Continuum Limit of Causal Fermion Systems

Conference on Causal Fermion Systems, 06 Oct 2025

Motivation

Goal:

Derivation of the Standard Model from the Causal Action Principle

Main references:

F. Finster, *The Continuum Limit of Causal Fermion Systems* (‘blue book’)

PF, F. Finster, *Construction of Currents in Causal Fermion Systems* (arXiv:2507.09633)

Motivation

Minkowski space

A critical point of the causal action

Perturbations

Causal Action Principle

$\Rightarrow$ Dynamical equations for the perturbations

Variational form of the Equations

$\Rightarrow$ Standard Model as effective Lagrangian

Recap of the Fundamental Objects

Local Correlation Operators $x \in M$

Physical Wave function $\psi^u(x) := \pi_x |u\rangle, \psi^u : M \to SM$ for $|u\rangle \in H$

Wave Evaluation Operator $\Psi(x) := \pi_x : \mathcal{H} \to S_xM \Rightarrow \Psi^* = -x|_{S_xM} : S_xM \to \mathcal{H}$.

Kernel of the Fermionic Projector $P(x,y) := \Psi(x)\Psi^*(y) : S_yM \to S_xM$

Closed Chain $A_{xy} := P(x,y)P(y,x) : S_xM \to S_xM$.

Recap of the Fundamental Objects

Local Correlation Operators $x \in M$

Physical Wave function $\psi^u(x) := \pi_x |u\rangle, \psi^u : M \to SM$ for $|u\rangle \in H$

Wave Evaluation Operator $\Psi(x) := \pi_x : \mathcal{H} \to S_xM \Rightarrow \Psi^* = -x|_{S_xM} : S_xM \to \mathcal{H}$.

Kernel of the Fermionic Projector $P(x,y) := \Psi(x)\Psi^*(y) : S_yM \to S_xM$

Closed Chain $A_{xy} := P(x,y)P(y,x) : S_xM \to S_xM$.

A_{xy} has the same, non-vanishing eigenvalues as xy,

CFS description in terms of $P(x,y)$

Restricted Euler-Lagrange Equation

Restrictred Euler-Lagrange Equation

EL equation:

$$l(x) := \int_M \mathcal{L}(x, y)\,\mathrm{d}\rho(y) - \mathfrak{r}\,\mathbf{tr}\,x - \mathfrak{s}$$

is minimal on M.

Restrictred Euler-Lagrange Equation

Restricted Euler-Lagrange equation $Dl(x) = 0$ for $x \in M$.

Restricted Euler-Lagrange equation $Dl(x) = 0$ for $x \in M$.

Variation $x \to x + \delta x$, equivalently $\Psi \to \Psi + \delta\Psi$

$$\begin{aligned}
\delta P(x,y) &= \delta\Psi(x)\Psi^*(y) \\
\delta P(y,x) &= \Psi(y)\delta\Psi^*(x) \\
\delta A_{xy} &= P(x,y)\delta P(y,x) + \delta P(x,y)P(y,x)
\end{aligned}$$

Restrictred Euler-Lagrange Equation

Restricted Euler-Lagrange equation $Dl(x) = 0$ for $x \in M$.

Variation $x \to x + \delta x$, equivalently $\Psi \to \Psi + \delta\Psi$

$$\begin{aligned} \delta P(x,y) &= \delta\Psi(x)\Psi^*(y) \\ \delta P(y,x) &= \Psi(y)\delta\Psi^*(x) \\ \delta A_{xy} &= P(x,y)\delta P(y,x) + \delta P(x,y)P(y,x) \end{aligned}$$

$$\Rightarrow \delta\mathcal{L}(x,y) = \operatorname{tr}_{S_xM}\left[\frac{\delta\mathcal{L}(x,y)}{\delta A_{xy}}\delta A_{xy}\right] = 2\operatorname{Re}\operatorname{tr}_{S_xM}\left[Q(x,y)\delta P(y,x)\right]$$

$$\delta \operatorname{tr} x = \delta\left(\Psi^*(x)\Psi(x)\right) = 2\operatorname{Re}\operatorname{tr}_{S_xM}\left[\Psi(x)\delta\Psi^*(x)\right]$$

Restrictred Euler-Lagrange Equation

$$
\begin{aligned}
l(x) &= \int_M \mathcal{L}(x,y)\,\mathrm{d}\rho(y) - \mathfrak{r}\,\mathrm{tr}\,x - \mathfrak{s} \\
Dl(x) &= 0
\end{aligned}
$$

Restrictred Euler-Lagrange Equation

$$
\begin{aligned}
l(x) &= \int_M \mathcal{L}(x,y)\mathrm{d}\rho(y) - \mathfrak{r}\,\mathrm{tr}\,x - \mathfrak{s} \\
Dl(x) &= 0
\end{aligned}
$$

$$
\delta l(x) = 2\,\mathrm{Re}\,\mathrm{tr}_{S_xM}\left[\left(\int Q(x,y)\Psi(y)\,\mathrm{d}\rho(y) - \mathfrak{r}\Psi(x)\right)\delta\Psi^*(x)\right]
$$

Restrictred Euler-Lagrange Equation

$$\begin{aligned} l(x) &= \int_M \mathcal{L}(x,y)\,\mathrm{d}\rho(y) - \mathfrak{r}\,\mathrm{tr}\,x - \mathfrak{s} \\ Dl(x) &= 0 \end{aligned}$$

$$\delta l(x) = 2\,\mathrm{Re}\,\mathrm{tr}_{S_xM}\left[\left(\int Q(x,y)\Psi(y)\,\mathrm{d}\rho(y) - \mathfrak{r}\Psi(x)\right)\delta\Psi^*(x)\right]$$

$$\Rightarrow \int_M Q(x,y)\Psi(y)\,\mathrm{d}\rho(y) = \mathfrak{r}\Psi(x) \quad \text{for all } x \in M$$

Minkowski Space

Minkowski Space

$\mathcal{H}$ as the negative energy solutions

$\prec \cdot | \cdot \succ_x$ Dirac inner product, $S_x M = \mathbb{C}^4$

Identification $F : \mathbb{R}^4 \to \mathcal{F}$

Minkowski Space

$\mathcal{H}$ as the negative energy solutions

$\prec \cdot | \cdot \succ_x$ Dirac inner product, $S_x M = \mathbb{C}^4$

Identification $F : \mathbb{R}^4 \to \mathcal{F}$

Measure $\rho = F^* \lambda$ (λ Lebesgue measure on $\mathbb{R}^4$)

Spacetime $M = \operatorname{supp} \rho = F(\operatorname{supp} \lambda) = F(\mathbb{R}^4)$

Physical wave function $\psi^u(x) = u(x)$

Minkowski Space

$$P(x,y) = \int_{\mathbb{R}^4} \frac{\mathrm{d}^4 k}{(2\pi)^4} (\gamma^\mu k_\mu + m)\delta(k^2 - m^2)\Theta(-k_0) e^{ik_\mu (y-x)}$$
$$= \left(i\gamma^\mu \partial_\mu^{(x)} + m \right) T(x,y)$$

Minkowski Space

$$
\begin{aligned}
P(x,y) &= \int_{\mathbb{R}^4} \frac{\mathrm{d}^4 k}{(2\pi)^4} (\gamma^\mu k_\mu + m)\delta(k^2 - m^2)\Theta(-k_0) e^{ik_\mu(y-x)} \\
&= \left(i\gamma^\mu \partial_\mu^{(x)} + m \right) T(x,y)
\end{aligned}
$$

By construction, $P(x,y)$ is bi-solution (negative frequency Wightman function)

$$
\left(i\gamma^\mu \partial_\mu^{(x)} - m \right) P(x,y) = 0
$$

Singular Structure

$P(x,y)$ is singular for $\xi^2 = 0, \xi := y - x$

$$T(x,y) = \frac{U(x,y)}{\xi^2} + \cdots = T^{(0)}(x,y), \quad T^{(n)}(x,y) := \frac{\partial^n}{\partial(m^2)^n} T(x,y)$$

$$P(x,y) = i\gamma^\mu \xi_\mu \frac{2U(x,y)}{\xi^4} + \cdots = \frac{i}{2}\gamma^\mu \xi_\mu T^{(-1)}(x,y) + mT^{(0)}(x,y)$$

Singular Structure and Perturbations

$$P(x,y) \text{ is singular for } \xi^2 = 0, \xi := y - x$$

$$T(x,y) = \frac{U(x,y)}{\xi^2} + \cdots = T^{(0)}(x,y), \quad T^{(n)}(x,y) := \frac{\partial^n}{\partial(m^2)^n} T(x,y)$$

$$P(x,y) = i\gamma^\mu \xi_\mu \frac{2U(x,y)}{\xi^4} + \cdots = \frac{i}{2}\gamma^\mu \xi_\mu T^{(-1)}(x,y) + mT^{(0)}(x,y)$$

Regularisation, e.g. $\xi^0 =: t \to t - i\varepsilon$, momentum cutoff

$$Q = 0$$

when evaluated weakly on the light-cone in the limit $\varepsilon \to 0$.

$\Rightarrow$ Minkowski space is a critical point of the causal action for $\mathfrak{r} = 0$.

Perturbations

Perturbations - General

Perturbations:

$$(i\gamma^\mu \partial_\mu^{(x)} + \mathcal{B} - m)\tilde{P}(x, y) = 0$$

with $\mathcal{B}$ linear, symmetric (w.r.t spin inner product) operator.

Perturbations:

$$(i\gamma^\mu \partial_\mu^{(x)} + \mathcal{B} - m)\tilde{P}(x,y) = 0$$

with $\mathcal{B}$ linear, symmetric (w.r.t spin inner product) operator.

- Differential operator
- Integral operator $\mathcal{B}\tilde{P}(x,y) = \int \mathrm{d}^4 z\, B(x,z)\tilde{P}(z,y)$
- Local vector potential $\mathcal{B}\tilde{P}(x,y) = \gamma^\mu A_\mu(x)\tilde{P}(x,y)$

Vector Potential:

$$\left(i\gamma^\mu \partial_\mu^{(x)} + \gamma^\mu A_\mu(x) - m\right) \tilde{P}(x,y) = 0$$

Vector Potential:

$$\left(i\gamma^\mu \partial_\mu^{(x)} + \gamma^\mu A_\mu(x) - m\right) \tilde{P}(x,y) = 0$$

Solve iteratively in orders of A:

$$\left(i\gamma^\mu \partial_\mu^{(x)} - m\right) P^{(0)}(x,y) = 0 \quad \text{(Vacuum)}$$
$$\left(i\gamma^\mu \partial_\mu^{(x)} - m\right) P^{(n)}(x,y) = -\gamma^\mu A_\mu(x) P^{(n-1)}(x,y)$$

Perturbations - Vector Potential

$$\tilde{P}(x, y) = P^{(0)}(x, y) + P^{(1)}(x, y) + \mathcal{O}(A^2)$$

Perturbations - Vector Potential

$$
\begin{aligned}
\tilde{P}(x,y) = &\frac{i}{2}\gamma^\mu \xi_\mu T^{(-1)}(x,y) \\
&+ \frac{1}{2}\gamma^\mu \xi_\mu \xi^\nu \int_0^1 A_\nu(x+\tau\xi)\,\mathrm{d}\tau\, T^{(-1)}(x,y) \\
&+ \frac{1}{2}\gamma^\mu \xi_\mu \xi^\nu \int_0^1 (\tau-\tau^2)(\partial^\sigma F_{\sigma\nu})(x+\tau\xi)\,\mathrm{d}\tau\, T^{(0)}(x,y) \\
&- \frac{1}{4}\gamma^\mu \xi_\mu \gamma^\nu \gamma^\sigma \int_0^1 F_{\sigma\nu}(x+\tau\xi)\,\mathrm{d}\tau\, T^{(0)}(x,y) \\
&- \xi^\mu \gamma^\nu \int_0^1 (1-\tau) F_{\mu\nu}(x+\tau\xi)\,\mathrm{d}\tau\, T^{(0)}(x,y) \\
&+ \gamma^\mu \int_0^1 (1-\tau)^2 (\partial^\nu F_{\nu\mu})(x+\tau\xi)\,\mathrm{d}\tau\, T^{(1)}(x,y) + \ldots,
\end{aligned}
$$

Perturbations - Vector Potential

$$\tilde{P}(x,y) \to \tilde{A}_{xy} \to \tilde{\lambda}_i^{xy} \to \tilde{Q}(x,y)$$

Perturbations - Vector Potential

$$\tilde{P}(x,y) \to \tilde{A}_{xy} \to \tilde{\lambda}_i^{xy} \to \tilde{Q}(x,y)$$

$$0 \overset{!}{=} \tilde{Q}(x,y) \propto \xi^\mu j_\mu(x) \Rightarrow j_\mu(x) = 0$$

with $j_\mu(x) = \partial^\nu F_{\nu\mu}(x)$ and $F = \mathrm{d}A$

Perturbations - Vector Potential

Causal action principle $\Rightarrow 0 = j_\mu(x) = \partial^\nu F_{\nu\mu}(x)$

Perturbations - Vector Potential

Causal action principle $\Rightarrow 0 = j_\mu(x) = \partial^\nu F_{\nu\mu}(x)$

$$\mathcal{L}_{\text{eff}}(\mathrm{d}A, A) \propto F_{\mu\nu}F^{\mu\nu}$$

Perturbations - Particles

$$\tilde{P}(x,y) = P^{(0)}(x,y) + \frac{1}{2\pi} v(x)\overline{v(y)}$$

where v is a positive-energy solution of the Dirac equation

Perturbations - Particles

$$\tilde{P}(x,y) = P^{(0)}(x,y) + \frac{1}{2\pi} v(x)\overline{v(y)}$$

where v is a positive-energy solution of the Dirac equation

$$0 \stackrel{!}{=} \tilde{Q}(x,y) \propto \xi^\mu \left(j_\mu(x) - c\, \overline{v(x)} \gamma_\mu v(x) \right)$$

Multiple Fermion Sectors, e.g

$$P(x,y) = P^{(e)}(x,y) \oplus P^{(\nu)}(x,y)$$

- Perturbation mixes different sectors
- Non-abelian terms at $\mathcal{O}(A^2)$

8 fermionic sectors ($U(8)$)

Perturbations - Non-Abelian Gauge Fields

8 fermionic sectors ($U(8)$)

Break Left-Right Symmetry
$\rightarrow SU(2)$-Block Formation

Perturbations - Non-Abelian Gauge Fields

8 fermionic sectors ($U(8)$)

Break Left-Right Symmetry
$\rightarrow SU(2)$-Block Formation

Strong interaction by remaining $SU(3)$

Perturbations - Gravity

$$\tilde{Q}(x,y) = \xi^\mu \xi^\nu \left(\frac{const}{\varepsilon^2} R_{\mu\nu} - T_{\mu\nu} \right)$$

Couples with $\varepsilon^2 \propto G \Rightarrow$ Gravity is a higher order effect (Weakness of gravity)

Perturbations - Additional Conditions

Three fermion generations

One uncharged fermionic sector (for two fermion Species + Photon)