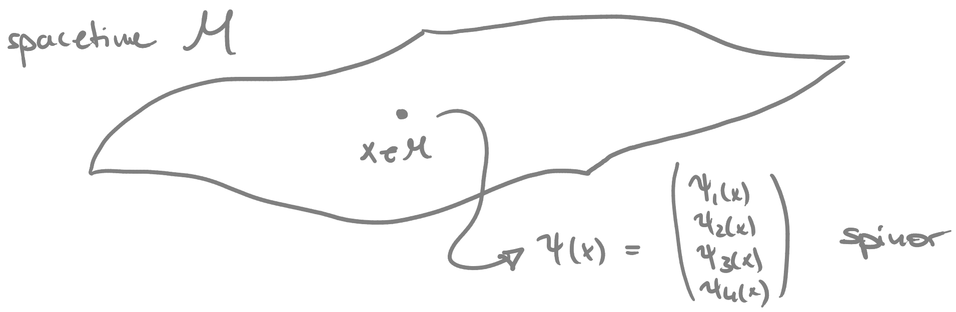
CAUSAL FERMION SYSTEMS 2025 (robert.jonsson@mau.se)

LECTURE Inherent structures I

References: "CFS - An Introduction ..." Ch 5.7 Basic Inherent Structures
arXiv 2411.06450
"Continuum Limit" Ch 1.1.2 - 1.1.4
arXiv 1605.04742

In this lecture, we begin to connect the abstract framework to the standard picture of fermions / relativistic spinor wavefunctions on spacetime.



Spinor wavefunctions can be viewed as section in the spinor bundle.
That is

spacetime point

$$\psi : \mathcal{M} \ni x \longmapsto \psi(x) \in S_x \cong \mathbb{C}^4$$

spinor space at x

spinor "attached" to x

The spinor inner product for Dirac spin-½ has signature $(++--)$

$$\prec \psi | \phi \succ_x = \psi(x)^\dagger \gamma^0 \phi(x)$$

Dirac matrix

How can we construct this picture from an abstract CFS?

$$(\mathcal{H}, \mathcal{F}, \rho)$$

universal measure

Hilbert space

$x \in \mathcal{F}$: self-adjoint, n pos. & neg. eigenvalues

Every vector in Hilbert space yields a wavefunction!

Def Let $u \in \mathcal{H}$, then the <u>physical wavefunction</u> ψ^u is the map

$$\begin{aligned} \psi^u : \quad \mathcal{M} &\longmapsto \mathcal{H} \\ x &\longmapsto \pi_x(u) \end{aligned}$$

$\mathcal{M} = \operatorname{supp} \rho$
spacetime

where π_x is the orthogonal projector on the image $x(\mathcal{H})$.

The spinors of all physical wavefunctions at x combined yield

Def The <u>spin space</u> at $x \in \mathcal{M}$ is

$$S_x := x(\mathcal{H}) = \pi_x(\mathcal{H})$$

And with x itself we equip this spin space with an indefinite inner product.

Def The <u>spin inner product</u> on spin space S_x is defined as

$\mathcal{H}$ scalar product

$$\begin{aligned} \prec . | . \succ_x : \quad S_x \times S_x &\longrightarrow \mathbb{C} \\ (u, v) &\longmapsto \prec u | v \succ_x = -\langle u | x v \rangle_{\mathcal{H}} \end{aligned}$$

Since x has (at most) n positive and n negative eigenvalues,
the spin inner product has signature (at most) (n, n).

Now we have equipped every point $x \in \operatorname{supp}\rho = \mathcal{M}$ in spacetime with a spin space with an indefinite inner product.
Next we need a natural map between different spin spaces:

Kernel of the Fermionic Projector $P(x,y)$

To this end we dress the projector π_x into some suggestive and fancy notation...

Def For every $x \in M$, define the wave evaluation operator as

$$\Psi(x) : (\mathcal{H}, \langle\,|\,\rangle) \to (S_x, \prec\,|\,\succ_x) \qquad \text{"}\Psi(x) = \pi_x\text{"}$$
$$u \mapsto \Psi(x)(u) = \psi^u(x) = \pi_x u$$

Then its adjoint is given by

$$\Psi(x)^* : (S_x, \prec\,|\,\succ_x) \to (\mathcal{H}, \langle\,|\,\rangle)$$
$$\Psi(x)^* = -x|_{S_x}$$

Pf $v \in S_x$, $u \in \mathcal{H}$

$$\prec v | \Psi(x) u \succ_x = \prec v | \pi_x u \succ_x = -\langle v | x u \rangle = \langle -xv | u \rangle \overset{!}{=} \langle \Psi^*(x) v | u \rangle$$

With the wave evaluation operator and its adjoint we can rewrite

$$-\Psi(x)^* \Psi(x) = x$$

Pf $u \in \mathcal{H}$: $\quad -\Psi(x)^* \Psi(x) u = -\Psi(x)^* \pi_x u = x \pi_x u = x u.$

More importantly, we can introduce a map between spin spaces!

Def Let $x, y \in M$. The kernel of the fermionic projector is

$$P(x,y) = -\Psi(x) \Psi(y)^* : S_y \to S_x$$

Note that it follows directly that

$$P(x,y)^* = -\Psi(y)\Psi(x)^* = P(y,x)$$

Two alternative ways to write and understand the map are

- in terms of spacetime operators:
  $$P(x,y) = -\Psi(x)\Psi(y)^* = \pi_x\, y|_{S_y}$$

- in bra-ket notation: let $(e_i)_{i \in \mathbb{N}}$ be an ONB of $\mathcal{H}$

$$P(x,y) = -\sum_i |\psi^{e_i}(x) \succ \prec \psi^{e_i}(y)| \qquad \text{"captures correlations in Dirac sea"}$$

Finally, we can also express the causal action in terms of $P(x,y)$.

<u>Def</u> For $x,y \in M$, define the closed chain

$$A_{xy} = P(x,y)\,P(y,x) : S_x \to S_x$$

"return trip from x to y and back"

The closed chain is isospectral to $A_{xy} \simeq xy \simeq yx$.

<u>Pf</u> $A_{xy} = P(x,y)\,P(y,x) = \Psi(x)\,\Psi(y)^* \,\Psi(y)\,\Psi(x)^*$

$$= \pi_x \, -y|_{S_y} \pi_y \, (-x|_{S_x}) = \pi_x \, y\, x|_{S_x}$$

Why is this isospectral to yx?

- yx maps a finite-dim subspace into itself.
  For example $I = \text{span}\{(xy)(H),\ \ker(xy)^\perp\}$

- eigenvalues of yx = roots of characteristic polynomial $yx|_I$

- its coefficients are functions of
  $\operatorname{tr}((yx)^p)$ $\quad (= \operatorname{tr}(xy)^p$ due to cyclicity $)$

but $\operatorname{tr}_{S_x}(A_{xy}^p) = \operatorname{tr}\big((\pi_x y x|_{S_x})^p\big) = \operatorname{tr}(\pi_x (yx)^p|_{S_x})$

$$= \operatorname{tr}((yx)^p \pi_x) = \operatorname{tr}((yx)^p) = \operatorname{tr}((xy)^p)$$

Hence $A_{xy} \simeq yx \simeq xy$ have the same eigenvalues.