Higher Codimensions Positive Energy Theorem

Working Seminar "Mathematical Physics" in the Summer Term 2025

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Structure of the talk

Geometric Set up

Spin Geometry for People in a Hurry

The Positive Energy Theorem

What Happen in Higher Codimensions?

The Positive Energy Theorem

Conjecture: *Positive Energy Theorem for Initial Data Set*, Schoen and Yau '79

Let (M^n, g, k) be an asymptotically large euclidean initial data set sitting inside some spacetime $(\mathcal{M}, g)$ satisfies the Dominant Energy Condition. Then $E \geq |P|_g$. If the ADM mass of one end is zero. Then $(\mathcal{M}, g)$ is flat (M^n, g, k) along.

The Positive Energy Theorem

Theorem: *Positive Energy Theorem for Initial Data Set for Spin Manifolds*,Witten, Parker & Taubes

Let (M^n, g, k) be a **Spin** asymptotically large euclidean initial data set sitting inside some spacetime $(\mathcal{M}, g)$ satisfies the Dominant Energy Condition. Then $E \geq |P|_g$. If the ADM mass of one end is zero. Then $(\mathcal{M}, g)$ is flat (M^n, g, k) along.

Why Generalize the Positive Energy

- The classical Positive Energy Theorem (PET) applies to **hypersurfaces**, i.e., spacelike submanifolds of codimension one.

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- The classical Positive Energy Theorem (PET) applies to **hypersurfaces**, i.e., spacelike submanifolds of codimension one.
- Can we extend the PET to initial data sets (M^r, g, k) **immersed** in a semi-Riemannian manifold $(N^{r+z}, \bar{g})$ with $z > 1$?

Why Generalize the Positive Energy

- The classical Positive Energy Theorem (PET) applies to **hypersurfaces**, i.e., spacelike submanifolds of codimension one.
- Can we extend the PET to initial data sets (M^r, g, k) **immersed** in a semi-Riemannian manifold $(N^{r+z}, \bar{g})$ with $z > 1$?
- What kind of geometric and analytical conditions must be imposed in this more general setting?

Goal of this talk

To present a generalization of the Positive Energy Theorem to higher codimension, following the spinorial approach of Witten, Hijazi, Zhang, and Daguang without assuming the condition on the parity of the codimension.

Geometric Set up

Asymptotically large Euclidean

Let $n \geq 3$. A Riemannian manifold (M^n, g) is said to be **asymptotically large Euclidean**(*ALE*) if there is a bounded set K such that $M \setminus K$ is a finite union of ends, $M \setminus K = \cup_{i=1}^{k} M_i$. Such that for each end M_l there exists a diffeomorphism, asymptotic charts,

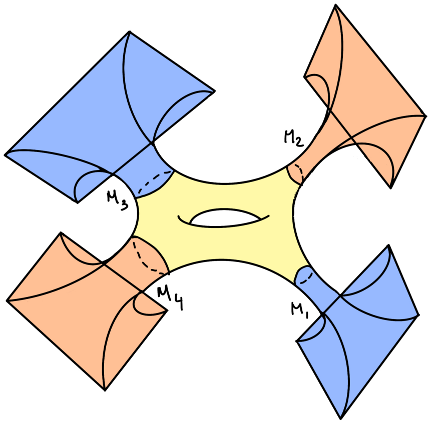
$$\Phi_l : M_l \mapsto \mathbb{R}^n \setminus \overline{B_1(O)}, \tag{1}$$

Then it will have coordinates $x_1, \ldots, x_n$ where

$$g_{ij}(x) = \delta_{ij} + \mathcal{O}_2(||x||^{-q}) \tag{2}$$

where $q \geq \frac{n-2}{2}$. Moreover, the scalar curvature is integrable $R_g \in L^1(M, g)$.

Asymptotically large Euclidean



Ilustration of an asymptotic large Euclidean manifold with multiple ends.

Initial data set

Let M^r be a smooth manifold.

- An **initial data set** ($I.D.S.$) on M is a pair (g, k) where g is a Riemannian metric on M and where k is a symmetric (0,2)-tensor, $k \in \Gamma(Sym^2(T^*M) \otimes \mathbb{R})$. Let $\mathcal{I}(M, \mathbb{R})$ be set of initial data sets for the line bundle.

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  - The *energy density* $\mu = R_g - |k|_g^2 + tr_g(k)^2$.
  - The *current density* $J := (div_g k)^\sharp - \nabla(tr_g k)$.

  If $\mu \geq |J|_g$ then we say that (M, g, k) satisfies the **Dominant Energy Condition** $(D.E.C)$.

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  If $\mu \geq |J|_g$ then we say that (M,g,k) satisfies the **Dominant Energy Condition** $(D.E.C)$.
- If $k \equiv 0$ then we have that $\mu = R_g \geq 0$.

Initial data set

Let $(\mathcal{M}, \mathbf{g})$ be a Lorentzian manifold. We say that $(M^n, g, k) \in \mathcal{I}(M, \mathbb{R})$ **sits** in $(\mathcal{M}^{n+1}, \mathbf{g})$

- The immersion $\iota : M \hookrightarrow \mathcal{M}$ is an isometric embedding with $T\mathcal{M}|_M = TM \oplus \mathbb{R}$
- The symmetric two tensor k is the second fundamental form of the embedding.

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Example

- Let $(\mathbb{R}^3, g_{\mathbb{E}}, 0) \in \mathcal{I}(M, \mathbb{R})$ and let $(\mathbb{M}, g_{\mathbb{M}})$. Then we can just consider $(\mathbb{R}^3, g_{\mathbb{E}}, 0)$ sits in $(\mathbb{M}, g_{\mathbb{M}})$.
- Let S^n and $(g_{st}, g_{st}) \in \mathcal{I}(\mathbb{S}^n, \mathbb{R})$. The (S^n, g_{st}, g_{st}) sits in $\mathbb{R}^{n+1}$

A.L.E. I.D.S.

Definition

An **A.L.E. I.D.S.** is a pair $(g, k) \in \mathcal{I}(M, \mathbb{R})$ such that (M, g) is an ALE manifold and $k_{ij}(x) \in O_1(|x|^{-q-1})$. The energy density and the norm of the current density are integrable, $\mu, |J|_g \in L^1$.

Let (M^r, g, k) be an asymptotically flat I.D.S.. Then we can define the *ADM*-mass or energy and the momentum

- ADM-mass $m_{ADM}(M_k, g) = \lim_{\rho\to\infty} \frac{1}{2(n-1)\omega_{n-1}} \int_{S_\rho} \sum_{i,j=1}^{r} (g_{ij,i} - g_{ii,j}) \frac{x^j}{|x^j|} dV_{S_\rho}$.

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- ADM-momentum, $P_i = \lim_{\rho\to\infty} \frac{1}{(r-1)\omega_{n-1}} \int_{S_\rho} \sum_{j=1}^{r} (k_{ij} - tr_g(k) g_{ij}) \nu^j d\mu_{S_\rho}$

for $1 \leq i, j \leq r$.

Weighted function Spaces

- **Weighted Hölder Spaces,** $C_s^{k,\alpha}(M)$

$$\|u\|_{C_s^{k,\alpha}} = \sup_{x \in M} r^{k-s+\alpha}(x) \|\nabla^k u(x)\|_{C^\alpha(B_{\frac{r}{2}}(x))} + \sum_{i=0}^{m} \sup_{x \in M} |r^{i-s}(x) \nabla^i u(x)|$$

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- **Weighted** L^2,

$$||u||_{L^2_q(M)} = \left(\int_M r^{-q2-n}|u|^2 dV_g\right)^{\frac{1}{2}}.$$

- **Weighted Sobolev space**

$$H^k_\delta(M) := \left\{ u \in H^k_{\text{loc}}(M) \mid \sum_{j=0}^{k} \int_M |\nabla^j u|^2 r^{-2\delta-n+2j}\, d\mu_g < \infty \right\}.$$

Spin Geometry for People in a Hurry

Set up for the Spin Bundle

Over (M^n, g, k) we can consider:

- The bundle $\mathbb{R} \oplus TM$ with the sections $e_0 = (1, 0)$ and $\{e_i\}_{i=1}^n$ any O.N.B. of TM.
- The Lorentzian metric $\mathbf{g}(e_\mu, e_\nu) = \delta_{\mu\nu} - 2\,\delta_\mu^0 \delta_\nu^0$.

Under this construction, we can just see that $(\mathbb{R} \oplus TM, \mathbf{g})$ is the pullback of $T\mathcal{M}^{n+1}$.

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Under this construction, we can just see that $(\mathbb{R} \oplus TM, \mathbf{g})$ is the pullback of $T\mathcal{M}^{n+1}$.
Then we can just consider the connection on $TM \oplus \mathbb{R}$ to be $\bar{\nabla}$ and the connection on M^n as ∇, then,

$$\bar{\nabla} = \nabla + k(,)e_0.$$

Spinor Bundle and Clifford Action

Let $S(M)$ be a spinor bundle over (M, g) with an action of $\mathrm{Cl}(TM)$. To define an action of $\mathrm{Cl}(\mathbb{R} \oplus TM)$, consider the extended bundle:

$$\widetilde{S}(M) = S(M) \oplus S(M),$$

with inner product given by the sum of the ones on each component.
Define the action as follows for $v \in T_pM$:

$$v \cdot (\psi_1, \psi_2) = (v \cdot \psi_1, -v \cdot \psi_2), \qquad e_0 \cdot (\psi_1, \psi_2) = (\psi_2, \psi_1).$$

This extends the Clifford action to $\mathrm{Cl}(\mathbb{R} \times TM)$ and satisfies the Clifford relations.

Spin connection

The connection in $\widetilde{S}(M)$ can be reconstructed as follows,

$$\tilde{\nabla}_i = \nabla_i + \frac{1}{2}\sum_{j=1}^{n} k(e_i, e_j)e_j \bullet e_0.$$

Then two Dirac operators can be defined,

- Classical Dirac operator,

$$\tilde{D} = \sum_{i=0}^{n} e_i \bullet \tilde{\nabla}_{e_i}$$

- Submanifold Dirac or Dirac-Witten operator,

$$D = \sum_{i=1}^{n} e_i \bullet \left(\nabla_i + \frac{1}{2}\sum_{j=1}^{n} k(e_i, e_j)e_j \bullet e_0 \right) = D^M - \frac{tr_g(k)}{2} e_0$$

Schrödinger-Lichnerowicz formula

Then the Schrödinger-Lichnerowicz formula for the Dirac operators can be defined,

- Classical Dirac operator,

$$\tilde{D} = \sum_{i=0}^{n} e_i \bullet \tilde{\nabla}_{e_i} \to \mathcal{D}(\phi) = \nabla^* \nabla(\phi) + \frac{R_g}{4}(\phi)$$

- Submanifold Dirac or Dirac-Witten operator,

$$D = D^M - \frac{tr_g(k)}{2} e_0 \to D^2(\phi) = \nabla^* \nabla(\phi) + \frac{1}{2}(\mu + Je_0)\phi.$$

The Dirac-Witten operator

Let (M, g, k) be a spin initial data set satisfying the strict dominant energy condition. Then,

- For all $u \in H^1_{-q}(\tilde{S}(M))$,

$$||u||_{H^1_{-q}} \leq C \cdot \left(||\tilde{D}(u)||_{L^2_{-q-1}} + ||u||_{L^2_{-q}}\right),$$

- Let $u \in L^2_{-q-1}(\tilde{S}(M))$ be such that, $(u, \tilde{D}(v))_{L^2_{-q-1}} = 0$ for all $v \in C^\infty_c(\tilde{S}(M))$. Then $u \in H^1_{-q}(\tilde{S}(M))$ and $\tilde{D}(u) = 0$.
- The Dirac-Witten operator is a **Fredholm operator of positive index**.
- $D : W^{1,2}_{-q}(\tilde{S}(M)) \to L^2_{-q-1}(\tilde{S}(M))$ is an **isomorphisim**.

The Positive Energy Theorem

The magic happens...

We want to find a Spinor $\phi \in \Gamma(\tilde{S}(M))$ such that,

$$\begin{cases} D\phi = 0, \\ \int_M ||\tilde{\nabla}\phi||^2 + \langle \phi, (\frac{1}{2}(\mu + Je_0))\phi \rangle - ||D(\phi)||^2 = \frac{(n-1)\omega_{n-1}}{2}(E - |P|). \end{cases}$$

The Witten boundary integral „reproduces" the ADM four momentum norm,

$$\int_M ||\tilde{\nabla}\phi|| + \langle \phi, (\frac{1}{2}(\mu + Je_0))\phi \rangle - ||D(\phi)||^2 =$$

$$\lim_{\rho_l \mapsto \infty} \int_{S_{\rho_l}} \sum_{i=1}^{n} \langle \phi, \sum_{j=1}^{n} (\delta_i^j + e_i \bullet e_j)\tilde{\nabla}_j \phi \rangle$$

Where $\{\rho_l\}$ is a sequence of radius such that the integral of the elements will decay properly.

Witten's boundary integral

Choosing $\phi \in \Gamma(\tilde{S}(M))$ such that it is:

- Asymptotically Constant with respect to the frame e_i.
- It is an eigenspinor of the action of $\sum_{i=1}^{r} P_i e_i e_0$ with eigenvalue $|P| = \sqrt{\sum_{i=1}^{r} P_i^2}$.

We obtain that,

$$
\begin{aligned}
\sum_{i=1}^{n} lim_{\rho_l \mapsto \infty} \int_{S_{\rho_l}} \langle \phi, \sum_{j=1}^{n} (\delta_i^j e_i \bullet e_j) \tilde{\nabla}_j \phi \rangle &= \sum_{i=1}^{n} \lim_{\rho_l \mapsto \infty} \int_{S_{\rho_l}} \langle \phi, \sum_{j=1}^{n} (\delta_i^j e_i \bullet e_j) \tilde{\nabla}_j \phi \rangle \\
&\quad + \lim_{\rho_l \mapsto \infty} \int_{S_{\rho_l}} \langle \phi, \sum_{j=1}^{n} (\delta_i^j e_i \bullet e_j) \left(\frac{1}{2} \sum_{i=1}^{r} k(e_i, e_j) e_0 \right) \phi \rangle . \\
&= \sum_{i=1}^{n} \lim_{\rho \to \infty} \int_{S_\rho} \sum_{j=1}^{r} (g_{ij,i} - g_{ii,j}) \nu^j \\
&\quad + \sum_{j=1}^{r} (k_{ij} - tr_g(k) g_{ij}) \nu^j \langle\langle \phi, e_j e_0 \phi \rangle\rangle d\mu_{S_\rho} \\
&= \frac{(n-1)\omega_{n-1}}{2} (E - |P|) |\phi|
\end{aligned}
$$

PET

Let (M^r, g, k) be a complement A.L.E. spin initial data set sitting in some spacetime $(\mathcal{M}^{r+1}, \mathbf{g})$ satisfying the dominant energy condition. Then we have that $E_l \geq |P|_g^2$.

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Proof.

Let $\psi_0 \in \Gamma(\tilde{S}(M))$A.C. eigenspinor of the action of $\sum_{i=1}^r P_i e_i e_0$ with eigenvalue $|P|$. Then there exists:

- $\eta = -D(\psi_0)$.
- Since D is an isomorphism there exists $\xi \in W^{1,2}_{-q}$ such that $\xi = D(\eta)$.

We define $\psi = \psi_0 + \xi$ such that $D(\psi) = D(\psi_0) + D(\xi) = -\eta + \eta = 0$. Therefore,

$$\lim_{i\to\infty} \frac{2}{(n-1)\omega_{n-1}} \int_{S_{\rho_i}} |\tilde{\nabla}\psi|^2 - |\tilde{D}\psi|^2 + \frac{1}{2}\langle\langle\psi, (\mu + J \bullet e_0)\psi\rangle\rangle d\mu_{S_{\rho_i}} = E - |P|$$

□

What Happen in Higher Codimensions?

The P.E.T. in Higher Codimensions

Theorem: *P.E.T. for Spin I.D.S. in higher codimensions*,Hijazi, Zhang and Daguang 2012

Let M^n be a **compact** spacelike spin submanifold of a **pseudo-Riemannian manifold** N^{n+m}, which has possibly finite number of generalized future or past apparent horizons Σ_i. Suppose that the normal bundle of M is spin and m **is odd**. If the generalized dominant energy condition holds, then $E_l \geq \sqrt{\sum_{k,A} P_{lkA}^2}$

The P.E.T. in Higher Codimensions

Theorem: *P.E.T. for Spin I.D.S. in higher codimensions*

Let (M^n, g, k) be an asymptotically large euclidean**spin** initial data set sitting inside some pseudo-Riemannian manifold $(\mathcal{M}, \mathbf{g})$. assumming that the normal bundle is spin. If the generalized dominant energy condition holds, then

$$E_l \geq \sqrt{\sum_{k,A} P_{lkA}^2}$$

Higher Codimensions Initial Data Set

Let M^r be a smooth manifold and E a vector bundle over it.

- An **initial data set** $(I.D.S.)$ on M is a pair (g, k) where g is a Riemannian metric on M and where k is a symmetric (0,2)-tensor, $k \in \Gamma(Sym^2(T^*M) \otimes E)$. Let $\mathcal{I}(M, E)$ be set of initial data sets for the vector bundle E.

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  - $\mu_G := \frac{1}{4}\sum_{A=1}^{z}\left(tr_g(k^A)^2 - |k^A|^2 + R_g\right)$
  - $J_A^G := div(k^A)^\sharp + \nabla(tr_g(k^A))$.

  If

$$\mu_G \geq \sum_{A=1}^{z}\left(|J_A^G|_g^2 + \sqrt{\sum_{j\neq k=1}^{r}\sum_{B\neq A=1}^{r} k_{ij}^{B\,2} k_{ik}^{A\,2}}\right)$$

  then we say that (M,g,k) satisfies the **Dominant Energy Condition** $(D.E.C)$.

Initial data set

Let $(\mathcal{M}, \mathbf{g})$ be a smooth manifold equipped with a pseudo-Riemannian metric with signature (r, z) We say that $(M^r, g, k) \in \mathcal{I}(M, E)$ **sits** in $(\mathcal{M}^{r+z}, \mathbf{g})$

- The immersion $\iota : M \hookrightarrow \mathcal{M}$ is an isometric embedding with $T\mathcal{M}|_M = TM \oplus E$
- The symmetric two tensor $k \in \Gamma(Sym^2(T^*M) \otimes E)$ is the second fundamental form of the embedding.

Generalized ADM quantities

Therefore we can define the ADM energy-momentum for the initial data set (M^r, g, k) of codimension z of an end of M to be,

$$
\begin{aligned}
E &= \lim_{\rho\to\infty} \frac{1}{2(r-1)\omega_{n-1}} \int_{S_\rho} \sum_{i,j=1}^{r} (g_{ij,i} - g_{ii,j})\nu^j d\mu_{S_\rho} \\
P^l_{iA} &= \lim_{\rho\to\infty} \frac{1}{(r-1)\omega_{n-1}} \int_{S_\rho} \sum_{j=1}^{r} (k^A_{ij} - tr_g(k^A) g_{ij})\nu^j d\mu_{S_\rho}
\end{aligned}
\tag{3}
$$

for any $i \in \{1, \ldots, r\}$ and $A \in \{1, \ldots z\}$.

The group $\mathrm{Spin}(r, z)$ and its maximal

If we consider $(\mathbb{R}^r \oplus \mathbb{R}^z, \langle , \rangle_{r,z})$ then the complexified Clifford algebra is,

$$\mathbb{C}l((\mathbb{R} \oplus \mathbb{R}^z, \langle , \rangle_{r,z})) \equiv \mathcal{T}(\mathbb{R}^{r+z})/\mathcal{I}_{(r,z)}.$$

Where $\mathcal{I}_{(r,z)}$ be the ideal generated by the elements of the form

$$(v, w) \otimes (v', w') + (v', w') \otimes (v, w) = -2\langle v, v'\rangle_{\mathrm{eu}} + 2\langle w, w'\rangle_{\mathrm{eu}}$$

Then the $Spin(r, z) \subseteq \mathbb{C}l_{(r,z)}$ is defined as,

$$Spin(r, z) = \{v_1 \cdot \ldots \cdot v_{2k} \in Cl^0_{r,z} : \langle v, v\rangle_{r,z} = 1 \wedge k \in \mathbb{N}\}$$

For $r, z > 0$ such that $r + z > 2$ then is *non-compact*. Topologically retracts onto its maximal compact subgroup.

$$K^+ = \{v_1 \cdot v_2 \cdot \ldots \cdot v_{2a} \cdot w_1 \cdot \ldots \cdot w_{2b} : v_i, w_j \in \mathbb{R}^{r,z}, |v_i| = 1 \wedge |w_j| = -1\}$$

Reduction of the structure group to

$$
\begin{tikzcd}
P^{K^+}(\mathcal{M}) \times K^+ \arrow[r, "\odot"] \arrow[d, "\bar{i}\times i"] & P^{K^+}(\mathcal{M}) \arrow[d, "\bar{i}"'] \arrow[dr] & \\
P^{Spin(r,z)}(\mathcal{M}) \times Spin(r,z) \arrow[r, "\odot"] \arrow[d, "\bar{\rho}\times\rho"'] & P^{Spin(r,z)}(\mathcal{M}) \arrow[r] \arrow[d, "\bar{\rho}"] & \mathcal{M} \\
P^{SO(r,z)}(\mathcal{M}) \times SO(r,z) \arrow[r, "\odot"] & P^{SO(r,z)}(\mathcal{M}) \arrow[ur] &
\end{tikzcd}
$$

The Inner product

In a $\mathbb{K}$-Clifford algebra, $\mathrm{Cl}(V, \beta)$, there are two standard scalar product,

$$(\phi, \psi) = (\phi \bullet \psi)_{\emptyset} \quad \& \quad (\phi, \psi)_{\Sigma} = \sum_{i=1}^{2^{r+z}} \phi_i \bar{\psi}_i$$

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To extend this to a **positive definite Hermitean scalar product** in the spinor bundle it can be used the complex Element, $\omega = i^{\frac{z(z-1)}{2}} \prod_{A=1}^{z} e_A$,

$$\langle \phi, \psi \rangle = (\omega \phi, \psi)_{\Sigma}$$

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$\omega = i^{\frac{z(z-1)}{2}} \Pi_{A=1}^{z} e_A$,

$$\langle \phi, \psi \rangle = (\omega\phi, \psi)_{\Sigma}$$

The behavior of the inner product it is going to be dominated by the codimension of the normal bundle.

- If z is odd, $e_i \bullet \omega = -\omega e_i$ and $e_A \bullet \omega = \omega \bullet e_A$. Moreover, $\langle e_i \bullet \phi, \psi \rangle = -\langle \phi, e_i \bullet \psi \rangle$ and $\langle e_A \bullet \phi, \psi \rangle = \langle \phi, e_A \bullet \psi \rangle$
- If z is even, $e_i \bullet \omega = \omega e_i$ and $e_A \bullet \omega = -\omega \bullet e_A$. Moreover, $\langle e_i \bullet \phi, \psi \rangle = \langle \phi, e_i \bullet \psi \rangle$ and $\langle e_A \bullet \phi, \psi \rangle = -\langle \phi, e_A \bullet \psi \rangle$

$*$-algebra in $\mathbb{R}^z \oplus \mathbb{R}^r$

Let $\mathbb{R}^r \oplus \mathbb{R}^z$ be a real vector space with an inner product of signature (r, z).
We define a $*$-involution by:

$$* : \mathbb{R}^r \oplus \mathbb{R}^z \to \mathbb{R}^r \oplus \mathbb{R}^z, \quad (u, v) \mapsto (u, v)^* := (-u, v).$$

This involution satisfies $(x^*)^* = x$, and flips the sign of vectors in the negative-definite part.

A new inner product

We can define the following map,

$$\begin{aligned} \langle\!\langle \bullet, \bullet \rangle\!\rangle : \mathbb{C}l_{r,z} \times \mathbb{C}l_{r,z} &\mapsto \mathbb{C} \\ (\phi, \psi) &\mapsto \langle\!\langle \phi, \psi \rangle\!\rangle := (\psi^* \phi)_{\emptyset} \end{aligned}$$

The previous map is in fact an indefinite sesquilinear form, i.e. complex-valued bilinear form which is antilinear in the second slot, and is not positive definite.

The new product

Let $\mathbb{C}l_{r,z}$ together with the inner product $\langle , \rangle$.

- The Clifford multiplication by vector from $e_i \in \mathbb{R}^r \oplus 0$ is **skew-symmetric,** $\langle\langle e_i \cdot \phi, \psi \rangle\rangle = -\langle\langle \phi, e_i \cdot \psi \rangle\rangle$.
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The new product

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Then we have the following properties for the Dirac operators,

- The Dirac operator $\bar{D}$ is formally self-adjoint with respect to the L^2_{-q} inner product.
- The Dirac-Witten operator $\tilde{D}$ is formally self-adjoint with respect to L^2_{-q}.

Theorem

Let (M, g, k) be a spin initial data set. For any $\psi \in C^\infty(\tilde{S}(M))$,

$$\tilde{D}^2(\phi) = \sum_{i=1}^{z} {}^{\mathcal{M}}\nabla^{\Sigma}_{e_i}{}^{*}\, {}^{\mathcal{M}}\nabla^{\Sigma}_{e_i}(\phi)$$
$$+ \frac{1}{2}\Bigg[\frac{1}{2}\sum_{A=s+1}\left(tr_g(k^A)^2 - |k^A|^2 + R_g\right)$$
$$- \frac{1}{2}\left(\sum_{j\neq k=1}^{r}\sum_{B\neq A=1+r}^{r+z} k^B_{ij}k^A_{ik} e_j \cdot e_k \cdot e_B \cdot e_A\right)$$
$$+ \left(div(k^A)^\sharp + \nabla(tr_g(k^A))\right)e_A\Bigg] \cdot \phi$$

PET in Higher Codimensions

Let (M^r, g, k) be a complete A.L.E. spin initial data set immersed in a spacetime $(\mathcal{M}^{r+z}, \mathbf{g})$ satisfying the Dominant Energy Condition.

Then: $E_l \geq \sqrt{\sum_{i=1}^{r} \sum_{A=r+1}^{r+z} (P_i^A)^2}$

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Sketch of the proof:

- Let $\psi_0 \in \Gamma(\widetilde{S}(M))$ be asymptotically constant in a frame $\{e_\alpha\}$, and an eigenspinor of $\sum_{i,A} P_{iA} e_i e_A$, with eigenvalue $|P|$.
- Define $\eta := -D(\psi_0)$. Since D is an isomorphism, there exists $\xi \in W^{1,2}_{-q}$ such that $\xi = D(\eta)$.
- Set $\psi := \psi_0 + \xi$, so that $D(\psi) = 0$.

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Then the spinorial identity yields:

$$\lim_{i\to\infty} \frac{2}{(n-1)\omega_{n-1}} \int_{S_{\rho_i}} \left(|\widetilde{\nabla}\psi|^2 - |\widetilde{D}\psi|^2 + \frac{1}{2}\langle\langle \psi, (\mu_G + \sum_{A=1}^{z} J_A \cdot e_A)\psi \rangle\rangle \right) d\mu = E - |P$$

Key Ideas

Key Ideas

- The Positive Energy Theorem extends to spacelike submanifolds of codimension $z > 1$, provided the **normal bundle is spin**.
- The dominant energy condition and the ADM Energy-momentum must be **generalized**.
- The Spin structure must be **adapted** to the case of pseudo-Riemannian manifolds.

Next work

- Make a rigidity statement.

Thanks for your attention!