma_t(B)\big)$ extends analytically to $0 < \operatorname{Im} z < \beta$ and satisfies

$$\omega\big(A\,\sigma_t(B)\big) = \omega\big(\sigma_{t+i\beta}(B)\,A\big).$$

Connection to Modular Theory:

- State $\omega(\cdot) = \langle\Omega| \cdot |\Omega\rangle$ is KMS for $\{\sigma_t\}$ at $\beta = 1$

Relative Modular Theory

Setup:

- Two cyclic and separating states: ω and ω'
- Corresponding vectors: $|\Omega\rangle$ and $|\Omega'\rangle$

Relative Tomita Operator:

- Definition: $S_{\omega',\omega}\, A\, |\Omega'\rangle = A^*|\Omega\rangle$ for $A \in \mathcal{M}$
- Polar decomposition: $S_{\omega',\omega} = J_{\omega',\omega}\Delta^{1/2}_{\omega',\omega}$

Properties:

- $J_{\omega',\omega}$: relative modular conjugation
- $\Delta_{\omega',\omega}$: relative modular operator
- $J_{\omega',\omega}\mathcal{M}J_{\omega',\omega} = \mathcal{M}'$

Araki-Uhlmann Relative Entropy

Definition:

$$\boxed{S_{\mathrm{rel}}(\omega', \omega) = -\langle \Omega | \log\big(\Delta_{\omega', \omega}\big) | \Omega \rangle}$$

Special Case - Unitarily Equivalent States:

- If $\omega(U \cdot U^{-1}) = \omega'(\cdot)$ for some unitary U
- Araki [1977]-Uhlmann [1977] formula:

$$S_{\mathrm{rel}}(\omega', \omega) = i \frac{d}{dt} \langle \Omega \big| U \Delta^{it} U^* \Delta^{-it} \Omega \rangle \big|_{t=0}$$

Applications of relative Entropy in QFT

Bisognano-Wichmann Theorem & KMS

Physical Context:

- Rindler wedge: $\mathcal{W}_R := \{x = (x_0, x_1, \ldots, x_n) \in \mathbb{R}^d : x_1 > |x_0|\}$
- Wedge algebra: local observables in the right wedge

Bisognano-Wichmann Theorem:

- Modular operator: $\Delta = e^{2\pi \boldsymbol{L_{01}}}$
- Boost generator: $L_{01} = \int d^{d-1}x\, x^1\, T_{00}(x)$

KMS Connection:

- Vacuum state $|\Omega\rangle$ is KMS for wedge algebra at $\beta = 2\pi$
- Modular group σ_t implemented by boosts: $\sigma_t(A) = U(\Lambda(-2\pi t))AU(\Lambda(2\pi t))$
- **Unruh effect**

Explicit Relative Entropy Calculation

Setup:

- Reference state: vacuum ω
- Excited state: ω' generated by $U|\Omega\rangle = e^{i\phi(f)}|\Omega\rangle$
- Operators localized in right wedge $\mathcal{W}_R$

Calculation:

$$
\begin{aligned}
S_0(\omega', \omega) &= i\frac{d}{dt}\langle\Omega|U\Delta^{it}U^*\Delta^{-it}\Omega\rangle\Big|_{t=0} \\
&= i\frac{d}{dt}\langle\Omega|e^{i\phi(f)}e^{2\pi itL_{\mathbf{01}}}e^{-i\phi(f)}e^{-2\pi itL_{\mathbf{01}}}\Omega\rangle\Big|_{t=0} \\
&= 2\pi\int_{x^{\mathbf{0}}=0\,,x^{\mathbf{1}}>0} x^1\; T_{00}\; dx
\end{aligned}
$$

T_{00} is the energy density smeared with f.

Recent Works on S_{rel}: Incomplete Chronological Overview

2019

- **Longo, R. (2019)** - "Entropy of Coherent Excitations"
  - Explicit formula for vacuum relative entropy of coherent states on wedge algebras
- **Hollands, S. (2019)** - "Relative entropy for coherent states in chiral CFT"
  - Result: Relative entropy between vacuum and exponentiated stress tensor equals c times Schwarzian action of diffeomorphisms
- **Hollands, S.; Ishibashi, A. (2019)** - "News vs Information"
  - Relative entropy in linearized quantum gravity around black holes
  - Main result: $\frac{d}{dt}(S + A/4) = 2\pi F$ (entropy-area-flux relation)

Recent Works on S_{rel}

- **Casini H., Grillo S., Pontello, D. (2019)** - "Relative entropy for coherent states from Araki formula"
  - Computes vacuum–coherent relative entropy in the Rindler wedge via Araki's formula, matching the canonical result.

2020

- **Faulkner T., Hollands S., Swingle B., Wang Y. (2020)**- "Approximate recovery and relative entropy I. general von Neumann subalgebras"

Recent Works on Relative Entropy: 2021

2021

- **Edoardo D'Angelo (2021)**- "Entropy for spherically symmetric, dynamical black holes from the relative entropy between coherent states of a scalar quantum field"
- **Kurpicz, F.; Pinamonti, N.; Verch, R. (2021)** - "Temperature and entropy-area relation of quantum matter near spherically symmetric outer trapping horizons"
- **Ciolli, F.; Longo, R.; Ranallo, A.; Ruzzi, G. (2021)** - "Relative entropy and curved spacetimes"
  - QNEC inequality for coherent states in curved spacetime

Some Works on Relative Entropy: 2023-2025

2022

- **Galanda, S.; AM ; Verch, R. (2023)** - "Relative Entropy of Fermion Excitation States on the CAR Algebra"
  - Counterpart to CCR results, uses self-dual CAR algebra

2024

- **Fröb, M.; AM ; Papadopoulos, K. (2024)** - "Relative Entropy in de Sitter is a Noether Charge"
  - Connection of relative entropy to Noether charge of modular flow translations

- **Fröb, M.; Sangaletti, L. (2024)** - "Petz-Rényi relative entropy in QFT from modular theory"

- **Finster F., Jonsson R., Lottner M. , Murro S., AM, (2024)**- Notions of Fermionic Entropies of a Causal Fermion System
  - Defines fermionic von Neumann, entanglement, and relative entropies for causal fermion systems via reduced one-particle density operator
  - connects to modular-theoretic computations of relative entropy

2025

- **Hollands, S.; Longo, R. (2025)** - "A New Proof of the QNEC"

- **Finster, F.; AM (2025)** - "The Relative Fermionic Entropy in Two-Dimensional Rindler Spacetime"
  - Fermionic relative entropy using modular theory and density operators
  - Application to non-unitary excitations in Rindler spacetime

Applications of relative Entropy in noncommutative QFT

Relative Entropy in QG - Intro NCQFT

[DFR01] QFT in a NC Minkowski-spacetime is represented on $\mathcal{V} \otimes \mathscr{F}_s(\mathscr{H})$, where $\mathcal{V}$ is the representation space of X

$$[X_\mu, X_\nu] = i\theta_{\mu\nu},$$

where $\mu, \nu = 0, \ldots, 3$ and θ is a skew-symmetric matrix

$$\theta_{\mu\nu} = \begin{pmatrix} 0 & \Theta & 0 & 0 \\ -\Theta & 0 & 0 & 0 \\ 0 & 0 & 0 & \Theta' \\ 0 & 0 & -\Theta' & 0 \end{pmatrix},$$

with $\Theta, \Theta' \in \mathbb{R}$ and $\Theta \geq 0$.

Application to NCQFT

[GL07] proved that $\phi_{\otimes}$ can be written on the Fock space $\mathscr{F}_s(\mathscr{H})$ by existence of a unitary map $\mathcal{U}$ from $\mathcal{U} : \mathcal{V} \otimes \mathscr{F}_s(\mathscr{H}) \to \mathscr{F}_s(\mathscr{H})$

$$
\begin{aligned}
\phi_\theta(f) :&= \int d^4x\, f(x)\, \phi_\theta(x) \\
&= \int \frac{d^3\mathrm{p}}{\omega_{\mathrm{p}}} \left(f^-(p)\, e^{-ip\theta P} a(p) + f^+(p)\, e^{ip\theta P} a^*(p) \right),
\end{aligned}
$$

with $\omega_p = +\sqrt{\vec{p}^2 + m^2}$ and $p = (\omega_p, \vec{p})$ and $f \in \mathscr{S}(\mathbb{R}^4)$ and P is the momentum operator.

Furthermore, the authors proved that ϕ_θ (and $\phi_{-\theta}$) is a wedge local field.

Solution

Theorem

The deformed relative entropy $S_\theta(\omega', \omega)$ is up to first order in Θ explicitly given by

$$
\begin{aligned}
S_\theta(\omega', \omega) &= i \frac{d}{dt} \langle \Omega | e^{i\phi_\theta(f_{\Theta'})} e^{2\pi i t L_{\mathbf{01}}} e^{-i\phi_\theta(f_{\Theta'})} e^{-2\pi i t L_{\mathbf{01}}} \Omega \rangle \Big|_{t=0} \\
&= S_0(\omega', \omega) + \frac{8\pi}{3} \Theta \left(\int d\mu(k)\, \omega_k \, |f^+_{\Theta'}(k)|^2 \right)^2 ,
\end{aligned}
$$

where $S_0(\omega', \omega)$ is the undeformed relative entropy.

The deformed version of the Beckenstein bound

$$\boxed{S \leq 2\pi\, R\, E + \frac{8\pi}{3}\,\Theta\, m^2.}$$

By coefficient comparison of the Beckenstein bound but not neglecting the m^2 term

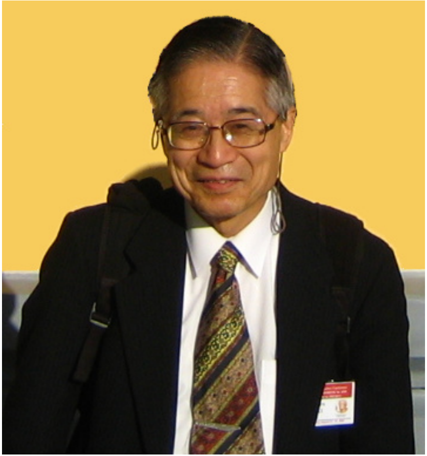
$$S \leq 2\pi\, R\, E + 4\pi\, G\, m^2,$$

we identify Θ with the Planck-length squared l_P^2, i.e.

$$\boxed{\Theta = \frac{3}{2} G = \frac{3}{2} l_P^2.}$$

Thank you for your Attention!

Armin Uhlmann 1971



Huzihiro Araki 2009