The stochastic gradient flow approach to QCD confinement and mass gap generation

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Stochastic quantization

Symmetries in GR

- **Generators of gauge symmetries** found from the **Hamiltonian** [1]

$$\mathcal{L}_{\rm ADM} = N\sqrt{h}\left[K^2 - K^{ij}K_{ij} - \bar{R}\right] \qquad \mathcal{H}_{\rm ADM} = \Pi\dot{N} + \Pi_i\dot{N}^i + \Pi^{ij}\dot{h}_{ij} - \mathcal{L}_{\rm ADM},$$

$$\boxed{\Pi = \frac{\partial\mathcal{L}_{\rm ADM}}{\partial\dot{N}} = 0, \qquad \Pi_i = \frac{\partial\mathcal{L}_{\rm ADM}}{\partial\dot{N}^i} = 0,}$$

$$[\Pi, \mathcal{H}_{\rm ADM}]_{\rm PB} = 0, \qquad [\Pi_i, \mathcal{H}_{\rm ADM}]_{\rm PB} = 0,$$

$$\mathcal{H} = G_{ijkl}\Pi^{ij}\Pi^{kl} - \sqrt{h}\bar{R} = 0, \qquad \mathcal{H}_i = 2h_{ij}\nabla^{(3)}_k\Pi^{kj} = 0,$$

Super-Hamiltonian **Super-Momentum**

Secondary constraints: eq. of motion

$$R^{0i} - \frac{1}{2}g^{0i}R = \frac{N\mathcal{H}^i + N^i\mathcal{H}}{2N^2\sqrt{h}} = 0,$$

$$R^{00} - \frac{1}{2}g^{00}R = -\frac{\mathcal{H}}{2N^2\sqrt{h}} = 0,$$

[1] Henneaux, Teitelboim, Quantization of gauge systems, Princeton U. Press

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Super-Hamiltonian **Super-Momentum**

Generators of gauge transformation: time, space diffeomorphisms

$$\xi\,[\mathcal{H}, h_{ij}]_{\rm PB} = \delta_t h_{ij} = \mathcal{L}_{\xi\mathbf{n}} h_{ij}$$

$$\xi^k\,[\mathcal{H}_k, h_{ij}]_{\rm PB} = \delta_{\boldsymbol{\xi}} h_{ij} = \mathcal{L}_{\boldsymbol{\xi}} h_{ij}$$

$$\boxed{\mathcal{H}_{\rm ADM} = \Pi\dot{N} + \Pi_i\dot{N}^i + N\mathcal{H} + N^i\mathcal{H}_i = 0,}$$ **The Hamiltonian vanishes**

[1] Henneaux, Teitelboim, Quantization of gauge systems, Princeton U. Press

Symmetries in GR

Path-integral Quantization [1]:

Path integral

$$= \int [Dx^\mu] \prod_{t,\alpha} \delta(\chi_\alpha) \prod_t (\text{sdet}\,[\chi_\alpha, \chi_\beta])^{1/2} \exp iS[x^\mu(t)]. \quad (16.5)$$

Delta on the constraints

$$\downarrow$$

We are asking the path integral not to fluctuate around the constraints
In GR constraints generate **external** symmetries

Symmetries are imposed in the quantization of GR rather than being **allowed to emerge**

[1] Henneaux, Teitelboim, Quantization of gauge systems, Princeton U. Press

Quantization Methods

All methods start from a classical description

1. **Canonical Quantization - Dynamic Perspective**
2. **Path-Integral Quantization - Ensemble Average Perspective**

Difficulties in handling Symmetries

Dynamics limited to a hypersurface in phase-space — Constraints Hypersurface $\chi_i = 0$

- **1. Dirac Prescription**
  define *physical states* as

$$\chi_i|\psi\rangle = 0$$

- **2. Gribov copies - Faddeev Popov determinant**

Path integral

$$= \int [Dx^\mu] \prod_{t,\alpha} \delta(\chi_\alpha) \prod_t (\text{sdet}\,[\chi_\alpha, \chi_\beta])^{1/2} \exp iS[x^\mu(t)]. \qquad (16.5)$$

[1] Henneaux, Teitelboim, Quantization of gauge systems, Princeton U. Press

Quantization Methods

1. Canonical Quantization - Dynamic Perspective
2. Path-Integral Quantization - Ensemble Average Perspective
3. Stochastic Quantization

Stochastic Dynamic Relaxation of the Ensemble Average

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PERTURBATION THEORY WITHOUT GAUGE FIXING

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ABSTRACT

We propose to formulate the perturbative expansion for field theory starting from the Langevin equation which describes the approach to equilibrium. We show that this formulation can be applied to gauge theories to compute gauge invariant quantities without fixing the gauge. A very simple example is worked out in detail. We also discuss the speed of approaching to equilibrium of the solution of the Langevin equation in the framework of perturbation theory.

Stochastic Quantization I

Introduce a “stochastic time” variable s and noise - Langevin dynamics

$$\frac{\partial}{\partial s}\phi_A\left(x^\mu, s\right) = -\frac{\delta S\left[\phi\right]}{\delta\phi_A} + \eta_A\left(x^\mu, s\right),$$

The noise is **additive** and **Gaussian**: higher-order (even) correlations are functions of $\langle\eta_{\mathrm{A}}\eta_{\mathrm{B}}\rangle$

$$\langle\eta_A\left(x, s\right)\eta_B\left(x', s'\right)\rangle = 2\alpha\delta_{AB}\delta\left(x - x'\right)\delta\left(s - s'\right) \qquad \langle\eta_{\mathrm{A}}\rangle = 0$$

$$p\left(\eta\right) \sim \left[D\eta\right]\exp - \frac{1}{4\alpha}\int \mathrm{d}^4x\,\mathrm{d}s\;\left[\eta\left(x, s\right)\right]^2$$

[1] Parisi and Wu, Scientia Sinica 24 (1981) 483

Stochastic Quantization II

Expectation values: (i) with respect to the noise, (ii) with respect to $P\left(\phi, s\right)$

Average over histories

$$\langle\phi_{\mathrm{A}_1,\eta}\left(x_1,s\right)\cdots\phi_{\mathrm{A}_N,\eta}\left(x_N,s\right)\rangle = \frac{\int\left[D\eta\right]\exp\left[-\frac{1}{4}\int\mathrm{d}^4x\,\mathrm{d}\sigma\,\eta^2\left(x,\sigma\right)\right]\phi_{\mathrm{A}_1,\eta}\left(x_1,s\right)\cdots\phi_{\mathrm{A}_N,\eta}\left(x_N,s\right)}{\int\left[D\eta\right]\exp\left[-\frac{1}{4}\int\mathrm{d}^4x\,\mathrm{d}\sigma\,\eta^2\left(x,\sigma\right)\right]}$$

Average over dynamic measure

$$\langle\phi_{\mathrm{A}_1,\eta}\left(x_1,s\right)\cdots\phi_{\mathrm{A}_N,\eta}\left(x_N,s\right)\rangle = \int\left[D\phi_{\mathrm{B}}\right]\,P\left(\phi,s\right)\,\phi_{\mathrm{A}_1}\left(x_1\right)\cdots\phi_{\mathrm{A}_N}\left(x_N\right)$$

The Fokker-Planck Eq., associated to the Langevin, dictates the dynamics of $P\left(\phi, s\right)$

Stochastic Quantization III

The associated Fokker-Planck ensures the correct "equilibrium" limit of $P\left(\phi, s\right)$

$$\frac{\partial F}{\partial s} = -\frac{\partial F}{\partial \phi_{\mathrm{A}}}\frac{\delta S[\phi]}{\delta \phi_{\mathrm{A}}} + \frac{1}{2}\frac{\partial^2 F}{\partial \phi_{\mathrm{A}}^2} + \frac{\partial F}{\partial \phi_{\mathrm{A}}}\eta_{\mathrm{A}}$$

Stochastic Calculus

$$\frac{\partial P\left(\phi, s\right)}{\partial s} = \frac{\partial}{\partial \phi_{\mathrm{A}}}\left[P\left(\phi, s\right)\frac{\delta S[\phi]}{\delta \phi_{\mathrm{A}}}\right] + \frac{1}{2}\frac{\partial^2 P\left(\phi, s\right)}{\partial \phi_{\mathrm{A}}^2}$$

Fokker-Planck

$$\frac{\partial P\left(\phi, s\right)}{\partial s} = 0 \longrightarrow \lim_{s\to\infty} P\left(\phi, s\right) = P\left(\phi\right) \sim \exp\left[-S\right]$$

The Euclidean Path-Integral measure is recovered at equilibrium

$$\lim_{s\to+\infty}\langle\phi_{\mathrm{A}}\left(x, s\right)\phi_{\mathrm{B}}\left(x', s\right)\rangle_{p(s)} = \langle\phi_{\mathrm{A}}\left(x\right)\phi_{\mathrm{B}}\left(x'\right)\rangle_{\mathrm{E}}$$

Stochastic quantization applied to gravity

Stochastic Quantization of General Relativity à la Ricci Flow

- Let's consider the presence of matter $$S=\frac{1}{2\kappa}\int \mathrm{d}^4x\sqrt{-g}R+\int \mathrm{d}^4x\sqrt{-g}\mathcal{L}_{\mathrm{M}},$$

We propose the Langevin equation [1] with scalar multiplicative noise

$$\frac{\partial}{\partial s}g_{\mu\nu}\left(x,s\right)=\imath\mathcal{G}_{\alpha\beta\mu\nu}\left(\lambda=-1\right)\frac{\delta S}{\delta g_{\alpha\beta}}+g_{\mu\nu}\left(x,s\right)e^{\imath\frac{\gamma}{2}}\sqrt{2\alpha}\,\tilde{\eta}\left(x,s\right)$$

$$\left\langle\tilde{\eta}\left(x,s\right)\tilde{\eta}\left(x',s'\right)\right\rangle=\delta\left(s-s'\right)\delta^{(4)}\left(x-x'\right)$$

- Starting point: seminal Rumpf work [2] with additive tensorial noise

$$\frac{\partial}{\partial s}g_{\mu\nu}\left(x,s\right)=\imath\mathcal{G}_{\alpha\beta\mu\nu}\left(\lambda\right)\frac{\delta S}{\delta g_{\alpha\beta}}+\eta_{\mu\nu}\left(x,s\right)$$

$$\left\langle\eta_{\alpha\beta}\left(x,s\right)\eta_{\mu\nu}\left(x',s'\right)\right\rangle=\mathcal{G}_{\alpha\beta\mu\nu}\left(\lambda\right)\delta\left(s-s'\right)\delta^{(4)}\left(x-x'\right)$$

- DeWitt Supermetric: special choice [2] $\lambda=-1$
- Strong link to Horava-Lifshitz gravity

$$\mathcal{G}^{\alpha\beta\mu\nu}\left(\lambda\right)=\frac{\sqrt{-g}}{2\kappa}\left[g^{\alpha\mu}g^{\beta\nu}+g^{\alpha\nu}g^{\beta\mu}-\lambda g^{\alpha\beta}g^{\mu\nu}\right]$$

$$\mathcal{G}_{\alpha\beta\mu\nu}\left(\lambda\right)=\frac{2\kappa}{\sqrt{-g}}\left[g_{\alpha\mu}g_{\beta\nu}+g_{\alpha\nu}g_{\beta\mu}-\frac{\lambda}{2\lambda+1}g_{\alpha\beta}g_{\mu\nu}\right]$$

[1] M. Lulli, **A. Marciano**, X. Shan, arXiv:2112.01490, to appear in Fortshritte der Physik

[2] H. Rumpf, Phys. Rev. D, 33, 4, 1986

Scalar multiplicative noise

- Let's consider the presence of matter $$S=\frac{1}{2\kappa}\int \mathrm{d}^4x\sqrt{-g}R+\int \mathrm{d}^4x\sqrt{-g}\mathcal{L}_{\mathrm{M}},$$

We propose the Langevin equation [1] with scalar multiplicative noise

$$\frac{\partial g_{\mu\nu}}{\partial s}=-2\imath\left[R_{\mu\nu}-\kappa\left(T_{\mu\nu}-\frac{1}{2}g_{\mu\nu}T\right)\right]+g_{\mu\nu}e^{\imath\frac{\gamma}{2}}\sqrt{2\alpha}\,\tilde{\eta}$$

$$\langle\tilde{\eta}\left(x,s\right)\tilde{\eta}\left(x',s'\right)\rangle=\delta\left(s-s'\right)\delta^{(4)}\left(x-x'\right)$$

- Zero-noise limit: looks like a Ricci-flow with a target fixed point
- Noise amplitude to be determined at the saddle-point of the equilibrium

[1] M. Lulli, **A. Marciano**, X. Shan, arXiv:2112.01490, to appear in Fortshritte der Physik
[2] H. Rumpf, Phys. Rev. D, 33, 4, 1986

Langevin equation and the Itô calculus

$$\boxed{\frac{\partial g_{\mu\nu}}{\partial s} = -2\imath\left[R_{\mu\nu} - \kappa\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)\right] + g_{\mu\nu}e^{\imath\frac{\gamma}{2}}\sqrt{2\alpha}\,\tilde{\eta},}$$

- We need to “interpret” the Langevin equation: Itô calculus [1]
- Different variables coupled to the same noise
- Need to compute Itô rules for this case: for a generic F

$$\boxed{\frac{\partial F}{\partial s} = -2\imath\frac{\partial F}{\partial g_{\mu\nu}}\left[R_{\mu\nu} - R^{\mathrm{T}}_{\mu\nu}\right] + \alpha e^{\imath\gamma}g_{\mu\nu}g_{\alpha\beta}\,\frac{\partial^2 F}{\partial g_{\mu\nu}\partial g_{\alpha\beta}} + e^{\imath\frac{\gamma}{2}}\sqrt{2\alpha}\frac{\partial F}{\partial g_{\mu\nu}}g_{\mu\nu}\tilde{\eta}}$$

- The associated Fokker-Planck reads

$$\boxed{\partial_s p = -\frac{\partial}{\partial g_{\mu\nu}}\left(-2\imath\left[R_{\mu\nu} - R^{\mathrm{T}}_{\mu\nu}\right]p\right) + \alpha e^{\imath\gamma}\frac{\partial^2}{\partial g_{\mu\nu}\partial g_{\alpha\beta}}\left(g_{\mu\nu}g_{\alpha\beta}\,p\right)}$$

[1] H. Rumpf, Phys. Rev. D, 33, 4, 1986

[2] M. Lulli, **A. Marciano**, X. Shan, arXiv:2112.01490, to appear in Fortshritte der Physik

Emergent Cosmological Constant

- We can look for the steady state solution of the Fokker-Planck

$$\partial_s p = -\frac{\partial}{\partial g_{\mu\nu}}\left(-2\imath\left[R_{\mu\nu} - R^{\mathrm{T}}_{\mu\nu}\right]p\right) + \alpha e^{\imath\gamma}\frac{\partial^2}{\partial g_{\mu\nu}\partial g_{\alpha\beta}}\left(g_{\mu\nu}g_{\alpha\beta}\,p\right)$$

- After setting $\partial_s p = 0$ one obtains

Number of independent metric components

$$p\left(g_{\mu\nu}\right) \simeq \mathcal{C}\exp\left\{\frac{1}{D\alpha e^{i\gamma}}\int^{g^{\mu\nu}}\mathcal{D}\bar{g}^{\mu\nu}\left[-2\imath\left(R_{\mu\nu} - R^{\mathrm{T}}_{\mu\nu}\right) - \alpha e^{\imath\gamma}\left(1+n_g\right)\bar{g}_{\mu\nu}\right]\right\}$$

- The saddle-point $\frac{\delta p}{\delta g^{\mu\nu}} = 0$ condition reads $R_{\mu\nu} - R^{\mathrm{T}}_{\mu\nu} - \frac{1}{2}\alpha\left(1+n_g\right)e^{\imath\left(\gamma+\frac{\pi}{2}\right)}g_{\mu\nu} = 0$

- After setting $\frac{1}{2}\alpha\left(1+n_g\right) = \Lambda, \qquad \alpha = \frac{2\Lambda}{1+n_g},$

The cosmological constant
appears as a consequence of the noise at the saddle point at equilibrium (!)

$$\boxed{R_{\mu\nu} - \Lambda e^{\imath\left(\gamma-\frac{\pi}{2}\right)}g_{\mu\nu} = \kappa\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)}$$

[1] M. Lulli, **A. Marciano**, X. Shan, arXiv:2112.01490, to appear in Fortshritte der Physik

ADM variables and Hamiltonian constraints analysis

Let us go from the metric tensor to the ADM variables

We need to follow the Itô transformation rule

$$\frac{\partial F}{\partial s} = -2\imath \frac{\partial F}{\partial g_{\mu\nu}} \left[R_{\mu\nu} - R^{\mathrm{T}}_{\mu\nu}\right] + \alpha e^{\imath\gamma} g_{\mu\nu} g_{\alpha\beta} \, \frac{\partial^2 F}{\partial g_{\mu\nu} \partial g_{\alpha\beta}} + e^{\imath\frac{\gamma}{2}} \sqrt{2\alpha} \frac{\partial F}{\partial g_{\mu\nu}} g_{\mu\nu} \tilde{\eta}$$

The final set of equations reads

$$\frac{\partial N}{\partial s} = -\frac{N}{2} \left[\imath \left(\frac{\mathcal{H}}{\sqrt{h}} + R\right) + \frac{\Lambda e^{-\imath\gamma}}{1+n_g} + e^{\imath\frac{\gamma}{2}} \sqrt{\frac{4\Lambda}{1+n_g}} \tilde{\eta}\right]$$

$$\boxed{\frac{\partial N^k}{\partial s} = \frac{\imath N \mathcal{H}^k}{\sqrt{h}}}$$

$$\frac{\partial h_{ij}}{\partial s} = \frac{1}{N} \mathcal{L}_{Nn} \left[\mathcal{H}, h_{ij}\right] + \left[\mathcal{H}, \left[\mathcal{H}, h_{ij}\right]\right] + \frac{h_{ij}\mathcal{H}}{2\sqrt{h}} - h_{ij} e^{\imath\frac{\gamma}{2}} \sqrt{\frac{4\Lambda}{1+n_g}} \tilde{\eta}$$

[1] M. Lulli, **A. Marciano**, X. Shan, arXiv:2112.01490, to appear in Fortshritte der Physik

Shift-vector and Navier-Stokes

- It reduces to a forced incompressible Navier-Stokes equation (for Euclidean signature)

$$\frac{\partial N^k}{\partial s} = \frac{\imath N \mathcal{H}^k}{\sqrt{h}}$$

$$\downarrow$$

$$\boxed{\partial_t v^i - \zeta \partial^2 v^i + \partial^i P + v^k \partial_k v^i = -\frac{\imath}{N} \frac{\partial N^i}{\partial s}}$$

$$r_c^{3/2} \partial_k T^{kt} = \partial_k v^k = 0$$

- **The noise term cancels out**
- The fields can enter a turbulent regime
- Turbulent fluctuations are **intermittent**
- Possibility to investigate the multi-fractal hypothesis [1] in GR

The only equation that loses noise responds with intermittency!

Possible role of turbulence in cosmological first-order phase transitions (!)

[1] U. Frisch, G. Parisi; R Benzi, G Paladin, G Parisi, A Vulpiani Journal of Physics A: Mathematical and General 17 (18), 3521

[2] M. Lulli, **A. Marciano**, X. Shan, arXiv:2112.01490, to appear in Fortshritte der Physik

Out-of-equilibrium dynamics of Black Holes

- Let us consider a "static" spherical symmetric metric $\mathrm{d}s^2 = -e^{\nu(r)}\mathrm{d}t^2 + e^{\mu(r)}\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\phi^2$
- After performing Itô transformation and taking the **quasi-equilibrium limit** $\mu = -\nu$

$$\frac{\partial \nu}{\partial s} = \frac{\mathrm{d}\nu}{\mathrm{d}g_{00}}\frac{\partial g_{00}}{\partial s} + e^{\imath\gamma}\alpha\, g_{00}^2\frac{\mathrm{d}^2\nu}{\mathrm{d}g_{00}^2}\,,$$

$$\frac{\partial \nu}{\partial s} = -\,\imath\left[\frac{\mathrm{d}^2\nu}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}\nu}{\mathrm{d}r} + \left(\frac{\mathrm{d}\nu}{\mathrm{d}r}\right)^2\right]e^{\nu} - \frac{1}{2}e^{\imath\frac{\gamma}{2}}\sqrt{2\alpha}\,\left(\tilde{\eta} + e^{\imath\frac{\gamma}{2}}\sqrt{2\alpha}\right)$$

- Let's normalise the equation to 2α

$$\nu = \nu_0(r) + \bar{\nu}(r,s) \qquad \frac{\bar{\nu}}{\sqrt{2\alpha}} = \bar{h} \qquad s\sqrt{2\alpha} = t \qquad \bar{\eta} = -\frac{1}{2}\frac{e^{\imath\frac{\gamma}{2}}}{\sqrt{2\alpha}}\tilde{\eta} \qquad h = \bar{h} + \frac{1}{2}e^{\imath\frac{\gamma}{2}}t$$

$$\frac{1}{2\alpha}\frac{\partial\bar{\nu}}{\partial s} = \frac{\partial\bar{h}}{\partial t} = \imath\left[\frac{e^{\nu_0}}{\sqrt{2\alpha}}\left(\frac{\mathrm{d}^2\bar{h}}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}\bar{h}}{\mathrm{d}r}\right) + e^{\nu_0}\left(\frac{\mathrm{d}\bar{h}}{\mathrm{d}r}\right)^2\right] + \bar{\eta} - \frac{1}{2}e^{\imath\frac{\gamma}{2}}$$

$$\frac{\partial h}{\partial t} = \imath\left[\nu_{\mathrm{KPZ}}(r,\alpha)\,\nabla^2 h(r,s) + \lambda_{\mathrm{KPZ}}(r)\left(\frac{\mathrm{d}h}{\mathrm{d}r}\right)^2\right] + \bar{\eta}$$

$$\nu_{\mathrm{KPZ}}(r,\alpha) = \frac{e^{\nu_0(r)}}{\sqrt{2\alpha}} \qquad \lambda_{\mathrm{KPZ}}(r) = e^{\nu_0(r)}$$

Kardar-Parisi-Zhang Eq. [1]
for a spherical surface

[1] M. Kardar, G. Parisi, Y.C. Zhang, PRL 56, 9, 889–892 (1986)
[2] M. Lulli, **A. Marciano**, X. Shan, arXiv:2112.01490, to appear in Fortshritte der Physik

Black Holes and KPZ

$$\frac{\partial h}{\partial t} = \imath \left[\nu_{\mathrm{KPZ}}(r,\alpha)\, \nabla^2 h(r,s) + \lambda_{\mathrm{KPZ}}(r) \left(\frac{\mathrm{d}h}{\mathrm{d}r}\right)^2 \right] + \bar{\eta}$$

Kardar-Parisi-Zhang Eq. [1]
for a spherical surface

$$\nu_{\mathrm{KPZ}}(r,\alpha) = \frac{e^{\nu_0(r)}}{\sqrt{2\alpha}} \qquad \lambda_{\mathrm{KPZ}}(r) = e^{\nu_0(r)}$$

- Gravitational waves good probe for near-horizon physics: **no screening**
- Black holes mergers: possibly use GW right before merger and final part of "ringdown" [2]
- Proper-time intermittency: include **BH spin**

$$\mathrm{d}s^2 \simeq -e^{\nu_0(r)}\left(1+\sqrt{2\alpha}h\right)\mathrm{d}t^2 + e^{\mu_0(r)}\left(1-\sqrt{2\alpha}h\right)\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\phi^2$$

- **Low dissipation**: far-from-horizon (linear) but need no large $\sqrt{2\alpha}$
- Possible accumulated effects during gravitational waves propagations [3]

[1] M. Kardar, G. Parisi, Y.C. Zhang, PRL 56, 9, 889–892 (1986)
[2] V. Cardoso, E. Franzin, P. Pani, PRL, 116, 171101 (2016)
[3] M. Lulli, **A. Marciano** and N. Yunes, work in progress

Topology change and out-of-equilibrium DOFs

Breakdown of the conformal symmetry and the scalaron

$$h_{\mu\nu} = h^{\perp}_{\mu\nu} + \partial_\mu a^{\perp}_\nu + \partial_\nu a^{\perp}_\mu + \left(\partial_\mu\partial_\nu - \frac{1}{4}\eta_{\mu\nu}\Box\right) a + \frac{1}{4}\eta_{\mu\nu}\varphi$$

$$\Phi = \varphi - \bar{\Box} a$$

Einstein-Hilbert expanded on dS or AdS backgrounds

$$\mathcal{S}^{(2)}_{\mathrm{EH}} = \frac{c^3}{16\pi G}\int d^4x\sqrt{-\bar{g}}\left[\frac{1}{4}h^{\mu\nu}_{\perp}\left(\bar{\Box} - \frac{\bar{R}}{6}\right)h^{\perp}_{\mu\nu} - \frac{3}{32}\Phi\left(\bar{\Box} + \frac{\bar{R}}{3}\right)\Phi\right]$$

Residual gauge transformation: conformal Killing vector and disappearance of the ghost

$$h^{\perp}_{\mu\nu} \to h^{\perp}_{\mu\nu} + \nabla_\mu\xi_\nu + \nabla_\nu\xi_\mu\,, \qquad \Phi \to \Phi + 2\nabla^\mu k_\mu$$

Projective connection & out-of-equilibrium torsional DOFs

$$\bar{\Gamma}^{\gamma}_{\alpha\beta} = \Gamma^{\gamma}_{\alpha\beta} + \mathcal{C}^{\gamma}_{\alpha\beta}$$

$$\mathcal{C}^{\gamma}_{\alpha\beta} = \lambda_1 \delta^{\gamma}_{\alpha} u_\beta + \lambda_2 u_\alpha \delta^{\gamma}_{\beta} + \lambda_3 w_{\alpha\beta} u^\gamma + \lambda_4 u_\alpha u_\beta u^\gamma$$

$$\sqrt{-g}\bar{R} = \sqrt{-g}R + \sqrt{-g}g^{\beta\delta}\left(\mathcal{C}^{\mu}_{\beta\delta}\mathcal{C}^{\alpha}_{\mu\alpha} - \mathcal{C}^{\mu}_{\beta\alpha}\mathcal{C}^{\alpha}_{\mu\delta}\right)$$

Cosmological term induced in the action

$$\sqrt{-g}g^{\beta\delta}\left(\mathcal{C}^{\mu}_{\beta\delta}\mathcal{C}^{\alpha}_{\mu\alpha} - \mathcal{C}^{\mu}_{\beta\alpha}\mathcal{C}^{\alpha}_{\mu\delta}\right) = \sqrt{-g}\left[\left(\lambda_2^2 + \lambda_3^2\right)(D-1)\, u_\mu u^\mu\right]$$

[1] M. Lulli, **A. Marciano**, X. Shan, arXiv:2112.01490, to appear in Fortshritte der Physik

Out-of-equilibrium torsional DOFs and topology changes

E_0 trivial $SU(3)$ bundle and E_1 non-trivial $SU(3)$ bundle:

$$a : E_0 \to E_1$$

Map between trivial and non-trivial bundles induces topology changes

$$c_2(E_1) = \frac{1}{8\pi^2} \int_M tr(F \wedge F) , \qquad c_2(E_0) = 0$$

Singularities of the maps produce non-trivial instantons $F = \pm * F$

map a induces singular gauge transformation g_a

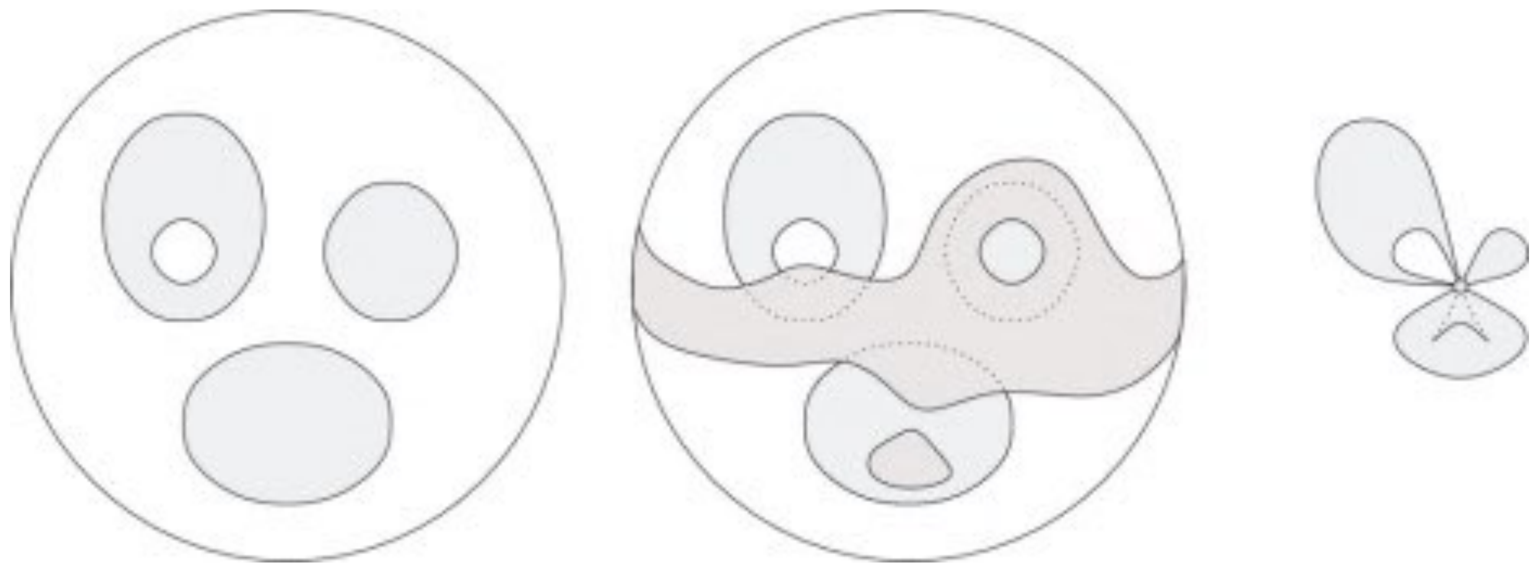
$$\oint_{\gamma = \partial S} A = \int_S dA = \int_S d\left(g_a^{-1} dg_a\right) = \int_S F$$

Stokes theorem and area-law for the holonomy

Berry phase description of confinement

Berry phase and stochastic gradient flows

The gauge-geometry flow allows for topology changes (fluctuations) out of equilibrium



Geometrical interpretation of ground-states

Changes of topologies through defects are induced by singularities in the stochastic gradient flow

Knot concordance as 4D picture to describe topology change [2]

[1] T. Asselmeyer-Maluga, M. Lulli, **A. Marciano**, R. Pasechnik, E. Zappala, arXiv:2408.15986
[2] T. Asselmeyer-Maluga, The wild fractal nature of spacetime, Wold Scientific 2025

Geometric phase and quark confinement

Intuition from notable example of U(1)

$$W_{\circ\gamma} = e^{\imath e \Phi_B}$$

Non-vanishing magnetic fluxes recovered from singular gauge transformations applied to vanishing fields

$$W_{\circ\gamma} \to e^{\pm \imath e \Phi_B} W_{\circ\gamma}$$

Holonomy defined as circuitation of gauge connection along path (Wilson loop when paths are closed)

$$W_{\alpha} = \mathcal{P} \exp\left[\imath g \oint dx^{\mu} A_{\mu}(x)\right]$$

Action of center symmetry

$$G_{\gamma(\tau_s)} = z G_{\gamma(\tau_t)} \qquad W_{\circ\gamma} \to (z^*)^l W_{\circ\gamma}$$

$\circ\gamma$ space-like loop and l linking number

For SU(N) the geometric phase is replaced by a center symmetry group element of $\mathbb{Z}_N$

$$G_{\gamma(\tau_s)} = z G_{\gamma(\tau_t)} \, . \qquad W_{\circ\gamma} \to (z^*)^l W_{\circ\gamma}$$

[1] T. Asselmeyer-Maluga, M. Lulli, **A. Marciano**, R. Pasechnik, E. Zappala, arXiv:2408.15986

Stochastic gauge-geometry flow

Consider the geometry back-reaction (fluctuation of topologies)

$$\mathcal{S}_{\mathrm{EYM}} = \quad +\frac{1}{2\kappa}\int dx^0_M d^{D-1}x\sqrt{-g}R[g_{\mu\nu}]$$
$$-\frac{1}{4g_s^2}\int dx^0_M d^{D-1}x\sqrt{-g}F^a_{\mu\nu}F^{a\,\mu\nu}$$

Stochastic flow complemented with noise multiplicative ansatz

$$\frac{\partial A^a_\mu(x,\tau)}{\partial\tau} = \quad \imath\nabla^\nu F^a_{\nu\mu}(x,\tau) + \xi^a_\mu(x,\tau)$$
$$\frac{\partial g_{\mu\nu}(x,\tau)}{\partial\tau} = -\imath\frac{\sqrt{-g}}{2\kappa}\left[R_{\mu\nu}(x,\tau) - \frac{1}{2}R(x,\tau)g_{\mu\nu}(x,\tau) \quad -\kappa T_{\mu\nu}(x,\tau)\right] + \xi_{\mu\nu}(x,\tau)$$

$$\xi^a_\mu(x,\tau) \equiv \xi(x,\tau)A^a_\mu(x,\tau)$$

$$\xi_{\mu\nu}(x,\tau) \equiv \xi'(x,\tau)g_{\mu\nu}(x,\tau)$$

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Topology changes

Stochastic noise induce fluctuations of the topology of the manifolds (hadronic ground states)

$$\bar{\Gamma}^{\alpha}_{\beta\gamma} = \Gamma^{\alpha}_{\beta\gamma} + \delta^{\alpha}_{\gamma} U_{\beta}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau} g_{\mu\nu}(\tau) = -2n^{\alpha}(\tau) U_{\alpha}(\tau) g_{\mu\nu}(\tau) = -2\lambda(\tau)\, g_{\mu\nu}(\tau)$$

Geometric flow and appearance of singularities at the geometry level

↕

Singularities of the gauge transformations and appearance of thin vortices

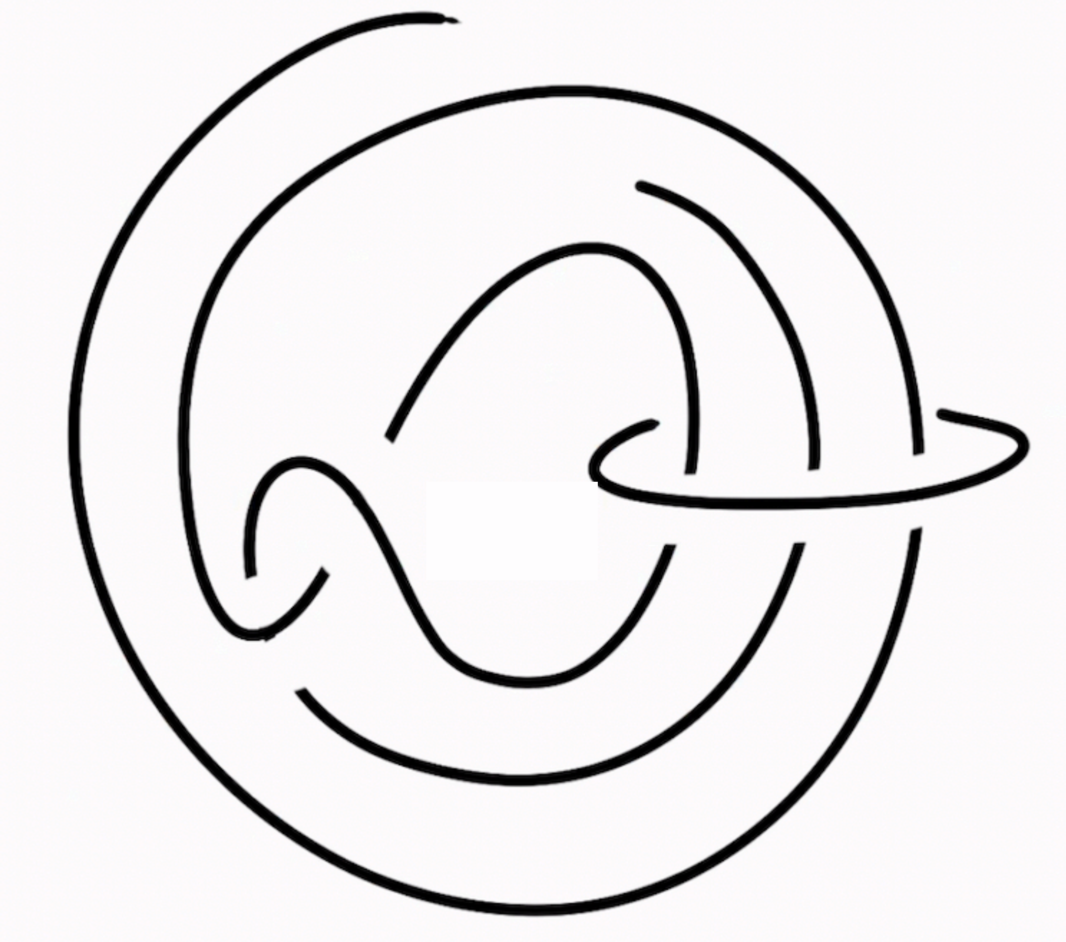
Dual fluxes are created dynamically in the stochastic time and source the area-law

[1] T. Asselmeyer-Maluga, M. Lulli, **A. Marciano**, R. Pasechnik, E. Zappala, arXiv:2408.15986

Quark Confinement

$$\langle W_{\gamma_1}(A) W_{\gamma_2}(A)\rangle = \frac{1}{Z}\int \mathcal{D}A\,\mathrm{Tr}\left[\mathcal{P}e^{\imath \oint_{\gamma_1} A_\mu dx^\mu}\right]\,\mathrm{Tr}\left[\mathcal{P}e^{\imath \oint_{\gamma_2} A_\mu dx^\mu}\right] e^{\imath S_{\mathrm{YM}}[A]}$$

VEV of the product of two Wilson loops



$$S_{\mathrm{YM}} = \int_M F \wedge *F \qquad \oint_{\gamma=\partial\Gamma} A = \int_\Gamma F$$

$$\langle W_{\gamma_1}(A) W_{\gamma_2}(A)\rangle \sim e^{\imath\,\mathrm{Area}(\Gamma_1)+\imath\,\mathrm{Area}(\Gamma_2)}\, e^{2\imath\, lk(\gamma_1,\gamma_2)}$$

Linking number between Wilson and 't Hooft loops

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Dimensional transmutation

Rescaling between leaf of the bundle with changes of topology

$$ds_T^2 = \frac{da_T^2}{a_{T^2}} = d\left(\frac{t}{L}\right)^2 \longleftrightarrow a_T(t, L) = a_0 \cdot \exp\left(\frac{t}{L}\right)$$

Level fo the Chen-Simons action and rescaling

$$\int_\Sigma tr\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right) = 8\pi^2 CS(\Sigma)$$

$$a = L_P \cdot \exp\left(\frac{3}{2 \cdot CS(\Sigma(2,5,7))}\right)$$

QCD scale dynamically generated in the stochastic time

$$a_1 = L_P \cdot \exp\left(\frac{140}{3}\right) \approx 7.5 \cdot 10^{-15} m$$

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Outlook and conclusions

Stochastic quantisation and out-of-equilibrium breakdown of symmetries

Geometric RG flow complemented with stochastic (multiplicative) noise

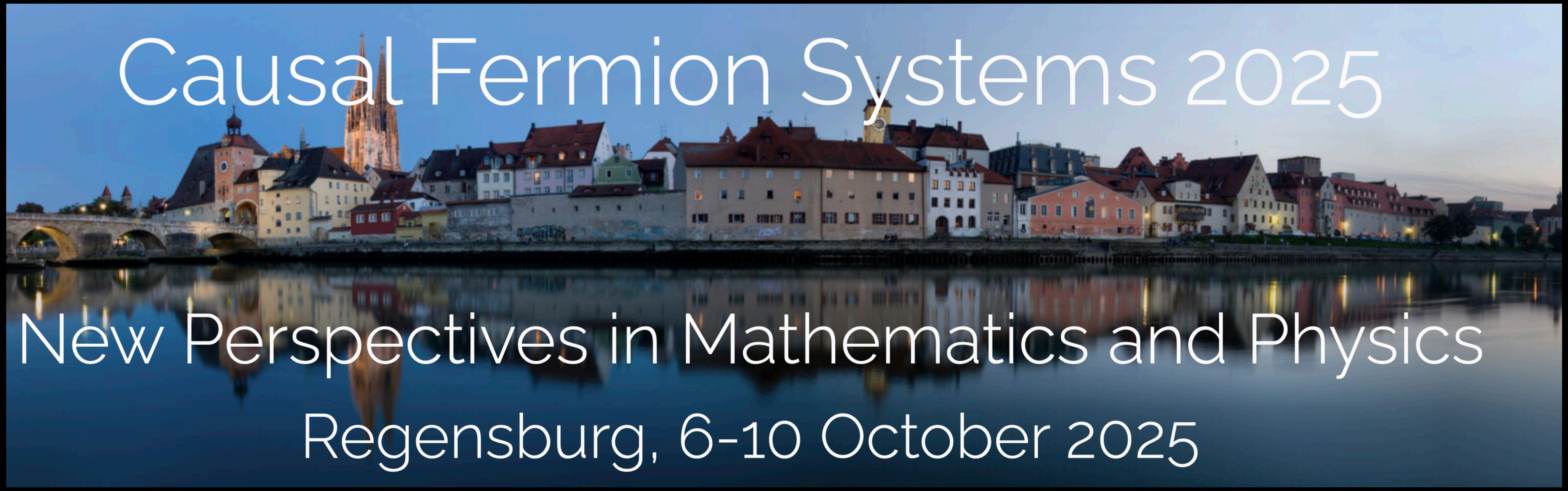
Consequences in astrophysics and cosmology

Consequences in particle physics: quark confinement and mass gap generation

Symmetry breaking & topology changes in out-of-equilibrium complex systems

谢谢

Thank you!



Grazie!

Danke!

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