How To Make Spectral Geometry Work

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Can One Hear Shapes?

- Goes back to Hermann Weyl ($>$ 100ys ago)
- **Setup:** Thin metal sheet, e.g., a "metal potato surface"
- Tap the metal, $\to$ record resonance frequencies.
- **Question:** Does the spectrum determine the shape?

Would it be important?

- Would bridge between Differential Geometry and Functional Analysis, i.e., between GR and QT.
- **Linear Regime:** Does the sound determine the shape?
- **Answer:** No. Isospectral shapes exist.
- **Question:** Can we gain more information from the sound?

Can We Gain More Information?

- **Answer:** Yes — tap strongly enough.
- Enter the **nonlinear regime.**
- **Linear:** Sum of solutions is a solution (eigenmodes oscillate harmonically).
- **Nonlinear:** Sum of solutions is **not** a solution (analog: interacting water waves).
- $\Rightarrow$ Eigenmodes start locally interacting.

What Happens in the Nonlinear Regime?

- Exciting a specific ω_n also excites other ω_m.
- **Result:** Sound makes sound!
- Mode interaction depends on how modes propagate.
- Propagation depends on curvature.
- **Key insight:** The "sound-makes-sound" effect encodes curvature information.

How to Calculate the Metric?

- **Experiment (A):** Tap lightly, record resonances ω_n.
- **Experiment (B):** Excite pairs (ω_n, ω_m) strongly.
  - Record newly excited resonances (u, w) (e.g., ϕ^4 theory).
  - Repeat for all pairs (ω_n, ω_m).
  - Record the **interaction matrix** $V(n, m, u, w)$.

What is the Matrix V?

- V is the **interaction Hamiltonian** $V = \lambda\phi^4$.
- Expressed in the **mode eigenbasis** $|v_n\rangle$:

$$V(n, m, u, w) = \langle v_n|\langle v_m|V|v_u\rangle|v_w\rangle$$

- This matrix V is the **hearable** outcome of Experiment (B).

How is the Shape Heard? (Step 1)

- **Goal:** Obtain the metric $g_{\mu\nu}(x)$.
- $g_{\mu\nu}$ is obtainable from **geodesic distance function** $\sigma(x, x')$.
- $\sigma(x, x') = \frac{1}{2}(\text{Geodesic Distance})^2$.

$$g_{\mu\nu}(x) = -\lim_{x' \to x} \frac{\partial^2 \sigma(x, x')}{\partial x^\mu \partial x'^\nu}$$

How is the Shape Heard? (Step 2)

- **Goal:** Obtain $\sigma(x, x')$.
- σ comes from the **wave propagator** $G(x, x')$.
- G satisfies the **Hadamard condition** near $x \to x'$:

$$G(x, y) = \frac{U(x, y)}{\sigma(x, y)} + V(x, y)\ln(\sigma(x, y)) + W(x, y) + \dots$$

- In 4D:

$$g_{\mu\nu}(x) \propto -\lim_{x' \to x} \partial_{x^\mu} \partial_{x'^\nu} \frac{1}{G(x, x')}$$

How is the Shape Heard? (Step 3)

- **Problem:** We cannot directly hear $G(x, x')$.
- **Solution:** Change basis from $|x\rangle$ to eigenbasis $|v_n\rangle$.

$$G(x, x') = \sum_{n,m} U^\dagger(x, v_n)\, G(v_n, v_m)\, U(x', v_m)$$

Here: $G(v_n, v_m) = \omega_n \delta_{n,m}$ because we chose eigenbasis

- ω_n are the resonance frequencies from Experiment (A).
- $U(x, v_n)$ is the unitary change of basis.

How is the Shape Heard? (Step 4)

- **Problem:** How to obtain $U(x, v_n)$?
- **Locality** $\Rightarrow$ In position basis $\{|x\}\rangle$, V is diagonal:

$$V(x, x', x'', x''') = f(x)\,\delta(x - x')\,\delta(x' - x'')\,\delta(x'' - x''')$$

- Therefore, $U(x, v_n)$ can be chosen to be any basis change that diagonalizes $V(n, m, u, w)$.

How is the Shape Heard? (Step 5)

- V is a diagonalizable 4-tensor $\Rightarrow$ Obtain diagonalizing U.
- Diagonalization is not unique — each choice yields a coordinate system.
- **Conclusion:** From ω_n (Experiment A) and $V(n, m, u, w)$ (Experiment B), we compute

$$U \;\Rightarrow\; G \;\Rightarrow\; \sigma \;\Rightarrow\; g_{\mu\nu}.$$

- **Final Result:** We can always determine the shape from the sound.

What Does This Mean for QFT on Curved Spacetime?

- The metal potato $\rightarrow$ A curved spacetime.
- **Experiment (A):** Excite free particles (keep them non-interacting) $\rightarrow$ particle spectrum.
- **Experiment (B):** Drive system into nonlinear regime $\rightarrow$ make 2 particles interact, measure outgoing particles.
- Do this for all particle pairs $\Rightarrow$ obtain the **S-matrix**, i.e., V.
- **New Result:** The Hearables, i.e., particle spectrum $+$ S-matrix determine the metric.

Physics and Math Take-Aways

- **Physics Take-Away:** The S-matrix completely encodes curvature.
- Encodes energy-momentum nonconservation.
- **Generalization of Noether's theorem:** Nonconservation reveals the full nonsymmetric geometry.
- **Application:** Accelerators can, in principle, measure spacetime curvature from the S-matrix.
- **Math Take-Away:** Geometry can be expressed purely in frequency data.
- $\Rightarrow$ A new kind of Fourier theory of geometry.
- **Quantum gravity?** PRL w. Barbara Šoda and Marcus Reitz.

Thank You

Questions?