Non-Riemannian Metric Independent Measures in Gravitation, Cosmology and Dynamical String Tension Theories

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Causal Fermion Systems

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Abstract: We first review and then in the last section present further new proposals and consequences concerning the use of non Riemannian, metric independent measures of integration and their application in the formulation of gravity theories, cosmology as well as applications to the construction of dynamical tension string theories. The well known unimodular gravity theory appears as a very special case of a gravity theory that uses non-canonical spacetime volume-forms . Concerning the most important issues in cosmology, we outline the construction of: (a) unified description of dark energy and dark matter as manifestations of a single entity – a special scalar field "darkon"; (b) quintessential models of universe evolution with a gravity-"inflaton"-assisted dynamical Higgs mechanism – dynamical suppression/generation of spontaneous electroweak gauge symmetry breaking in the "early"/"late" universe; (c) mechanism for suppression of 5-th force without fine-tuning; through spontaneously broken scale invariant theories (d) construction of stable non singular Emergent universe solutions Concerning application of non-canonical . (e) construction of models that can address the H0 problem. (f) the construction of holomorphic gravity, possible through the use of a Non Riemannian Measure. (g) the construction of dynamical tension string theories, where the string tension appears as an integration constant, (h) the construction of brane world scenarios in the context of dynamical tension string theories (i) we discuss avoidance of the Hagedorn temperature possible relaxation of string swampland constraints in dynamical tension string theories

1. Introduction: the origin of the cosmological constant problem in General Relativity from the use of the Riemannianian integration measure and the idea of Non Gravitating vacuum energy

We are motivated by the cosmological constant problem. On the mathematical level this boils down to an asymmetry between the matter sector and the gravitational sectors of the theory concerning the role of an origin for the energy density. As it is well known, in non-gravitational physics, like in particle mechanics, for example, the origin from which we measure energy is not important. In mathematical terms that means that the equations of motion are invariant under addition of a constant to the matter Lagrangian L_m

$$L_m \longrightarrow L_m + C\,.$$

In general coordinate invariant theories we woork with a lagrangian density however, which we will denote as $\mathcal{L}_m$. However, when the lagrangian density Lagrangian $\mathcal{L}_m$ is integrated with the standard Riemannian measure of integration, the square root of the determinant of the metric $\sqrt{-g}$, as in,

$$\int d^4x\,\sqrt{-g}\mathcal{L}_m\,,$$

The shift of the lagrangian density,

$$\mathcal{L}_m \longrightarrow \mathcal{L}_m + C\,.$$

when the lagrangian density Lagrangian $\mathcal{L}_m$ is integrated with the standard Riemannian measure of integration, the square root of the determinant of the metric $\sqrt{-g}$, is not a symmetry of the theory now, indeed, the gravitational equations of motion derived from the variation of the metric get an extra contribution of the form $Cg_{\mu\nu}$ after the above shift. This implies the choice of measure of integration has a crucial effect on the cosmological constant problem, and in particular that for the standard measure of integration, the vacuum energy gravitates. This motivates us to search for alternative measures of integrations, or what is the same alternative volume forms, as we discuss next where a shift of the Lagrangian is a symmetry. This could be achieved if the measure is a total derivative, then the shift above would not change the equations of motion of the theory and we could talk then, at least in some sense of a non gravitating vacuum energy model, as we will see, this possibility is present when we go on and discuss the Non-Riemannian Volume-Form Formalism . Invariance under a shift of the lagrangian density we call Non Gravitating vacuum energy.

2. Extension of Gravity theories by allowing Non-Riemannian Measures

Volume-forms are fairly basic objects in differential geometry – they exist on arbitrary differentiable manifolds and define covariant (under general coordinate reparametrizations) integration measures. It is important to stress that the existence of volume-forms is *completely independent* of the presence or absence of additional geometric structures on the manifold – Volume forms are defined [7] by nonsingular maximal rank differential forms ω:

$$\begin{aligned}
\int_{\mathcal{M}} \omega(\ldots) &= \int_{\mathcal{M}} dx^D \, \Omega(\ldots) \;, \\
\omega &= \frac{1}{D!}\omega_{\mu_1\ldots\mu_D} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_D} \;, \\
\omega_{\mu_1\ldots\mu_D} &= -\varepsilon_{\mu_1\ldots\mu_D}\Omega \;,
\end{aligned} \tag{1}$$

The volume element density Ω transforms as scalar density under general coordinate reparametrizations. Notice also that Ω is a total derivative as well, therefore adding a constant to any lagrangian that multiplies Ω will give rise to a total derivative that will not give a modification of the equations of motion and no generation of a cosmological constant, according to our expressed goal to have some sense of a non gravitating vacuum energy principle.

In standard generally-covariant theories (with action $S = \int d^D x \sqrt{-g}\mathcal{L}$) the Riemannian spacetime volume-form is defined through the "D-bein" (frame-bundle) canonical one-forms $e^A = e^A_\mu dx^\mu$ ($A = 0, \ldots, D-1$):

$$\omega = e^0 \wedge \ldots \wedge e^{D-1} = \det \| e^A_\mu \| \, dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_D} \longrightarrow \quad \Omega = \det \| e^A_\mu \| \, d^D x = \sqrt{-\det \| g_{\mu\nu} \|} \, d^D x \tag{2}$$

Instead of, or alongside with, $\sqrt{-g}$ we can employ one or several different alternative *non-Riemannian* volume elements as in (1) given by non-singular *exact* D-forms $\omega^{(j)} = dB^{(j)}$ where:

$$B^{(j)} = \frac{1}{(D-1)!} B^{(j)}_{\mu_1 \ldots \mu_{D-1}} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu-1} \longrightarrow \quad \Omega^{(j)} \equiv \Phi(B^{(j)}) = \frac{1}{(D-1)!} \varepsilon^{\mu_1 \ldots \mu_D} \, \partial_{\mu_1} B^{(j)}_{\mu_2 \ldots \mu_D} \tag{3}$$

In other words, the non-Riemannian volume elements are defined in terms of the dual field-strengths of auxiliary rank $D-1$ tensor gauge fields $B^{(j)}_{\mu_1 \ldots \mu_{D-1}}$.

Most of our research on "non-Riemannian measures" has still involved a spacetime manifold with a standard Riemannian geometric structure, and a torsionless affine connection $\Gamma^{\lambda}{}_{\mu\nu}$ either independent of $g_{\mu\nu}$ (first-order metric-affine / Einstein-Palatini formalism) or as a Levi-Civita connection w.r.t. $g_{\mu\nu}$ (second-order purely metric / Einstein-Hilbert formalism), but more recently we have also introduced theories with torsion and non Riemannian measures [8]

The generic form of modified gravity actions involving (one or more) non-Riemannian volume-elements, called for short actions, read (henceforth $D = 4$, and we will use units with $16\pi G_{Newton} = 1$):

$$S = \int d^4x\, \Phi(B^{(1)})\left(R + \mathcal{L}^{(1)}\right) + \int d^4x \sum_{j\geq 2} \Phi(B^{(j)})\, \mathcal{L}^{(j)} + \int d^4x\, \sqrt{-g}\mathcal{L}^{(0)} , \tag{4}$$

where R is the scalar curvature. The equations of motion of (4) w.r.t. the auxiliary tensor gauge fields $B^{(j)}_{\mu\nu\kappa}$ according to (3) imply:

$$\partial_\mu\left(R + \mathcal{L}^{(1)}\right) = 0 \;\; , \;\; \partial_\mu \mathcal{L}^{(j)} = 0 \;\; (j \geq 2) \; , \longrightarrow \;\; R + \mathcal{L}^{(1)} = M_1 \quad , \quad \mathcal{L}^{(j)} = M_j \; , \tag{5}$$

where all M_j $(j \geq 1)$ are *free integration constants* not present in the original NRVF gravity action (4).

A characteristic feature of the NRVF gravitational theories (4) is that when starting in the first-order (Palatini) formalism all non-Riemannian volume-elements $\Phi(B^{(j)})$ yield almost *pure-gauge* degrees of freedom, additional physical (field-propagating) gravitational degrees of freedom except for few discrete degrees of freedom with conserved canonical momenta appearing as the arbitrary integration constants M_j in (5). The reason is that the NRVF gravity action (4) in Palatini formalism is linear w.r.t. the velocities of some of the components of the auxiliary gauge fields $B^{(j)}_{\mu\nu\kappa}$ defining the non-Riemannian volume-element densities, and does not depend on the velocities of the rest of auxiliary gauge field components.

However, in the second-order formalism (where $\Gamma_{\mu\nu}{}^{\lambda}$ is the usual Levi-Civita connection w.r.t. $g_{\mu\nu}$) the first non-Riemannian volume form $\Phi(B^{(1)})$ in (4) is *not* any more pure-gauge. The reason is that the scalar curvature R (in the metric formalism) contains *second-order* (time) derivatives (the latter amount to a total derivative in the ordinary case $S = \int d^4x\sqrt{-g}R + \ldots$). Now defining $\chi_1 \equiv \Phi(B^{(1)})/\sqrt{-g}$, the latter field becomes physical degree of freedom as seen from the equations of motion of (4) w.r.t. $g^{\mu\nu}$:

$$R_{\mu\nu} + \frac{1}{\chi_1}\left(g_{\mu\nu}\Box\chi_1 - \nabla_\mu\nabla_\nu\chi_1\right) + \ldots = 0\,. \tag{6}$$

3. Generally Covariant Formulation of Unimodular Theory as an example of the use of Non-Riemannian Measures

Let us note that the well-known covariant formulation of unimodular gravity [9] can be viewed as a simple particular case within the general class (4) of modified gravity actions based on the non-Riemannian volume-form formalism. Indeed, the action of unimodular gravity when expressed in a generally covariant form[9] reads:

$$\mathcal{S} = \int d^4x \sqrt{-g}(R + 2\Lambda + \mathcal{L}_m) - \int d^4x\, \Phi\, 2\Lambda \tag{7}$$

with Λ being a dynamical field, and $\Phi \equiv \partial_\mu F^\mu$ where the vector density F^μ can be written as Hodge-dual $F^\mu \equiv \frac{1}{3!}\varepsilon^{\mu\nu\kappa\lambda} B_{\nu\kappa\lambda}$ w.r.t. rank 3 auxiliary gauge field $B_{\nu\kappa\lambda}$ (cf. (3) for $D = 4$). Variation w.r.t. F^μ implies $\Lambda = const$, whereas variation w.r.t. Λ yields $\Phi = \sqrt{-g}$, in what follows, for general NRVF gravity models (4) the field ratio χ_1 is either a non-trivial algebraic function of the matter fields in $\mathcal{L}^{(j)}$ within the first-order (Palatini) formalism (cf. Eq.(49) below), or it becomes a new dynamical scalar field within the second-order (metric) formalism (cf. Eq.(6)).

4. Jackiw Teitlboim Gravity

In two dimensions, where Einstein gravity becomes trivial because the Einstein tensor vanishes identically, a single modified measure provides an acceptable theory of gravity. That is, we can consider the Non Riemannian formulation of the Jackiw Teitlboim Gravity,

$$S = \int_M R(g)\,\Phi\, d^2x$$

where

$$\Phi(A) = \frac{1}{6}\,\varepsilon^{\alpha\beta}\partial_\alpha A_\beta\,,$$

and where the variation with respect to the gauge field A_β leads to the condition that the curvature $R = M$, where M is a constant. This is, as anticipated in fact the Jackiw-Teitelboim model [10].

5. The Constrained Volume of the Modified Measure formulation

In our case the traditional measure is replaced by a density which is independent of the metric, this density we choose it to be a total derivative, for example in 4 dimensions, the curl of a totally antisymmetric 3 index tensor, the variation with respect to totally antisymmetric 3 index tensor leads to the result that the Lagrangian it multiplies is a constant. Notice also that the constraint that the volume be a constant is automatically satisfied, since the integral of the density which is a total derivative is a surface term, so the volume in unchanged by local variations that leave the surface unchanged. To make this more explicit, let us consider

$$S = \int d^4x\, \Phi(1)\,(R + \mathcal{L}^{(1)}) + \int d^4x \sum_{j\geq 2} \Phi(j)\, \mathcal{L}^{(j)} + \int d^4x\, \sqrt{-g}\mathcal{L}^{(0)} + S^{(constraint)}\,, \tag{8}$$

where

$$S^{(constraint)} = \lambda_1\Big(\int d^4x\, \Phi(1) - c_1\Big) + \sum_{j\geq 2} \lambda_j\Big(\int d^4x \Phi(j) - c_j\Big)$$

We take now the measures $\Phi(1)$ and $\Phi(j)$ to be independent variables. The variation of the global lagrange multipliers λ_1, λ_j implies that the total volume determined by each of this measures is constrained to be a constant c_1, c_j. Finally, the variation with respect of the measures $\Phi(1)$ and $\Phi(j)$ gives us the result that each of the lagragian densities are constants, in fact equal to the global lagrange multipliers, $R + \mathcal{L}^{(1)} = -\lambda_1$, $\mathcal{L}^{(j)} = -\lambda_j$. After this point, all the theory follows the same structure as we had with the other formalism and the two formalism gives rise to exactly the same physical consequences.

5.1. The Central Charge Action or Perelman Entropy for RG flow

An example of the constrained Volume of the Modified Measure formulation is the Central Charge Action or Perelman Entropy for RG flow where it has been found that the RG equations can be obtained from an action principle of that kind. In the central charge action construction, for example in the paper On sigma model RG flow, 'central charge' action and Perelman's entropy, see for example [11] the variation with respect to a scalar ϕ is subjected to the constraint that the volume is fixed, which gives the result that that the lagrangian is a constant , as we discussed in a general way before.

To get a non-trivial functional $\Sigma(G)$ Perelman [12] suggested to minimize an action $S(G,\phi)$ in ϕ while restricting ϕ to satisfy a unit volume condition:

$$V = \int d^{\Delta}x \sqrt{G}\, e^{-2\phi} = 1\,. \tag{9}$$

Imposing this condition, the action with a global Lagrange multiplier λ that was proposed which is exactly a particular case of the formulation of the modified measure formalism in the constrained volume approach. The variation with respect to the measure implies again that the lagrangian, which is identified as β, And the result is [11] , [12], $\beta = -\lambda = constant$

pointed out to me by E.Witten

THE MEASURE $\Omega^{(j)} \equiv \Phi(B^{(j)}) = \frac{1}{(D-1)!}\varepsilon^{\mu_1\ldots\mu_D}\,\partial_{\mu_1} B^{(j)}_{\mu_2\ldots\mu_D}$

- Automatically satisfies one of the axioms required in the causal fermion systems, which is the the integral of the measure is kept fixed in the variation of the action, this is true because this measure is a total derivative , so the integral of this measure becomes a Surface term because of Gauss theorem, and of course in the variational principle we do not vary at the boundaries. For a detailed connection between modified measure theory and Causal Fermion Systems, see,
- Modified measures as an effective theory for causal fermion systems

Felix Finster, Eduardo Guendelman, Claudio F. Paganini, Class.Quant.Grav. 41 (2024) 035007 • e-Print: 2303.16566 [gr-qc]

5.2. *Causal Fermion Systems*

Another example of a Theory that requires a measure, in this case a universal measure, is the Causal Fermion systems [34] , in this case also the total volume defined by this measure has to be constant under variations and the correspondence with the Modified Measures Formulation has been argued and studied in [35] .

6. Simple Model of Unified Dark Energy and Dark matter

A simple NRVF gravity model providing a unified description of dark energy and dark matter defined by an action, particular representative of the class (4), was proposed in [13] and then generalized in [14] where also intriging symmetries were also discovered which we will review. We consider,

$$S = \int d^4x \left[\sqrt{-g}\left(R + X - V_1(\phi)\right) + \Phi(B)\left(X - V_2(\phi)\right) \right], \tag{10}$$

or equivalently:

$$S = \int d^4x \sqrt{-g}\left(R - U(\phi)\right) + \left(\sqrt{-g} + \Phi(B)\right)\left(X - V(\phi)\right) \tag{11}$$

using the notations: $V \equiv V_2$, $U \equiv V_1 - V_2$, $X \equiv -\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$, and $\Phi(B) \equiv 1/3!\varepsilon^{\mu\nu\kappa\lambda}\partial_\mu B_{\nu\kappa\lambda}$ (cf. (3)). Variation of the action (11) w.r.t. auxiliary gauge field $B_{\nu\kappa\lambda}$ yields (cf. the general Eqs.(5)):

$$X - V(\phi) = -2M_0, \tag{12}$$

where M_0 is free integration constant. The variation of (11) w.r.t. scalar field ϕ can be written in the following suggestive form:

$$\nabla_\mu J^\mu = -\sqrt{2X} U'(\phi) \,, \tag{13}$$

$$J_\mu \equiv -(1+\chi)\sqrt{2X}\partial_\mu \phi \;\;,\;\; \chi \equiv \frac{\Phi(B)}{\sqrt{-g}} \,. \tag{14}$$

The dynamics of ϕ is entirely determined by the dynamical constraint (12), completely independent of the potential $U(\phi)$. On the other hand, the ϕ-equation of motion written in the form (13) is in fact an equation determining the dynamics of χ. The energy-momentum tensor $T_{\mu\nu}$ in the Einstein equations can be written in a relativistic hydrodynamical form as:

$$T_{\mu\nu} = \rho_0 u_\mu u_\nu + g_{\mu\nu} \widetilde{p} \;\;,\quad J_\mu = \rho_0 u_\mu \tag{15}$$

where u_μ is a fluid velocity unit vector:

$$u_\mu \equiv -\frac{\partial_\mu \phi}{\sqrt{2X}} \quad (\text{note } u^\mu u_\mu = -1 \,) \,, \tag{16}$$

and the energy density $\widetilde{\rho}$ and pressure $\widetilde{p}$ are given as:

$$\widetilde{\rho} = \rho_0 + 2M_0 + U(\phi), \quad \widetilde{p} = -2M_0 - U(\phi) \tag{17}$$

with $\rho_0 \equiv (1+\chi)2X = \widetilde{\rho} + \widetilde{p}$. Energy-momentum conservation $\nabla^\nu T_{\mu\nu} = 0$ implies:

$$\nabla^\mu(\rho_0 u_\mu) = -\sqrt{2X}\, U'(\phi), \quad u_\nu \nabla^\nu u_\mu = 0 \,, \tag{18}$$

the last Eq.(18) meaning that the matter fluid flows along geodesics. In Eqs.(15), (17) the quantity $\rho_{\rm DE} \equiv 2M_0 + U(\phi) = -\tilde{p}$ has the interpretation as dark energy density, whereas ρ_0 is the dark matter energy density. For $U(\phi) = \text{const}$ or $U(\phi) = 0$ the model (11) possesses a non-trivial hidden nonlinear Noether symmetry under:

$$\delta_\epsilon \phi = \epsilon\sqrt{X}, \quad \delta_\epsilon g_{\mu\nu} = 0\,, \quad \delta_\epsilon \mathcal{B}^\mu = -\epsilon \frac{1}{2\sqrt{X}} \phi^{,\mu}\left(\Phi(B) + \sqrt{-g}\right)\,, \tag{19}$$

where $\mathcal{B}^\mu \equiv \frac{1}{3!}\varepsilon^{\mu\nu\kappa\lambda} B_{\nu\kappa\lambda}$, with a Noether conserved current $J^\mu = \rho_0 u_\mu$ according to (14): $\nabla_\mu J^\mu = 0$. Specifically, for Friedmann-Lemaitre-Robertson-Walker metric with Friedmann scale factor $a(t)$ Eq.(14) with $U(\phi) = 0$ implies: $\rho_0 = c_0/a^3$, c_0 being a free integration constant.

Thus, according to (15), (17) the model provides an exact description of ΛCDM model, and for a non-trivial potential $U(\phi)$, breaking the hidden Noether symmetry (19), we have interacting dark energy and dark matter.

The above interpretation justifies the alias "darkon" for the scalar field ϕ. Let us specifically emphasize that both dark energy and dark matter components of the energy density (17) have been *dynamically* generated thanks to the non-Riemannian volume element construction – both due to the appearance of the free integration constant M_0 and of the hidden nonlinear Noether symmetry (19) ("darkon" symmetry). In Ref.[15] the correspondence between ΛCDM model and the "darkon" Noether symmetry was exhibited up to linear order w.r.t. gravity-matter perturbations and the implications of the "darkon" symmetry breaking for possible explanation of the cosmic tensions was briefly discussed. [16] confront some potential with the late accelerated expansion data.

7. Scale Invariance, avoidance of Fifth Force in Fermionic and Dust Matter Models and Flat Potentials

Continuing with the issue of Dark matter formulated in the context of modified measure theories but now with spontaneously broken scale invariance, we show now that in this class of theories the fifth force problem can be solved.

more explicitly, let us consider the action containing a non Riemannian measure Φ which is a total divergence and is invariant under the global scale transformations:

$$g_{\mu\nu} \to e^{\theta} g_{\mu\nu}, \quad \Gamma^{\mu}_{\alpha\beta} \to \Gamma^{\mu}_{\alpha\beta}, \quad \phi \to \phi - \frac{M_p}{\alpha}\theta, \quad \Phi \to e^{2\theta}\Phi \tag{20}$$

where $\theta = const$. It is convenient to represent the action in the following form:

$$\begin{aligned} S &= S_g + S_\phi + S_m \qquad (21) \\ S_g &= -\frac{1}{\kappa}\int (\Phi + b_g\sqrt{-g}) R(\Gamma, g) e^{\alpha\phi/M_p} d^4x\,; \\ S_\phi &= \int e^{\alpha\phi/M_p}\left[(\Phi + b_\phi\sqrt{-g})\frac{1}{2}g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - (\sqrt{-g}V_2 - \Phi V_1)e^{\alpha\phi/M_p}\right] d^4x\,; \\ S_m &= \int (\Phi + b_m\sqrt{-g}) L_m d^4x\,, \end{aligned}$$

where the Lagrangian for the matter, as collection of particles, which provides the scale invariance of S_m reads

$$L_m = -m\sum_i \int e^{\frac{1}{2}\alpha\phi/M_p}\sqrt{g_{\alpha\beta}\frac{dx_i^\alpha}{d\lambda}\frac{dx_i^\beta}{d\lambda}}\,\frac{\delta^4(x - x_i(\lambda))}{\sqrt{-g}}d\lambda \tag{22}$$

where λ is an arbitrary parameter. For simplicity we consider the collection of the particles with the same mass parameter m. We assume in addition that $x_i(\lambda)$ do not participate in the scale transformations (20). We will assume that $d\vec{x}_i/d\lambda \equiv 0$ for all particles. It is convenient to proceed in the frame where $g_{0l} = 0, \; l = 1,2,3$. Then the particle density is defined by

$$n(\vec{x}) = \sum_i \frac{1}{\sqrt{-g_{(3)}}} \delta^{(3)}(\vec{x} - \vec{x}_i(\lambda)) \tag{23}$$

where $g_{(3)} = \det(g_{kl})$ and

$$S_m = -m \int d^4x (\Phi + b_m \sqrt{-g})\, n(\vec{x})\, e^{\frac{1}{2}\alpha\phi/M_p} \tag{24}$$

It turns out that when working with the new metric (ϕ remains the same)

$$\tilde{g}_{\mu\nu} = e^{\alpha\phi/M_p}(\chi + b_g) g_{\mu\nu}, \tag{25}$$

here $\chi \equiv \Phi/\sqrt{-g}$, and we call this metric the Einstein frame metric, the connection becomes Riemannian. Notice that $\tilde{g}_{\mu\nu}$ is invariant under the scale transformations (20). The transformation (25) causes the transformation of the particle density

$$\tilde{n}(\vec{x}) = (\chi + b_g)^{-3/2}\, e^{-\frac{3}{2}\alpha\phi/M_p}\, n(\vec{x}) \tag{26}$$

The Lagrangian density coupled to the Measure Φ has to be a constant, as we have discussed in general. Let us call this constant M

After the change of variables to the Einstein frame (25) and some simple algebra, the gravitational equations take the standard GR form

$$G_{\mu\nu}(\tilde{g}_{\alpha\beta}) = \frac{\kappa}{2} T^{eff}_{\mu\nu} \tag{27}$$

where $G_{\mu\nu}(\tilde{g}_{\alpha\beta})$ is the Einstein tensor in the Riemannian space-time with the metric $\tilde{g}_{\mu\nu}$. The components of the effective energy-momentum tensor are as follows:

$$
\begin{aligned}
T_{00}^{eff} &= \frac{\zeta + b_\phi}{\zeta + b_g}\left(\dot{\phi}^2 - \tilde{g}_{00}X\right) \\
&+ \tilde{g}_{00}\left[V_{eff}(\phi;\zeta,M) - \frac{\delta \cdot b_g}{\chi + b_g}X + \frac{3\chi + b_m + 2b_g}{2\sqrt{\zeta + b_g}}\, m\,\tilde{n}\right]
\end{aligned}
\tag{28}
$$

$$
\begin{aligned}
T_{ij}^{eff} &= \frac{\chi + b_\phi}{\zeta + b_g}(\phi_{,k}\phi_{,l} - \tilde{g}_{kl}X) \\
&+ \tilde{g}_{kl}\left[V_{eff}(\phi;\zeta,M) - \frac{\delta \cdot b_g}{\zeta + b_g}X + \frac{\zeta - b_m + 2b_g}{2\sqrt{\zeta + b_g}}\, m\,\tilde{n}\right]
\end{aligned}
\tag{29}
$$

Here the following notations have been used:

$$
X \equiv \frac{1}{2}\tilde{g}^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta} \qquad \textit{and} \qquad \delta = \frac{b_g - b_\phi}{b_g}
\tag{30}
$$

and the function $V_{eff}(\phi;\zeta)$ is defined by

$$
V_{eff}(\phi;\chi) = \frac{b_g\left[Me^{-2\alpha\phi/M_p} + V_1\right] + V_2}{(\chi + b_g)^2}
\tag{31}
$$

The dilaton ϕ field equation in the Einstein frame is as follows

$$
\begin{aligned}
&\frac{1}{\sqrt{-\tilde{g}}}\partial_\mu\left[\frac{\chi + b_\phi}{\chi + b_g}\sqrt{-\tilde{g}}\tilde{g}^{\mu\nu}\partial_\nu\phi\right] - \frac{\alpha}{M_p}\,\frac{(\chi + b_g)Me^{-2\alpha\phi/M_p} - (\chi - b_g)V_1 + 2V_2 - \delta b_g(\chi + b_g)X}{(\chi + b_g)^2} \\
&= \frac{\alpha}{M_p}\,\frac{\chi - b_m + 2b_g}{2\sqrt{\chi + b_g}}\, m\,\tilde{n}
\end{aligned}
\tag{32}
$$

In the above equations, the scalar field χ is determined as a function $\chi(\phi, X, \tilde{n})$ by means of the following constraint:

$$\frac{(b_g - \chi)\left(Me^{-2\alpha\phi/M_p} + V_1\right) + 2V_2}{(\chi + b_g)^2} - \frac{\delta \cdot b_g X}{\chi + b_g} = \frac{\chi - b_m + 2b_g}{2\sqrt{\chi + b_g}}\, m\, \tilde{n} \tag{33}$$

One should now pay attention to the interesting result that the explicit $\tilde{n}$ dependence involving **the same form of χ dependence**

$$\frac{\chi - b_m + 2b_g}{2\sqrt{\chi + b_g}}\, m\, \tilde{n} \tag{34}$$

appears simultaneously in the dust contribution to the pressure (through the last term in Eq. (39)), in the effective dilaton to dust coupling (in the r.h.s. of Eq. (32)) and in the r.h.s. of the constraint (44).

Let us analyze consequences of this wonderful coincidence in the case when the matter energy density (modeled by dust) is much larger than the dilaton contribution to the dark energy density in the space region occupied by this matter. Evidently this is the condition under which all tests of Einstein's GR, including the question of the fifth force, are fulfilled. if the dust is in the normal conditions there is a possibility to provide the desirable feature of the dust in GR: it must be -. This is realized provided that in normal conditions (n.c.) the following equality holds with extremely high accuracy:

$$\chi^{(n.c.)} \approx b_m - 2b_g \tag{35}$$

Remind that we have assumed $b_m > b_g$. Then $\chi^{(n.c.)} + b_g > 0$, and the transformation (25) and the subsequent equations in the Einstein frame are well defined. Inserting (35) in the last term of Eq. (38) we obtain the effective dust energy density in normal conditions

$$\rho_m^{(n.c.)} = 2\sqrt{b_m - b_g}\, m\tilde{n} \tag{36}$$

When we get only a slight deviation of from ζ from $b_m - 2b_g$, when the matter energy density is many orders of magnitude larger than the dilaton contribution to the dark energy density, we obtain an effective 5th force coupling f. For this look at the ϕ-equation in the form (32) and estimate the Yukawa type coupling constant in the r.h.s. of this equation. In fact, using the constraint (44) and representing the particle density in the form $\tilde{n} \approx N/v$ where N is the number of particles in a volume v, one can make the following estimation for the effective dilaton to matter coupling "constant" f defined by the Yukawa type interaction term $f\tilde{n}\phi$ (if we were to invent an effective action whose variation with respect to ϕ would result in Eq. (32)):

$$f \equiv \alpha \frac{m}{M_p} \frac{\chi - b_m + 2b_g}{2\sqrt{\zeta + b_g}} \approx \alpha \frac{m}{M_p} \frac{\chi - b_m + 2b_g}{2\sqrt{b_m - b_g}} \sim \frac{\alpha}{M_p} \frac{\rho_{vac}}{\tilde{n}} \approx \alpha \frac{\rho_{vac} v}{N M_p} \tag{37}$$

becomes less than the ratio of the "mass of the vacuum" in the volume occupied by the matter to the Planck mass. The model yields this kind of "Archimedes law" without any especial (intended for this) choice of the underlying action and without fine tuning of the parameters. The model not only explains why all attempts to discover a scalar force correction to Newtonian gravity were unsuccessful so far but also predicts that in the near future there is no chance to detect such corrections in the astronomical measurements as well as in the specially designed fifth force experiments on intermediate, short (like millimeter) and even ultrashort (a few nanometer) ranges. This prediction is alternative to predictions of other known models.

7.1. Flat potentials in the absence of matter, consequences for DE or Inflation

If we set To clarify what the theory is telling us only in the gravity, scalar field sector, we set the dust matter momentum tensor are as follows:

$$
\begin{aligned}
T_{00}^{eff} &= \frac{\chi + b_\phi}{\chi + b_g}\left(\dot{\phi}^2 - \tilde{g}_{00} X\right) \\
&+ \tilde{g}_{00}\left[V_{eff}(\phi;\chi,M) - \frac{\delta \cdot b_g}{\zeta + b_g} X\right]
\end{aligned}
\tag{38}
$$

$$
\begin{aligned}
T_{ij}^{eff} &= \frac{\chi + b_\phi}{\chi + b_g}\left(\phi_{,k}\phi_{,l} - \tilde{g}_{kl} X\right) \\
&+ \tilde{g}_{kl}\left[V_{eff}(\phi;\zeta,M) - \frac{\delta \cdot b_g}{\chi + b_g} X\right]
\end{aligned}
\tag{39}
$$

Here the following notations have been used:

$$
X \equiv \frac{1}{2}\tilde{g}^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta} \qquad \textit{and} \qquad \delta = \frac{b_g - b_\phi}{b_g}
\tag{40}
$$

and the function $V_{eff}(\phi;\chi)$ is defined by

$$
V_{eff}(\phi;\chi) = \frac{b_g\left[Me^{-2\alpha\phi/M_p} + V_1\right] + V_2}{(\chi + b_g)^2}
\tag{41}
$$

The dilaton ϕ field equation in the Einstein frame is as follows

$$
\begin{aligned}
&\frac{1}{\sqrt{-\tilde{g}}}\partial_\mu\left[\frac{\chi + b_\phi}{\chi + b_g}\sqrt{-\tilde{g}}\tilde{g}^{\mu\nu}\partial_\nu\phi\right] - \frac{\alpha}{M_p}\,\frac{(\zeta + b_g)Me^{-2\alpha\phi/M_p} - (\zeta - b_g)V_1 + 2V_2 - \delta b_g(\chi + b_g)X}{(\chi + b_g)^2} \\
&=
\end{aligned}
\tag{42}
$$

In the above equations, the scalar field χ is determined as a function $\chi(\phi, X,)$ by means of the following constraint:

In the above equations, the scalar field χ is determined as a function $\chi(\phi, X,)$ by means of the following constraint:

$$\frac{(b_g - \chi)\left(Me^{-2\alpha\phi/M_p} + V_1\right) + 2V_2}{(\chi + b_g)^2} - \frac{\delta \cdot b_g X}{\chi + b_g} = 0 \tag{43}$$

which now becomes, after some simplifications a linear equation. The simplest case, resulting from the choice $b_g = 0$, studied first in [2] already gives a very simple result

$$\chi = \frac{2V_2}{e^{-2\alpha\phi/M_p} + V_1} \tag{44}$$

which then inserted into (41), gives

$$V_{eff}(\phi) = \frac{(e^{-2\alpha\phi/M_p} + V_1)^2}{4V_2} \tag{45}$$

which generalizes the effective potential used by Starobinsky for example. It also displays an infinite flat region, that can be used for inflation, but it does not display Dark energy in that case.

As we have seen, the matter modifies the scalar field dynamic, this will have very important consequences in the discussion of the tensions in cosmology in some more complete models.

Furthermore , in the case $b_g \neq 0$, K essence terms appear, this will also will be explored later in this paper in more comprehensive treatments.

8. Actions with full Lagrangian Densities Shift Invariance for all lagrangian densities

Our original motivation to introduce a modified measure different than $\sqrt{-g}$ was to obtain invariance under under the shift of the lagrangian densities that couple to each measure $\Phi(j)$, as in

$$\mathcal{L}^j \longrightarrow \mathcal{L}^j + C^j \,.$$

but in (4) we have still left a lagrangian density $\mathcal{L}^0$ that couples to $\sqrt{-g}$, so the shift symmetry does not hold for $\mathcal{L}^0$.

It is possible nevertheless to introduce $\sqrt{-g}$ in the equations of motion and not directly as a measure in the action by introducing a term in the action of the form

$$\Phi_2(B)\frac{\Phi_0(C)}{\sqrt{-g}}$$

where $\Phi_0(C)$ is a total derivative, whose variation implies that $\Phi_2(B)$ is proportional to $\sqrt{-g}$, that is

$$\frac{\Phi_2(B)}{\sqrt{-g}} = \chi_2$$

where χ_2 is a new constant that will affect the strength of the effective potential as we will explain in several examples below.

9. Quintessential Inflationary Model with Dynamical Higgs Effect in Metric-Affine Formulation

The starting point is the following specific NRVF gravity action from the class (4) involving coupling to a scalar "inflaton" φ and to the bosonic sector of the standard electroweak particle model where, following Bekenstein's idea from 1986 [17] about gravity-assisted dynamical spontaneous symmetry breakdown, the Higgs-like $SU(2) \times U(1)$ iso-doublet scalar σ_a enters with a standard positive mass-squared and without self-interaction in sharp distinction w.r.t. standard particle model. The pertinent NRVF action reads explicitly [19], [18],

$$S = \int d^4x\, \Phi_1(A) \Big[R(g,\Gamma) - 2Ł_0 \frac{\Phi_1(A)}{\sqrt{-g}} + L^{(1)}(\varphi,\sigma) \Big] + \int d^4x\, \Phi_2(B) \Big[f_2 e^{2\alpha\varphi} + L_{\mathrm{EW-gauge}} - \frac{\Phi_0(C}{\sqrt{-g}}$$

with notations:

- $\Phi_1(A) = \frac{1}{3!}\varepsilon^{\mu\nu\kappa\lambda}\, \partial_\mu A_{\nu\kappa\lambda}$ and similarly for $\Phi_2(B)$, $\Phi_0(C)$ according to (3);
- The scalar curvature $R(g,\Gamma) = g^{\mu\nu} R_{\mu\nu}(\Gamma)$ is given in terms of the Ricci tensor $R_{\mu\nu}(\Gamma)$ in the first-order (Palatini) formalism;
- The matter Lagrangian reads:

$$L^{(1)}(\varphi,\sigma) = X_\varphi + f_1 e^{\alpha\varphi} + X_\sigma - m_0^2 \sigma_a^* \sigma_a e^{\alpha\varphi}, \quad X_\varphi \equiv -\frac{1}{2} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi \;,\;\; X_\sigma \equiv -g^{\mu\nu} \nabla_\mu \sigma_a^* \nabla_\nu \sigma_a$$

- $L_{\mathrm{EW-gauge}}$ denotes the Lagrangian of the $SU(2) \times U(1)$ gauge fields.
- $Ł_0$ is small dimension full constant which will be identified in the sequel with the "late" universe cosmological constant in the dark energy dominated accelerated expansion's epoch.

The equations of motion w.r.t. auxiliary tensor gauge fields in $\Phi_1(A)$, $\Phi_2(B)$ and $\Phi_1(C)$ yield (cf. (5)):

$$g^{\mu\nu}\Big(R_{\mu\nu}(\Gamma) - \frac{1}{2}\partial_\mu\varphi\partial_\nu\varphi - \nabla_\mu\sigma_a^*\nabla_\nu\sigma_a\Big) - 4Ł_0\frac{\Phi_1(A)}{\sqrt{-g}} + (f_1 - m_0^2\sigma_a^*\sigma_a)e^{\alpha\varphi} = M_1, \tag{47}$$

$$f_2 e^{-2\alpha\phi} + L_{\rm EW-gauge} - \frac{\Phi_0(C)}{\sqrt{-g}} = -M_2, \; \frac{\Phi_2(B)}{\sqrt{-g}} = \chi_2 \tag{48}$$

where $M_{1,2}, \chi_2$ are integration constants. The $g^{\mu\nu}$-equations of motion together with (47)-(48) imply that the ratio $\chi_1 \equiv \frac{\Phi(A)}{\sqrt{-g}}$ is an algebraic function of the matter fields:

$$\chi_1(\varphi,\sigma) \equiv \frac{\Phi_1(A)}{\sqrt{-g}} = \frac{2\chi_2(f_2 e^{2\alpha\varphi} + M_2)}{M_1 + (m_0^2\sigma_a^*\sigma_a - f_1)e^{\alpha\varphi}}. \tag{49}$$

The equation of motion w.r.t. $\Gamma^\mu_{\nu\lambda}$ yields a solution for $\Gamma^\mu_{\nu\lambda}$ as the Levi-Civita connection w.r.t. to a *Weyl-conformally rescaled* metric:

$$\bar{g}_{\mu\nu} = \chi_1(\varphi,\sigma)\, g_{\mu\nu} \tag{50}$$

with $\chi_1(\varphi,\sigma)$ as in (49). The conformal transformation $g_{\mu\nu} \to \bar{g}_{\mu\nu}$ via (50) on the NRVF action (46) converts the latter into the physical Einstein-frame action (objects in the Einstein-frame are indicated by a bar):

$$S_{\rm EF} = \int d^4x\, \sqrt{-\bar{g}}\Big[R(\bar{g}) - \frac{1}{2}\bar{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \bar{g}^{\mu\nu}\nabla_\mu\sigma_a^*\nabla_\nu\sigma_a - U_{\rm eff}(\varphi,\sigma) + L_{\rm EW-gauge}(\bar{g})\Big]\,. \tag{51}$$

Here the interesting object is the effective Einstein-frame scalar potential:

$$U_{\rm eff}(\varphi,\sigma) = \frac{\Big[M_1 + e^{\alpha\varphi}(m_0^2\sigma_a^*\sigma_a - f_1)\Big]^2}{4\chi_2(f_2 e^{2\alpha\varphi} + M_2)} + 2Ł_0\,, \tag{52}$$

which is entirely *dynamically generated* due to the appearance of the free integration constants $M_{1,2}$ and χ_2 (47)-(48). Fig 1 shows the qualitative U_{eff} shape. $U_{\rm eff}(\varphi,\sigma)$ exhibits a number of remarkable features:

- $U_{\rm eff}(\varphi,\sigma)$ possesses two (infinitely) large flat regions as a function of φ at $\sigma_a =$ fixed.
- The first one – the (-) flat "inflaton" region for large negative values of φ (and σ_a – finite) corresponds to the "slow-roll" inflationary evolution of the "early" universe driven by φ where:

$$U_{\rm eff}(\phi,\sigma) \simeq U_{(-)} = \frac{M_1^2}{4\chi_2\, M_2} + 2Ł_0\,, \tag{53}$$

  independent of the finite value of σ_a, which is energy scale of the inflationary epoch. Thus, in the "early" universe there is *no spontaneous breaking* of electroweak $SU(2)\times U(1)$ symmetry. Moreover, σ_a does not participate in the "slow-roll" inflationary evolution, so σ stays constant there equal to the "false"vacuum value $\sigma = 0$
- The second flat region is the (+) flat "inflaton" region for large positive values of φ (and σ_a – finite) which corresponds to the evolution of the post-inflationary ("late") universe. Here:

$$U_{\rm eff}(\varphi,\sigma) \simeq U_{(+)}(\sigma) = \frac{\left(m_0^2\,\sigma_a^*\sigma_a - f_1\right)^2}{4\chi_2 f_2} + 2Ł_0 \tag{54}$$

  becomes a *dynamically induced* $SU(2)\times U(1)$ spontaneous symmetry breaking Higgs-like potential with a Higgs "vacuum" at $|\sigma_{\rm vac}| = \frac{1}{m_0}\sqrt{f_1}$.

10. Emergent Universe Solutions, followed by inflation, in the the case of one quintessential inflation field

Let us also note that Ref. [21] (for an earlier version, see [20]) exhibits an explicit realization of the cosmological "seesaw" mechanism through the NRVF formulation, as well as it yields an additional "emergent universe" cosmological solution without a "Big-Bang" initial singularity. For a brief illustration of the latter effects let us consider the "inflaton-only" NRVF action studied in [21] (for simplicity we skip the R^2 term):

$$S = \int d^4x\, \Phi_1(A)\Big[R + X_\varphi - f_1 e^{-\alpha\varphi}\Big] + \int d^4x\, \Phi_2(B)\Big[-b\, e^{-\alpha\varphi} X_\varphi + f_2 e^{-2\alpha\varphi} - \frac{\Phi_0(C)}{\sqrt{-g}}\Big]\,, \quad (55)$$

where b is an additional dimensionless parameter.

The "inflaton" potential in the Einstein frame (analog of (68)) is:

$$U_{\rm eff}(\varphi) = \frac{1}{4\chi_2}\left(f_1 e^{-\alpha\varphi} + M_1\right)^2 \left(f_2 e^{-2\alpha\varphi} + M_2\right)^{-1} \quad (56)$$

so that on the (-) and (+) "inflaton" flat regions $U_{\rm eff}(\varphi)$ reduces to: $U_{(-)} \simeq \frac{f_1^2}{4\chi_2 f_2}$ and $U_{(+)} \simeq \frac{M_1^2}{4\chi_2 M_2}$, accordingly. Therefore, choosing $f_1 \sim f_2 \sim 10^{-8} M_{Pl}^4$ conforming to the inflationary scale, and taking $M_1 \sim M_{EW}^4$ and $M_2 \sim M_{Pl}^4$ we achieve $U_{(+)} \sim 10^{-122} M_{Pl}^4$ vastly smaller than $U_{(-)}$. If we take $\alpha \to -\alpha$ in (55) the roles of $f_{1,2}$ and $M_{1,2}$ are interchanged.

Similar "seesaw" effect is found in Refs.[23], [24] where the scalar potential is extracted from the slow-roll parameters. [1]. Furthermore, the NRVF model (55) yields in EInstein-frame "emergent universe" solution for the range of the b-parameter: $-4(2-\sqrt{3}) < b\frac{f_1}{f_2} < -1$.

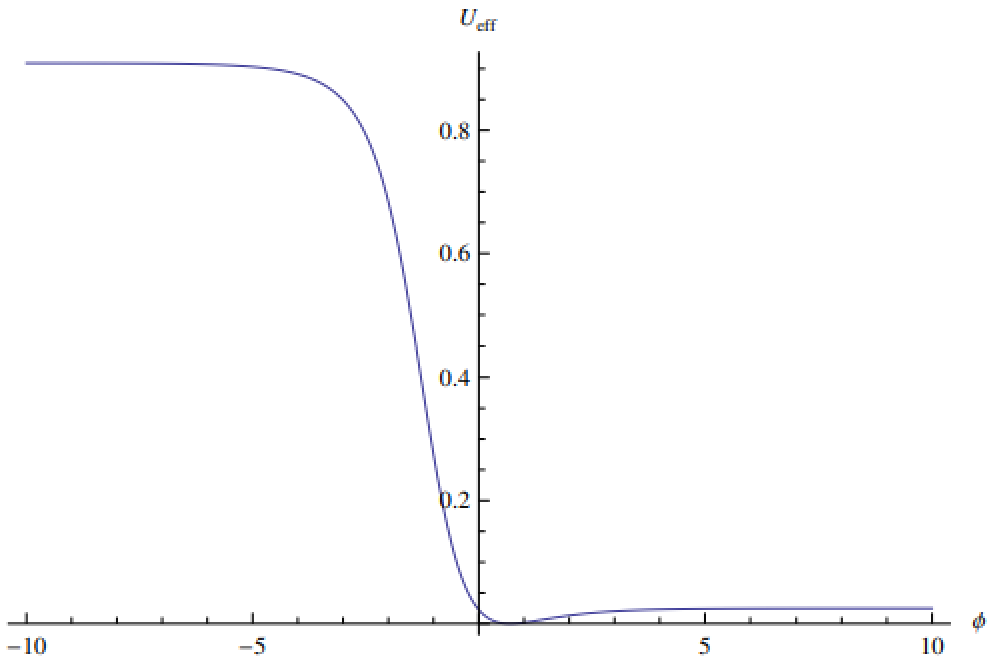


Figure 2. *Qualitative shape of the one-dimensional plot for the effective scalar potential* U_{eff}.

12. General shift invariant theory using Two Independent Non-Riemannian Measures and two scalar fields and potentials with three flat regions

We shall assume the action General form with lagrangian densities shift invariace involving two independent non-metric integration measure densities generalizing the model analyzed in [21] is given by

$$S = \int d^4x\, \Phi_1(A)\Big[R + L^{(1)}\Big] + \int d^4x\, \Phi_2(B)\Big[L^{(2)} + \epsilon R^2 + \frac{\Phi(H)}{\sqrt{-g}}\Big] . \tag{57}$$

Here the following definitions are used:

- The quantities $\Phi_1(A)$ and $\Phi_2(B)$ are two densities and these are independent non-metric volume-forms defined in terms of field-strengths of two auxiliary 3-index antisymmetric tensor gauge fields

$$\Phi_1(A) = \frac{1}{3!}\varepsilon^{\mu\nu\kappa\lambda}\partial_\mu A_{\nu\kappa\lambda} \quad , \quad \Phi_2(B) = \frac{1}{3!}\varepsilon^{\mu\nu\kappa\lambda}\partial_\mu B_{\nu\kappa\lambda} . \tag{58}$$

- The scalar curvature $R = g^{\mu\nu}R_{\mu\nu}(\Gamma)$ and the Ricci tensor $R_{\mu\nu}(\Gamma)$ are defined in the first-order (Palatini) formalism, in which the affine connection $\Gamma^{\mu}_{\nu\lambda}$ is *a priori* independent of the metric $g_{\mu\nu}$. Let us recall that $R + R^2$ gravity within the second order formalism was originally developed in [37].

- The two different Lagrangians $L^{(1,2)}$ correspond to two scalar matter fields φ_1 and φ_2 such that

$$L^{(1)} = -\frac{1}{2}g^{\mu\nu}\partial_\mu\varphi_1\partial_\nu\varphi_1 - \frac{1}{2}g^{\mu\nu}\partial_\mu\varphi_2\partial_\nu\varphi_2 - V(\varphi_1,\varphi_2), \tag{59}$$

$$L^{(2)} = -\frac{b_1}{2}e^{-\alpha_1\varphi_1}g^{\mu\nu}\partial_\mu\varphi_1\partial_\nu\varphi_1 - \frac{b_2}{2}e^{-\alpha_2\varphi_2}g^{\mu\nu}\partial_\mu\varphi_2\partial_\nu\varphi_2 + U(\varphi_1,\varphi_2), \tag{60}$$

  where the potentials $V(\varphi_1,\varphi_2)$ and $U(\varphi_1,\varphi_2)$ are defined as

$$V(\varphi_1,\varphi_2) = f_1 \exp\{-\alpha_1\varphi_1\} + g_1 \exp\{-\alpha_2\varphi_2\}, \text{ and } U(\varphi_1,\varphi_2) = f_2 \exp\{-2\alpha_1\varphi_1\} + g_2 \exp \tag{61}$$

  Here $\alpha_1, \alpha_2, f_1, g_1, f_2, \epsilon$ and g_2 are positive parameters, whereas b_1 and b_2 are dimensionless and their signs are to be discussed. The parameters α_1 and α_2 have dimensions of M_{Pl}^{-1}, instead the parameters f_1, f_2, g_1 and g_2 have units of M_{Pl}^4 and the parameter ϵ has units of M_{Pl}^{-2}. Let us recall that since we are considering units in which $G_{\text{Newton}} = 1/16\pi$ then the Planck mass $M_{Pl} = \sqrt{2} = \sqrt{1/8\pi G_{\text{Newton}}}$.

Notice that since both $\Phi_1(A)$ and $\Phi_2(B)$ are total derivatives, the shift of these lagrangians by constants $C^{(1)}$ and $C^{(2)}$ are symmetries

$$L^{(1)} \to L^{(1)} + C^{(1)}$$

and

$$L^{(2)} \to L^{(2)} + C^{(2)}$$

notice that these symmetries, as well as the scale symmetry will be spontaneouly broken by the integration of the equations of motion.

12.1. Resulting three Flat Regions from an Effective Scalar Potential

After going through the procedure of defining the Einstein frame, etc, we find, k-essence terms and an effective potential for the two scalars. Upon substituting expression (**??**) into (**??**) we arrive at the explicit form for the Einstein-frame scalar Lagrangian:

$$L_{\rm eff} = A_1(\varphi_1,\varphi_2)X_1 + A_2(\varphi_1,\varphi_2)X_2 + B_1(\varphi_1,\varphi_2)X_1^2 + B_2(\varphi_1,\varphi_2)X_2^2 + B_{12}(\varphi_1,\varphi_2)X_1X_2 - U_{\rm eff}(\varphi_1, \tag{62}$$

where the functions $A_1(\varphi_1,\varphi_2)$ and $A_2(\varphi_1,\varphi_2)$ are given by

$$A_1(\varphi_1,\varphi_2) = 1 + \left[\frac{1}{2}b_1 e^{-\alpha_1\varphi_1} - \epsilon(V-M_1)\right]\frac{V-M_1}{U+M_2+\epsilon(V-M_1)^2} = 1+$$
$$\left[\frac{1}{2}b_1 e^{-\alpha_1\varphi_1} - \epsilon\big(f_1 e^{-\alpha_1\varphi_1} + g_1 e^{-\alpha_2\varphi_2} - M_1\big)\right]\frac{f_1 e^{-\alpha_1\varphi_1} + g_1 e^{-\alpha_2\varphi_2} - M_1}{f_2 e^{-2\alpha\varphi_1} + g_2 e^{-2\alpha\varphi_2} + M_2 + \epsilon(f_1 e^{-\alpha\varphi_1} + g_1 e^{-\alpha_2\varphi_2} - } \tag{63}$$

and

$$A_2(\varphi_1,\varphi_2) = 1 + \left[\frac{1}{2}b_2 e^{-\alpha_2\varphi_2} - \epsilon(V-M_1)\right]\frac{V-M_1}{U+M_2+\epsilon(V-M_1)^2}$$
$$= 1 + \left[\frac{1}{2}b_2 e^{-\alpha_2\varphi_2} - \epsilon\big(f_1 e^{-\alpha\varphi_1} + g_1 e^{-\alpha_2\varphi_2} - M_1\big)\right]\frac{f_1 e^{-\alpha_1\varphi_1} + g_1 e^{-\alpha_2\varphi_2} - M_1}{f_2 e^{-2\alpha\varphi_1} + g_2 e^{-2\alpha\varphi_2} + M_2 + \epsilon(f_1 e^{-\alpha\varphi_1} + g_1 e^{-\alpha_2} } \tag{64}$$

The coefficient $B_1(\varphi_1,\varphi_2)$ is defined as

$$B_1(\varphi_1,\varphi_2) = \chi_2 \frac{\epsilon\left[U + M_2 + (V-M_1)b_1 e^{-\alpha_1\varphi_1}\right] - \frac{1}{4}b_1^2 e^{-2\alpha\varphi_1}}{U+M_2+\epsilon(V-M_1)^2}$$
$$= \chi_2 \frac{\epsilon\left[f_2 e^{-2\alpha\varphi_1} + g_2 e^{-2\alpha\varphi_2} + M_2 + (f_1 e^{-\alpha\varphi_1} + g_1 e^{-\alpha_2\varphi_2} - M_1)b_1 e^{-\alpha_1\varphi_1}\right] - \frac{1}{4}b_1^2 e^{-2\alpha_1\varphi_1}}{f_2 e^{-2\alpha_1\varphi_1} + g_2 e^{-2\alpha_2\varphi_2} + M_2 + \epsilon(f_1 e^{-\alpha_1\varphi_1} + g_2 e^{-\alpha_2\varphi_2} - M_1)^2}, \tag{65}$$

and for the function $B_2(\varphi_1, \varphi_2)$ we obtain

$$B_2(\varphi_1, \varphi_2) = \chi_2 \frac{\epsilon\left[U + M_2 + (V - M_1)b_2 e^{-\alpha_2\varphi_2}\right] - \frac{1}{4}b_2^2 e^{-2\alpha\varphi_2}}{U + M_2 + \epsilon(V - M_1)^2}$$

$$= \chi_2 \frac{\epsilon\left[f_2 e^{-2\alpha\varphi_1} + g_2 e^{-2\alpha\varphi_2} + M_2 + (f_1 e^{-\alpha\varphi_1} + g_1 e^{-\alpha_2\varphi_2} - M_1)b_2 e^{-\alpha_2\varphi_2}\right] - \frac{1}{4}b_2^2 e^{-2\alpha_2\varphi_2}}{f_2 e^{-2\alpha_1\varphi_1} + g_2 e^{-2\alpha_2\varphi_2} + M_2 + \epsilon(f_1 e^{-\alpha_1\varphi_1} + g_2 e^{-\alpha_2\varphi_2} - M_1)^2}, \quad (66)$$

and the coefficient $B_{12}(\varphi_1, \varphi_2)$ becomes

$$B_{12}(\varphi_1, \varphi_2) = \chi_2 E_0\Big[- E_1 b_2 e^{-\alpha_2\varphi_2} - E_2 b_1 e^{-\alpha_1\varphi_1} + 2E_0 E_1 E_2[(M_2 + U) +$$

$$\epsilon(M_1 - V)^2] + 2\epsilon[(1/E_0) - (E_1 + E_2)(M_1 - V)]\Big], \quad (67)$$

where the quantities E_0, E_1 and E_2 are defined as

$$E_0 = \frac{(V - M_1)}{2\chi_2[U + M_2 + \epsilon(V - M_1)^2]}, \quad E_1 = \chi_2\left[\frac{b_1 e^{-\alpha_1\varphi_1}}{V - M_1} - 2\epsilon\right], \quad \text{and} \quad E_2 = \chi_2\left[\frac{b_2 e^{-\alpha_2\varphi_2}}{V - M_1} - 2\epsilon\right]$$

The effective scalar field potential as a function of the scalar fields yields

$$U_{\text{eff}}(\varphi_1, \varphi_2) = \frac{(V - M_1)^2}{4\chi_2\left[U + M_2 + \epsilon(V - M_1)^2\right]} =$$

$$\frac{(f_1 e^{-\alpha_1\varphi_1} + g_1 e^{-\alpha_2\varphi_2} - M_1)^2}{4\chi_2\left[f_2 e^{-2\alpha_1\varphi_1} + g_2 e^{-2\alpha_2\varphi_2} + M_2 + \epsilon(f_1 e^{-\alpha_1\varphi_1} + g_1 e^{-\alpha_2\varphi_2} - M_1)^2\right]}, \quad (68)$$

where we have used for V and U, the expressions given by Eq.(61).

The crucial feature of $U_{\text{eff}}(\varphi_1, \varphi_2)$ is the presence of three infinitely large flat regions. In this sense, we have one for large positive values of the scalar fields φ_1 and φ_2 and two others for the limits $\varphi_1 \to -\infty$ and $\varphi_2 \to -\infty$.

For large negative values of φ_1, which we will choose to describe the very early phase of the universe, meaning the emergent phase and inflation we have for the effective potential and the coefficient functions in the Einstein-frame scalar Lagrangian (62)-(68):

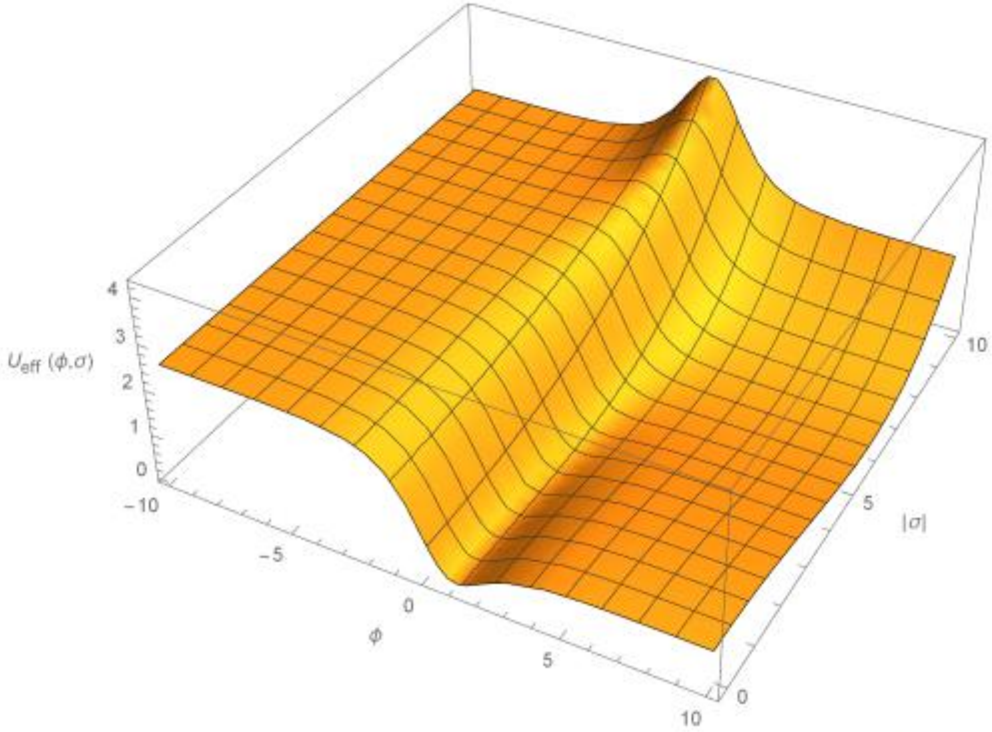


Figure 1. *Qualitative shape of the two-dimensional plot for the effective scalar potential* U_{eff}.

For large negative values of φ_1, which we will choose to describe the very early phase of the universe, meaning the emergent phase and inflation we have for the effective potential and the coefficient functions in the Einstein-frame scalar Lagrangian (62)-(68):

$$U_{\text{eff}}(\varphi_1, \varphi_2) \simeq U_{\text{eff}}(-\infty, \varphi_2) = U_{eff} = \frac{f_1^2/f_2}{4\chi_2(1+\epsilon f_1^2/f_2)}\,, \tag{69}$$

$$A_1(\varphi_1, \varphi_2) \simeq A_1(-\infty, \varphi_2) = A_1 = \frac{1+\frac{1}{2}b_1 f_1/f_2}{1+\epsilon f_1^2/f_2}\,, \quad B_1(\varphi_1, \varphi_2) \simeq B_1(-\infty, \varphi_2) = B_1 = -\chi_2 \frac{b_1^2/4f_2}{} \tag{70}$$

For the terms A_2 and B_2 in the limit in which $\varphi_1 \to -\infty$ we have

$$A_2(\varphi_1, \varphi_2) \simeq A_2(-\infty, \varphi_2) = A_2 = \frac{1}{1+\epsilon f_1^2/f_2}, \quad \text{and} \quad B_2(\varphi_1, \varphi_2) \simeq B_2(-\infty, \varphi_2) = B_2 = \frac{\chi_2\epsilon}{1+\epsilon f_1^2/} \tag{71}$$

For the coefficient $B_{12}(\varphi_1, \varphi_2)$ in the limit in which the scalar field $\varphi_1 \to -\infty$ becomes

$$B_{12}(\varphi_1, \varphi_2) \simeq B_{12}(-\infty, \varphi_2) = B_{12} = \chi_2 \tilde{E}_0 \Big[- \tilde{E}_2 b_1 + \tilde{E}_0 f_2 + \frac{2\epsilon}{\tilde{E}_0} + \epsilon \tilde{E}_0 f_1^2 + 2\epsilon f_1 (\tilde{E}_1 + \tilde{E}_2) \Big], \tag{72}$$

where

$$\tilde{E}_0 = \frac{f_1}{2\chi_2[f_2 + \epsilon f_1^2]}, \quad \tilde{E}_1 = \chi_2 \left[\frac{b_1}{f_1} - 2\epsilon \right], \text{ and } \tilde{E}_2 = -2\chi_2\epsilon.$$

For the second flat region we will consider that the scalar field $\varphi_2 \to -\infty$ such that the effective potential in the second flat region is given by

$$U_{\text{eff}}(\varphi_1, \varphi_2) \simeq U_{\text{eff}}(\varphi_1, -\infty) = U_{\text{eff g}} = \frac{g_1^2/g_2}{4\chi_2(1+\epsilon g_1^2/g_2)}\,, \tag{73}$$

and the kinetic coefficients A_2 and B_2 in the limit in which $\varphi_2 \to -\infty$ result

$$A_2(\varphi_1, -\infty) = A_{2g} = \frac{1+\frac{1}{2}b_2 g_1/g_2}{1+\epsilon g_1^2/g_2}, \quad B_2(\varphi_1, -\infty) = B_{2g} = -\chi_2 \frac{b_2^2/4g_2 - \epsilon(1+b_2 g_1/g_2)}{1+\epsilon g_1^2/g_2}\,, \tag{74}$$

and the terms A_1 and B_1 in this limit result

$$A_1(\varphi_1, \varphi_2) \simeq A_1(\varphi_1, -\infty) = A_{1g} = \frac{1}{1+\epsilon g_1^2/g_2}\,, \quad B_1(\varphi_1, \varphi_2) \simeq B_1(\varphi_1, -\infty) = B_{1g} = \frac{\chi_2\epsilon}{1+\epsilon g_1^2/g_2} \tag{75}$$

For the coefficient $B_{12}(\varphi_1, \varphi_2)$ in the limit in which the scalar field $\varphi_2 \to -\infty$ we have

$$B_{12}(\varphi_1, \varphi_2) \simeq B_{12}(\varphi_1, -\infty) = B_{12g} = \chi_2 \tilde{E}_3 \Big[-\tilde{E}_2 b_2 + \tilde{E}_3 g_2 + \frac{2\epsilon}{\tilde{E}_3} + \epsilon \tilde{E}_3 g_1^2 + 2\epsilon g_1(\tilde{E}_4 + \tilde{E}_2) \Big], \tag{76}$$

where

$$\tilde{E}_3 = \frac{g_1}{2\chi_2[g_2 + \epsilon g_1^2]}, \quad \text{and} \quad \tilde{E}_4 = \chi_2 \left[\frac{b_2}{g_1} - 2\epsilon \right].$$

In the third flat region for large positive φ_1 and also φ_2, we find that the effective potential reduces to

$$U_{\text{eff}}(\varphi_1, \varphi_2) \simeq U_{\text{eff}}(+\infty, +\infty) = U_{\text{eff}(+)} = \frac{M_1^2/M_2}{4\chi_2(1+\epsilon M_1^2/M_2)}\,, \tag{77}$$

and the kinetic coefficients are

$$A_1(\varphi_1,\varphi_2) = A_2(\varphi_1,\varphi_2) \simeq A_{(+)} \equiv \frac{M_2}{M_2+\epsilon M_1^2} \quad , \quad B_1(\varphi_1,\varphi_2) = B_2(\varphi_1,\varphi_2) \simeq B_{(+)} \equiv \epsilon\chi_2 \frac{M}{M_2+} \tag{78}$$

and for this limit we find that the coefficient $B_{12}(\varphi_1,\varphi_2)$ becomes

$$B_{12}(\varphi_1,\varphi_2) = B_{12}(+\infty,+\infty) = B_{12(+)} = 2\epsilon\chi_2 \frac{M_2}{M_2+\epsilon M_1^2} = 2\,B_{(+)}. \tag{79}$$

We will consider that the flat region (69) corresponds to the evolution of the early universe (emergent and inflation). On the other hand, the flat regions (73) and (77) concern to the evolution of the late universe with a two phase structure.

In particular, if we assume the order of magnitude of the coupling parameters in the effective potential given by Eq.(69), are $f_1 \sim f_2 \sim (10^{-2}M_{Pl})^4$, then the order of magnitude of the vacuum energy density of the early universe during the inflationary epoch yields

$$U_{\rm eff}(-\infty,\varphi_2) = U_{eff} \sim f_1^2/f_2 \sim 10^{-8}M_{Pl}^4\,, \tag{80}$$

here we have assumed that the parameter ϵ is small and the integration constant $\chi_2 \sim \mathcal{O}(1)$.

In order to study the evolution of the universe from an emergent and inflationary scenarios to dark epoch, we consider that the standard Friedman-Lemaitre-Robertson-Walker space-time metric is given by

$$ds^2 = -dt^2 + a^2(t)\Big[\frac{dr^2}{1-Kr^2} + r^2(d^2 + \sin^2 d\phi^2)\Big], \tag{81}$$

where $a(t)$ denotes the scale factor and K corresponds to the space curvature.

By assuming that the matter is described by a perfect fluid with an energy density and pressure ρ and p, we have that the associated Friedmann equations are

$$\frac{\ddot{a}}{a} = -\frac{1}{12}(\rho + 3p), \quad H^2 + \frac{K}{a^2} = \frac{1}{6}\rho, \quad \text{and} \quad \dot{\rho} + 3H(\rho + p) = 0, \tag{82}$$

where $H = \frac{\dot{a}}{a}$ is the Hubble parameter. Also, here the energy density and pressure associated to the scalar fields $\varphi_1 = \varphi_1(t)$ and $\varphi_2 = \varphi_2(t)$ are defined as

$$\rho = A_1(\varphi_1, \varphi_2)X_1 + A_2(\varphi_1, \varphi_2)X_2 + 3B_1(\varphi_1, \varphi_2)X_1^2 + 3B_2(\varphi_1, \varphi_2)X_2^2 + 3B_{12}(\varphi_1, \varphi_2)X_1X_2 + U_{\text{eff}}(\varphi_1, \tag{83}$$

and

$$p = A_1(\varphi_1, \varphi_2)X_1 + A_2(\varphi_1, \varphi_2)X_2 + B_1(\varphi_1, \varphi_2)X_1^2 + B_2(\varphi_1, \varphi_2)X_2^2 + B_{12}(\varphi_1, \varphi_2)X_1X_2 - U_{\text{eff}}(\varphi_1, \varphi_2) \tag{84}$$

Henceforth the dots indicate derivatives with respect to the time t and we have assumed that the scalar fields are homogeneous.

Using this more generic structure we showed that the non linear K essence allows for the existence of a non singular Emergent universe for the Early Universe. This phase, during an infinite time, is followed by an inflationary phase, which then decays first to an early dark energy phase and then to a late dark energy phase, in the late universe the the Dark Matter can be explained in terms of the Kessence terms for the scalar fields, displayed in the above equations.

12.2. Different roles for the three flat regions. Emphasis on the H0 Tension

13. Dynamical Generation of Inflation in Metric Formulation

Let us now consider a substantially truncated version of the model (46) without any matter fields, involving few non-Riemannian volume elements [28]

$$S = \int d^4x \Big\{ \Phi_1(A) \Big[R(g) - 2Ł_0 \frac{\Phi_1(A)}{\sqrt{-g}} \Big] + \Phi_2(B) \frac{\Phi_0(C)}{\sqrt{-g}} \Big\} \, , \tag{85}$$

where now unlike (46) $R(g) \equiv g^{\mu\nu} R_{\mu\nu}(\Gamma(g))$ is the scalar curvature in the second order (metric) formalism ($\Gamma^{ł\mu\nu}(g)$ being the Levi-Civita connection w.r.t. $g_{\mu\nu}$).

The equations of motion w.r.t. auxiliary tensor gauge fields $A_{\mu\nu ł}$, $\Phi_2(B)$ and $\Phi_1(C)$ are special cases of the dynamical constraint Eqs.(47)-(48) with all matter field terms being zero, which again introduce the three free integration constants $M_{1,2}$, χ_2.

Passage to the physical Einstein frame is again realized via the conformal transformation (50), however this time we have to use the well-known formulas for conformal transformations within the metric formalism ; bars indicate magnitudes in the $\bar{g}_{\mu\nu}$-frame) $\chi_1 \equiv \frac{\Phi_1(A)}{\sqrt{-g}}$.and re defining a new scalar field through χ_1 as $\chi_1 = \exp\big(u/\sqrt{3}\big)$ we find that we are able to write the Einstein-frame NRVF action in the form [28]:

$$S_{\rm EF} = \int d^4x \sqrt{-\bar{g}} \Big[R(\bar{g}) - \frac{1}{2} \bar{g}^{\mu\nu} \partial_\mu u \partial_\nu u - U_{\rm eff}(u) \Big] \, , \tag{86}$$

$$U_{\rm eff}(u) = 2Ł_0 - M_1 \exp\big(-\frac{u}{\sqrt{3}}\big) + \chi_2 M_2 \exp\big(-2\frac{u}{\sqrt{3}}\big) \, . \tag{87}$$

Thus, from the original pure-gravity NRVF action (85) we derived a physical Einstein-frame action (86)-(87) containing a *dynamically created* scalar field u with a non-trivial effective scalar potential $U_{\rm eff}(u)$ (87) entirely *dynamically generated* by the initial non-Riemannian volume elements in (85) because of the appearance of the free integration constants $M_{1,2}$, χ_2 in their respective equations of motion. There are two main features of the effective potential

(87) which are relevant for cosmological applications with the dynamically created field u as an "inflaton".

- $U_{\text{eff}}(u)$ (87) possesses one flat region for large positive values of u where $U_{\text{eff}}(u) \simeq 2Ł_0$, which corresponds to "early" universe' inflationary evolution with energy scale $2Ł_0$.
- $U_{\text{eff}}(u)$ (87) has a stable minimum for a small finite value $u = u_*$ where $e^{-u_*/\sqrt{3}} = M_1/(2\chi_2 M_2)$.
- The region around the stable minimum at $u = u_*$ correspond to "late" universe' evolution where the minimum value of the potential:

$$U_{\text{eff}}(u_*) = 2Ł_0 - \frac{M_1^2}{4\chi_2 M_2} \equiv 2Ł_{\text{DE}} \tag{88}$$

is the dark energy density value.

In Ref.[28] a thorough analysis has been performed of the slow-roll inflationary dynamics driven by the dynamically created "inflaton" u with its dynamically generated effective potential (87), including explicit calculation of the standard slow-roll parameters ϵ and η, as well we have obtained explicit expressions for the tensor-to-scalar ratio r and the scalar spectral index n_s of density perturbations as functions of the number of e-folds $\mathcal{N} = \log a$ (a being the Friedmann scale factor):

$$r \simeq \frac{12}{\left[\mathcal{N} + \frac{\sqrt{3}}{4} u_i(\mathcal{N}) + c_0\right]^2}, \quad n_s \simeq 1 - \frac{r}{4} - \sqrt{\frac{r}{3}}, \tag{89}$$

with $c_0 \equiv \frac{\sqrt{3}}{2} - \frac{3}{4}\log\Big(2\big(1 + 2/\sqrt{3}\big)\Big)$. $u_i(\mathcal{N})$ is the value of the "unflaton" at the start of inflation as function of $\mathcal{N}$.

For a plausible assumption about the scales of $M_{1,2}$, χ_2 and taking $\mathcal{N} = 60$ e-folds till end of inflation the observables are predicted to be: $n_s \approx 0.969$, $r \approx 0.0026$, which conform to the *PLANCK* constraints [22] ($0.95 < n_s < 0.97$, $r < 0.064$).

14. The construction of holomorphic gravity is possible through the use of a Non Riemannian Measure.

14.1. Non Holomorphic Structure of the Acion in General Relativity and similar theories with the Riemannian Measure

The action of GR, and other theories that use the standard Riemannian volume element $d^4x\sqrt{-g}$ is of the form,

$$S = \int d^4x\sqrt{-g}L \tag{90}$$

where L is a generally coordinate invariant lagrangian. Now notice the non holomorphic structure due to the appearance of $\sqrt{-g}$, that under a general coordinate transformation, even when holomorphic,

$$d^4x \rightarrow Jd^4x$$

, while $\sqrt{-g}$ needs to be defined, for example if J is real and negative, we can define

$$\sqrt{-g} \rightarrow \mid J \mid^{-1} \sqrt{-g}$$

where J is the jacobian of the transformation and $\mid J \mid$ is the absolute value of the transformation. Therefore $d^4x\sqrt{-g} \rightarrow \frac{J}{|J|}d^4x\sqrt{-g}$, so invariance is achieved only for $J = \mid J \mid$, that is if $J > 0$, that is signed general coordinate transformations are problematic, and ill defined. One could argue that when taking the square root of the determinant of the metric one may choose the negative solution when it suit us, but this would be an arbitrary procedure if no specific rule is given to choose the positive or the negative root. We choose instead to declare that $\sqrt{-g}$ is always positive and replace it in the measure by something else whose sign is well defined.

Recall that 0 is a branch point of the square root function. Suppose $w = \sqrt{z}$, and z starts at 4 and moves along a circle of radius 4 in the complex plane centered at 0. When we move in the complex circle from 4 we start with 2, but after we do the full circle we get -2, not 2. Obviously one of the definitions of the function will not leave the volume element invariant.

Of course we need to define a measure that will transform holomorphically , the Riemannian measure, which makes use of the square root is not acceptable. Of course complex coordinate transformations will be even more problematic than just real and with a negative jacobian.

14.2. Theory using Metric Independent non-Riemannian Volume-Forms, holomorphic cases

For another equation that will be invariant under holomorphic general coordinate invariant transformations, we must avoid $\sqrt{-g}$, such an equation which will be,

$$\frac{\Omega^2}{(-g)} \equiv \chi = K = \text{const.} \tag{91}$$

The resulting action that replaces (58) is,

$$S = \int d^4x\, \Omega \Big[R + L\Big] + \int d^4x\, \Omega^2 \Big[\frac{\Phi(H)}{(-g)}\Big]\,. \tag{92}$$

the density $\Phi(H)$ is defined from eq.

$$\Phi(H) = \frac{1}{3!}\varepsilon^{\mu\nu\kappa\lambda}\partial_\mu H_{\nu\kappa\lambda} \tag{93}$$

so the integration obtained from the variation of the H gauge field is eq. (91) now. The solution of eq. (91) are

$$\frac{\Omega}{\sqrt{(-g)}} = \pm\sqrt{K}. \tag{94}$$

where the sign in (94) will be dynamically determined, The measure Ω could have a small imaginary part. Of course this imaginary part can be set to zero by initial conditions, but that is not mandatory.

where the sign in (94) will be dynamically determined, The measure Ω could have a small imaginary part. Of course this imaginary part can be set to zero by initial conditions, but that is not mandatory.

Notice that although the theory has the complex holomophic invariance, a particular solution (which here means choosing between the plus or minus) does not have to be, although the space of all solution is holomorphic invariant. The restriction to some sign breaks the holomorphic invariance.

15. The determinant of the vierbein is a non invariant measure under signed Local Lorentz Transformations

Another possibility for a measure that would transform like the the jacobian of the coordinate transformation, not the absolute value of the jacobian, would be the determinant of the vierbein. This will destroy however (up to a sign) the invariance of the theory under signed local Lorentz transformation of the vierbeins. that is Local Lorentz transformations with negative determinants, so, it is not a solution, rather we trade one asymmetry for another.

15.1. The four scalars as integration manifold

Notice that using the volume element converts the the integration over coordinates in the action into integration over scalar fields, since

$$\Phi d^4 x = d\varphi_1 d\varphi_2 d\varphi_3 d\varphi_4$$

. The scalars are complex as the original coordinates. The mapping of the coordinates to the scalars may not be one to one. The scalar integration manifold existing in the

15.2. Gravitational Equations of motion

Here we review [54]. We start by considering the equation that results from the variation of the degrees of freedom that define the measure Ω, that is the scalar fields φ_a, these are,

$$A^{\mu}_{a}\partial_{\mu}(R+L+2\Omega\frac{\Phi(H)}{(-g)})=0 \tag{95}$$

Notice that the determinant of $A^{\mu a}$ is proportional to Ω^3, so if the measure is not vanishing, the matrix $A^{\mu a}$ is non singular and therefore $\partial_{\mu}(R+L+2\Omega\frac{\Phi(H)}{(-g)})=0$, so that,

$$R+L+2\Omega\frac{\Phi(H)}{(-g)}=M=constant \tag{96}$$

The variation with respect to the metric $g^{\mu\nu}$, we obtain.

$$\Omega(R_{\mu\nu}+\frac{\partial L}{\partial g^{\mu\nu}})+g_{\mu\nu}\Omega^2\frac{\Phi(H)}{(-g)}=0 \tag{97}$$

solving $\Omega\frac{\Phi(H)}{(-g)}$ from (96) and inserting into (97), we obtain,

$$R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R+\frac{1}{2}Mg_{\mu\nu}+\frac{\partial L}{\partial g^{\mu\nu}}-\frac{1}{2}g_{\mu\nu}L=0 \tag{98}$$

which gives exactly the form of Einstein equation with the canonical energy momentum defined from L

$$T_{\mu\nu}=g_{\mu\nu}L-2\frac{\partial}{\partial g^{\mu\nu}}L\,. \tag{99}$$

The equations of motion of the connection (in the first order formalism) implies that the connection is the Levi Civita connection. L can describe a scalar field with the potential and the term $\frac{1}{2}M$ can be interpreted as a shift of the scalar field potential by a constant or a floating contribution to the cosmological constant. Notice that there no way to introduce an explicitly a cosmological constant term in the action.

16. String Theories with dynamical tension and associated Braneworld scenarios

The standard world sheet string sigma-model action using a world sheet metric is [70], [71], [72]

$$S_{sigma-model} = -T \int d^2\sigma \frac{1}{2} \sqrt{-\gamma} \gamma^{ab} \partial_a X^\mu \partial_b X^\nu g_{\mu\nu}. \tag{100}$$

Here γ^{ab} is the intrinsic Riemannian metric on the 2-dimensional string worldsheet and $\gamma = det(\gamma_{ab})$; $g_{\mu\nu}$ denotes the Riemannian metric on the embedding spacetime. T is a string tension, a dimension full scale introduced into the theory by hand.

Now instead of using the measure $\sqrt{-\gamma}$, on the 2-dimensional world-sheet, in the framework of this theory two additional worldsheet scalar fields $\varphi^i\, (i = 1, 2)$ are considered. A new measure density is introduced:

$$\Phi(\varphi) = \frac{1}{2} \epsilon_{ij} \epsilon^{ab} \partial_a \varphi^i \partial_b \varphi^j. \tag{101}$$

There are no limitations on employing any other measure of integration different than $\sqrt{-\gamma}$. The only restriction is that it must be a density under arbitrary diffeomorphisms (reparametrizations) on the underlying spacetime manifold. Then the modified bosonic string action is (as formulated first in [55] and latter discussed and generalized also in [56])

$$S = -\int d^2\sigma\Phi(\varphi)(\frac{1}{2}\gamma^{ab}\partial_a X^\mu \partial_b X^\nu g_{\mu\nu} - \frac{\epsilon^{ab}}{2\sqrt{-\gamma}}F_{ab}(A)), \tag{102}$$

where F_{ab} is the field-strength of an auxiliary Abelian gauge field A_a: $F_{ab} = \partial_a A_b - \partial_b A_a$. To check that the new action is consistent with the sigma-model one, let us derive the equations of motion of the action (102). The variation with respect to φ^i leads to the following equations of motion:

$$\epsilon^{ab}\partial_b\varphi^i\partial_a(\gamma^{cd}\partial_c X^\mu \partial_d X^\nu g_{\mu\nu} - \frac{\epsilon^{cd}}{\sqrt{-\gamma}}F_{cd}) = 0. \tag{103}$$

since $det(\epsilon^{ab}\partial_b\varphi^i) = \Phi$, assuming a non degenerate case ($\Phi \neq 0$), we obtain,

$$\gamma^{cd}\partial_c X^\mu \partial_d X^\nu g_{\mu\nu} - \frac{\epsilon^{cd}}{\sqrt{-\gamma}}F_{cd} = M = const. \tag{104}$$

The equations of motion with respect to γ^{ab} are

$$T_{ab} = \partial_a X^\mu \partial_b X^\nu g_{\mu\nu} - \frac{1}{2}\gamma_{ab}\frac{\epsilon^{cd}}{\sqrt{-\gamma}}F_{cd} = 0. \tag{105}$$

One can see that these equations are the same as in the sigma-model formulation .

Taking the trace of (105) we get that $M = 0$. By solving $\frac{\epsilon^{cd}}{\sqrt{-\gamma}} F_{cd}$ from (104) (with $M = 0$) we obtain the standard string eqs. The emergence of the string tension is obtained by varying the action with respect to A_a:

$$\epsilon^{ab}\partial_b(\frac{\Phi(\varphi)}{\sqrt{-\gamma}}) = 0. \tag{106}$$

Then by integrating and comparing it with the standard action it is seen that

$$\frac{\Phi(\varphi)}{\sqrt{-\gamma}} = T. \tag{107}$$

That is how the string tension T is derived as a world sheet constant of integration opposite to the standard equation (100) where the tension is put ad hoc. Let us stress that the modified measure string theory action does not have any *ad hoc* fundamental scale parameters. associated with it. This can be generalized to incorporate super symmetry, see for example [56], [59], [58] , [60]. For other mechanisms for dynamical string tension generation from added string world sheet fields, see for example [68] and [69]. However the fact that this string tension generation is a world sheet effect and not a universal uniform string tension generation effect for all strings has not been sufficiently emphasized before. Notice that Each String in its own world sheet determines its own tension. Therefore the tension is not universal for all strings.

Introducing Background Fields including a New Background Field, The Tension Field

Schwinger [79], [80] had an important insight and understood that all the information concerning a field theory can be studied by understanding how it reacts to sources of different types. This has been discussed in the text book by Polchinski for example [74] . Then the target space metric and other external fields acquire dynamics which is enforced by the requirement of zero beta functions. However, in addition to the traditional background fields usually considered in conventional string theory, one may consider as well an additional scalar field that induces currents in the string world sheet and since the current

Introducing world sheet currents that couple to the internal gauge fields

If to the action of the string we add a coupling to a world-sheet current j^a, i.e. a term

$$S_{\text{current}} = \int d^2\sigma A_a j^a, \tag{108}$$

then the variation of the total action with respect to A_a gives

$$\epsilon^{ab}\partial_a\left(\frac{\Phi}{\sqrt{-\gamma}}\right) = j^b. \tag{109}$$

We thus see indeed that, in this case, the dynamical character of the brane is crucial here.

How a world sheet current can naturally be induced by a bulk scalar field, the Tension Field

Suppose that we have an external scalar field $\phi(x^\mu)$ defined in the bulk. From this field we can define the induced conserved world-sheet current

$$j^b = e\partial_\mu\phi\frac{\partial X^\mu}{\partial\sigma^a}\epsilon^{ab} \equiv e\partial_a\phi\epsilon^{ab}, \tag{110}$$

where e is some coupling constant. The interaction of this current with the world sheet gauge field is also invariant under local gauge transformations in the world sheet of the gauge fields $A_a \to A_a + \partial_a\lambda$.

For this case, (109) can be integrated to obtain

$$T = \frac{\Phi}{\sqrt{-\gamma}} = e\phi + T_i, \tag{111}$$

or equivalently

$$\Phi = \sqrt{-\gamma}(e\phi + T_i), \tag{112}$$

The constant of integration T_i may vary from one string to the other. Notice that the interaction is metric independent since the internal gauge field does not transform under the the conformal transformations. This interaction does not therefore spoil the world sheet conformal transformation invariance in the case the field ϕ does not transform under this transformation. One may interpret (112) as the result of integrating out classically (through integration of equations of motion) or quantum mechanically (by functional integration of the internal gauge field, respecting the boundary condition that characterizes the constant of integration T_i for a given string). Then replacing $\Phi = \sqrt{-\gamma}(e\phi + T_i)$ back into the remaining terms in the action gives a correct effective action for each string. Each string is going to be quantized with each one having a different T_i. The consequences of an independent quantization of many strings with different T_i covering the same region of space time will be studied in the next section.

The Tension field from World Sheet Quantum Conformal Invariance

The case of two different string tensions

If we have a scalar field coupled to a string or a brane in the way described in the sub section above, i.e. through the current induced by the scalar field in the extended object, according to eq. (112), so we have two sources for the variability of the tension when going from one string to the other: one is the integration constant T_i which varies from string to string and the other the local value of the scalar field, which produces also variations of the tension even within the string or brane world sheet. As we discussed in the previous section, we can incorporate the result of the tension as a function of scalar field ϕ, given as $e\phi + T_i$, for a string with the constant of integration T_i by defining the action that produces the correct equations of motion for such string, adding also other background fields, the anti symmetric two index field $A_{\mu\nu}$ that couples to $\epsilon^{ab}\partial_a X^\mu \partial_b X^\nu$ and the dilaton field φ .

$$S_i = -\int d^2\sigma (e\phi + T_i)\frac{1}{2}\sqrt{-\gamma}\gamma^{ab}\partial_a X^\mu \partial_b X^\nu g_{\mu\nu} + \int d^2\sigma A_{\mu\nu}\epsilon^{ab}\partial_a X^\mu \partial_b X^\nu + \int d^2\sigma \sqrt{-\gamma}\varphi R. \tag{113}$$

Notice that if we had just one string, or if all strings will have the same constant of integration $T_i = T_0$. We will take cases where the dilaton field is a constant or zero, and the antisymmetric two index tensor field is pure gauge or zero, then the demand of conformal invariance for $D = 26$ becomes the demand that all the metrics

$$g^i_{\mu\nu} = (e\phi + T_i)g_{\mu\nu} \tag{114}$$

will satisfy simultaneously the vacuum Einstein´s equations. The interesting case to consider is when there are many strings with different T_i, let us consider the simplest case of two strings, labeled 1 and 2 with $T_1 \neq T_2$, then we will have two Einstein´s equations, for $g^1_{\mu\nu} = (e\phi + T_1)g_{\mu\nu}$ and for $g^2_{\mu\nu} = (e\phi + T_2)g_{\mu\nu}$,

$$R_{\mu\nu}(g^1_{\alpha\beta}) = 0 \tag{115}$$

and , at the same time,

$$R_{\mu\nu}(g^2_{\alpha\beta}) = 0 \tag{116}$$

These two simultaneous conditions above impose a constraint on the tension field ϕ, because the metrics $g^1_{\alpha\beta}$ and $g^2_{\alpha\beta}$ are conformally related, but Einstein´s equations are not conformally invariant, so the condition that Einstein´s equations hold for both $g^1_{\alpha\beta}$ and $g^2_{\alpha\beta}$ is highly non trivial. Then for these situations, we have,

$$e\phi + T_1 = \Omega^2(e\phi + T_2) \tag{117}$$

which leads to a solution for $e\phi$

$$e\phi = \frac{\Omega^2 T_2 - T_1}{1 - \Omega^2} \tag{118}$$

which leads to the tensions of the different strings to be

$$e\phi + T_1 = \frac{\Omega^2(T_2 - T_1)}{1 - \Omega^2} \tag{119}$$

and

$$e\phi + T_2 = \frac{(T_2 - T_1)}{1 - \Omega^2} \tag{120}$$

Both tensions can be taken as positive if $T_2 - T_1$ is positive and Ω^2 is also positive and less than 1.

16.0.1. Flat space in Minkowski coordinates and flat space after a special conformal transformation

The flat spacetime in Minkowski coordinates is,

$$ds_1^2 = \eta_{\alpha\beta} dx^\alpha dx^\beta \tag{121}$$

where $\eta_{\alpha\beta}$ is the standard Minkowski metric, with $\eta_{00} = 1$, $\eta_{0i} = 0$ and $\eta_{ij} = -\delta_{ij}$.

We now consider the conformally transformed metric

$$ds_2^2 = \Omega(x)^2 \eta_{\alpha\beta} dx^\alpha dx^\beta \tag{122}$$

where conformal factor coincides with that obtained from the special conformal transformation

$$x\prime^\mu = \frac{(x^\mu + a^\mu x^2)}{(1 + 2a_\nu x^\nu + a^2 x^2)} \tag{123}$$

for a certain D vector a_ν. which gives $\Omega^2 = \frac{1}{(1+2a_\mu x^\mu + a^2 x^2)^2}$ In summary, we have two solutions for the Einstein´s equations, $g^1_{\alpha\beta} = \eta_{\alpha\beta}$ and

$$g^2_{\alpha\beta} = \Omega^2 \eta_{\alpha\beta} = \frac{1}{(1 + 2a_\mu x^\mu + a^2 x^2)^2} \eta_{\alpha\beta} \tag{124}$$

We can then study the evolution of the tensions using $\Omega^2 = \frac{1}{(1+2a_\mu x^\mu + a^2 x^2)^2}$. We will consider the cases where $a^2 \neq 0$.

The homogeneous and isotropic Universe in Dynamical String Tension Theories

We now consider the case when a^μ is not light like and we will find that for $a^2 \neq 0$, irrespective of sign, i.e. irrespective of whether a^μ is space like or time like, we will have thick Braneworlds where strings can be constrained between two concentric spherically symmetric bouncing higher dimensional spheres and where the distance between these two concentric spherically symmetric bouncing higher dimensional spheres approaches zero at large times. The string tensions of the strings one and two are given by

$$e\phi + T_1 = \frac{(T_2 - T_1)(1 + 2a_\mu x^\mu + a^2x^2)^2}{(1 + 2a_\mu x^\mu + a^2x^2)^2 - 1} = \frac{(T_2 - T_1)(1 + 2a_\mu x^\mu + a^2x^2)^2}{(2a_\mu x^\mu + a^2x^2)(2 + 2a_\mu x^\mu + a^2x^2)} \tag{125}$$

$$e\phi + T_2 = \frac{(T_2 - T_1)}{(1 + 2a_\mu x^\mu + a^2x^2)^2 - 1} = \frac{(T_2 - T_1)}{(2a_\mu x^\mu + a^2x^2)(2 + 2a_\mu x^\mu + a^2x^2)} \tag{126}$$

Let us by consider the case where a^μ is time like, then without loosing generality we can take $a^\mu = (A, 0, 0, ..., 0)$. Now, in order to get homogeneous and isotropic cosmological solutions we must consider the limit $A \to 0$ and $(T_2 - T_1) \to 0$, in such a way that $\frac{(T_2 - T_1)}{A} = K$, where K is a constant. In that case the spatial dependence in the tensions (125) and (126) drops out and we get,

$$e\phi + T_1 = e\phi + T_2 = \frac{K}{4t} \tag{127}$$

The embedding metric can now be solved.

$$g_{\mu\nu} = \frac{1}{(e\phi + T_1)} g^1_{\mu\nu} = \frac{4t}{K}\eta_{\mu\nu} \tag{128}$$

which is not a vacuum metric, as opposed to $\eta_{\mu\nu}$ because of the conformal factor $\frac{4t}{K}$.

One should notice that the homogeneous and isotropic solution has been obtained only in the limit $A \to 0$ and $(T_2 - T_1) \to 0$, in such a way that $\frac{(T_2-T_1)}{A} = K$, where K is a constant. If A and $T_2 - T_1$ are small but finite, then for large times, of the order of $1/A$. We can formulate this as an uncertainty principle,

$$(T_2 - T_1)\Delta t \approx constant \tag{129}$$

where we have used that A is of the order of $(T_2 - T_1)$. So a small uncertainty in the tension $(T_2 - T_1)$ leads to a long lived homogeneous and isotropic phase, while a big uncertainty in the tension $(T_2 - T_1)$ leads to short lived homogeneous and isotropic phase.

In fact in these situations, for finite $(T_2 - T_1)$ and A, it is the case that the string tensions can only change sign by going first to infinity and then come back from minus infinity. We can now recognize at those large times the locations where the string tensions go to infinity, which are determined by the conditions

$$2a_\mu x^\mu + a^2 x^2 = 0 \tag{130}$$

or

$$2 + 2a_\mu x^\mu + a^2 x^2 = 0 \tag{131}$$

Let us start by considering the case where a^μ is time like, then without loosing generality we can take $a^\mu = (A, 0, 0, ..., 0)$. In this case the denominator in (125) , (126) is

$$(2a_\mu x^\mu + a^2x^2)(2 + 2a_\mu x^\mu + a^2x^2) = (2At + A^2(t^2 - x^2))(2 + 2At + A^2(t^2 - x^2)) \quad (132)$$

The condition (130), if $A \neq 0$ implies then that

$$x_1^2 + x_2^2 + x_3^2..... + x_{D-1}^2 - (t + \frac{1}{A})^2 = -\frac{1}{A^2} \quad (133)$$

if $A \to 0$, it is more convenient to write this in the form

$$A(x_1^2 + x_2^2 + x_3^2..... + x_{D-1}^2) - At^2 - 2t = 0 \quad (134)$$

which for the limit $A \to 0$ gives us the single singular point $t = 0$, which is the origin of the homogeneous and isotropic cosmological solution.

The other boundary of infinite string tensions is, (131) is given by,

$$x_1^2 + x_2^2 + x_3^2..... + x_{D-1}^2 - (t + \frac{1}{A})^2 = \frac{1}{A^2} \quad (135)$$

This has no limit for $A \to 0$, all these points disappear from the physical space (they go to infinity).

This has no limit for $A \to 0$, all these points disappear from the physical space (they go to infinity).

For $A \neq 0$ we see that (135) represents an exterior boundary which has an bouncing motion with a minimum radius $\frac{1}{A}$ at $t = -\frac{1}{A}$, The denominator (132) is positive between these two bubbles. So for $T_2 - T_1$ positive the tensions are positive and diverge at the boundaries defined above.

The internal boundary (133) exists only for times t smaller than $-\frac{2}{A}$ and bigger than 0, so in the time interval $(-\frac{2}{A}, 0)$ there is no inner surface of infinite tension strings. This inner surface collapses to zero radius at $t = -\frac{2}{A}$ and emerges again from zero radius at $t = 0$.

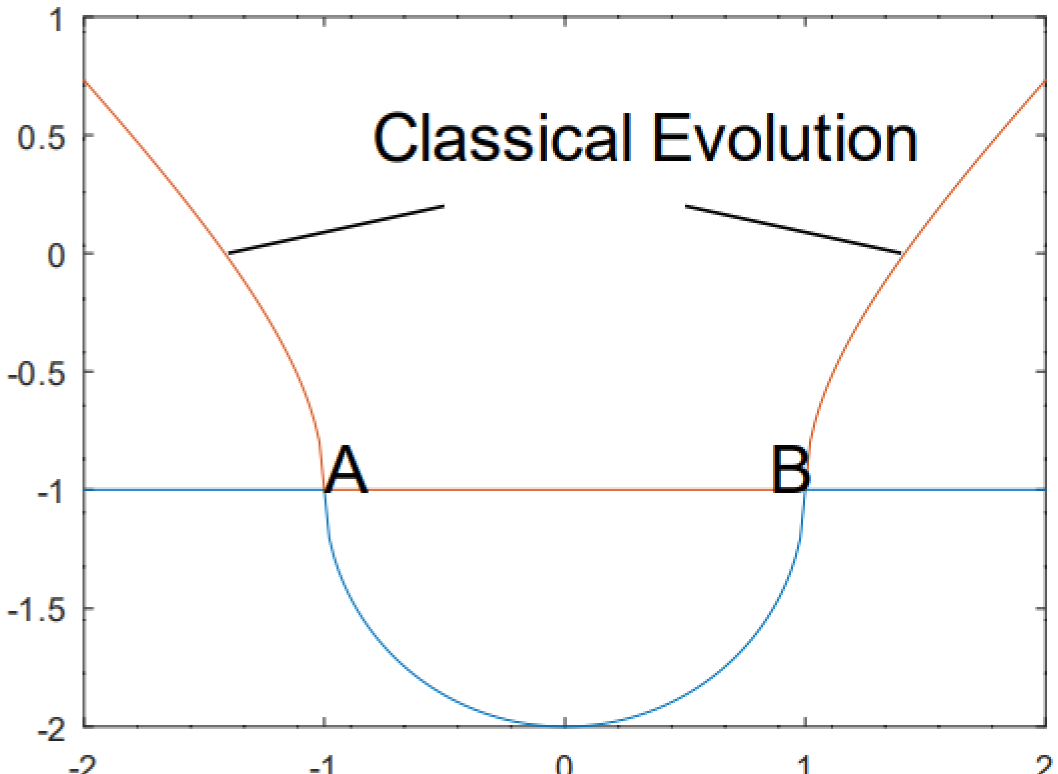
For large positive or negative times, the difference between the upper radius and the lower radius goes to zero as $t \to \infty$

$$\sqrt{\frac{1}{A^2} + (t + \frac{1}{A})^2} - \sqrt{-\frac{1}{A^2} + (t + \frac{1}{A})^2} \to \frac{1}{tA^2} \to 0 \tag{136}$$

of course the same holds $t \to -\infty$. This means that for very large early or late times the segment where the strings would be confined (since they will avoid having infinite tension) will be very narrow and the resulting scenario will be that of a brane world for late or early times, while in the bouncing region the inner surface does not exist. Notice that this kind of braneworld scenario is very different to the ones previously studied, in particular both gravity (closed strings) and gauge fields (open strings) are treated on the same footing, since the mechanism that confines the strings between the two surfaces relies only on the string tension becoming very big.

Quantum creation, by analitically continuing the time to imaginary time, we could match the2 branches of the classical evolution in

$$x_1^2 + x_2^2 + x_3^2 \ldots + x_{D-1}^2 - (t + \frac{1}{A})^2 = \frac{1}{A^2}$$



19. Possible Applications for the construction of a Hagedorn Temperature free String Model of the Strong interactions.

In [83], Andreev has discussed the need to avoid the Hagedorn temperature [82] in order to obtain a behavior more in accordance to that of QCD, since in the real world there is no phase transition but an analytic crossover. If strings are indeed relevant for QCD then one has to show that a stringy description is also valid for high T and he has shown that this can still be achieved but in the context of string models, but then these string models have to be multi tension string models, so it appears that our ideas and those of Andreev go in the same direction.

The approach by Andreev is more phenomenological than ours and he introduces strings with different tensions and then gives a prescription concerning the string tensions that one should be allowed to contribute at a given temperature. Such type of prescriptions will probably not be necessary in our approach, no need to eliminate states with certain string tensions depending of the temperature, instead we would rely on the the dynamical effect where the tensions grow and suppress the Hagedorn temperature. Much work in this direction is needed.

24. Can Dynamical Tension String Theory recover the Swampland?

The standard string theory is argued generates a space of acceptable theories and a ¨swampland¨ a space of theories that cannot be correct [87].

In a general setting, there are a few statements made where the Planck scale appears [88]:

1. distance conjecture: the statement into the requirement that trans-Planckian excursions can not be allowed for any fields present in the cosmological evolution.

$$\Delta\phi / M_P < O(1)$$

with M_P being the reduced Planck mass

2, Due to the difficulties of consistently constructing the meta-stable de-Sitter vacua at the heart of cosmology it has been further proposed a requirement on possible field potentials of theories in the Landscape [88], given by either

$$M_P \frac{dV/d\phi}{V} > O(1) \qquad\qquad -M_P^2 \frac{d^2V/d\phi^2}{V} > O(1)$$

But in dynamical string tension implies a dynamical Planck scale that can go to infinity

- At the points in space time where the dynamical Planck scale goes to infinity the Swampland constraints dissapear
- Tensions going to infinity represent target scale invariant states, so it is reasonable they will appear in some phases of the evolution of the universe.
- Other ideas concerning Dark Matter, Dark Energy .

When talking to Professor Maldacena, on strings with different tensions he observed that ordinary string interactions decouple for strings with different tension, which lead me to the idea that there are strings that constitute the visible matter and then strings with a different tension are the Dark Matter, Furthermore, both the visible matter and the dark strings share the same space time and in particular the same compactification, and it is the compactification what is organizing the particle structure, since for the visible sector, we obtain the Standard Model, then the Dark Strings should give rise to dark copies of the Standard Model.

Strings with a different tension as dark matter E.I. Guendelman Eur.Phys.J.C 85 (2025) 6, 671 and Addendum that discusses the idea that the dark matter must be copies of the standard model

Strings with a different tension producing dark copies of the Standard Model. Guendelman, E.I. , *Eur. Phys. J. C* **85**, 1079 (2025). https://doi.org/10.1140/epjc/s10052-025-14777-8 see comments

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More comments on social media of Guendelman, E.I. Dynamical string tension theories with target space scale invariance SSB and restoration. *Eur. Phys. J. C* **85**, 276 (2025). https://doi.org/10.1140/epjc/s10052-025-13966-9

- **Out of the string theory swampland**
- EurekAlert!
- **Escaping String Theory Swampland**
- Mirage News
- **Emerging from the String Theory Swampland: A Breakthrough Discovery**
- ScienMag

Emerging from the String Theory Swampland: A Breakthrough Discovery
Bioengineer.org
Stringtheorie steckt im Sumpf fest: Neuer Ansatz wagt den Ausbruch
Ingenieur.de

String Theory's "Swampland" Problem May Have a Solution todayheadline
Today Headline
Out of the string theory swampland: New models may resolve problem that conflicts with dark energy
Phys.org……etc., etc.

Other issues related to non commutativity in dynamical string tension: the string tension

determines a non Commutativity parameter of the coordinates that in terms determines a minimum length, this non commutativity parameter is now local, so the mínimum length and also the zero point enegy density becomes local. ISSUES BEING EXPLORED NOW WITH DOUGLAS SINGLETON.

THANK YOU FOR YOUR ATTENTION