A New Proof of the Classical Minkowski Inequality via a Divergence Identity

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May 30, 2025

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Historical Background

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Hermann Minkowski in his work 'Volumen und Oberflächen' from 1903 proved an inequality for convex bodies in $n = 3$ dimensions.

Theorem (Minkowski 1903)

Let $\Omega \subset \mathbb{R}^n$ be a convex body[a] and B a ball of radius 1. Denote by $|\partial\Omega|$ the surface area of Ω and denote by $|\mathbb{S}^2|$ the surface are of B, then

$$\left(\frac{|\partial\Omega|}{|\mathbb{S}^2|}\right)^{\frac{1}{2}} \leq \frac{1}{|\mathbb{S}^2|}\int_{\partial\Omega}\frac{H}{2}d\sigma$$

with equality if and only if Ω is a ball[b].

[a] Compact convex set with non-empty interior.

[b] This is the rigidity case.

Understanding the Minkowski Inequality

This Minkowski inequality asserts

> *Among all convex bodies with the same surface area, balls alone minimize the integral of mean curvature. - Minkowski, 1903*

$$\left(\frac{|\partial\Omega|}{|\mathbb{S}^2|}\right)^{\frac{1}{2}} \leq \frac{1}{|\mathbb{S}^2|}\int_{\partial\Omega}\frac{H}{2}d\sigma$$

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Theorem (Classical Minkowski Inequality)

If $\Omega \Subset \mathbb{R}^n$ with $n \geq 3$ is a convex domain with smooth, boundary and H the mean curvature of $\partial\Omega$ computed with respect to the outward unit normal, then

$$\left(\frac{|\partial\Omega|}{|\mathbb{S}^{n-1}|}\right)^{\frac{n-2}{n-1}} \leq \frac{1}{|\mathbb{S}^{n-1}|}\int_{\partial\Omega} \frac{H}{n-1} d\sigma,$$

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- **Application**: Harvie-Wang '24 prove a black hole uniqueness theorem based on the work of McCormick using the Minkowski inequality.
- **Application**: Liu-Yau '05 prove positivity of their definition of quasi-local mass in time orientable spacetimes using the classical Minkowski inequality for convex bodies.

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Setting

Let $(\mathbb{R}^n, \delta)$ be the n-dimensional Euclidean space. Let $\Omega \subset \mathbb{R}^n$ for $n \geq 3$ be a bounded and convex domain with smooth boundary $\partial\Omega$. We consider weak solutions $u : \mathbb{R}^n \backslash \Omega \to \mathbb{R}$, called *p-capacitary potentials* of Ω (for short: *p-potentials*), to the problem

$$\begin{cases} \Delta_p u := \operatorname{div}\left(|Du|^{p-2} Du\right) = 0 & \text{in} \quad \mathbb{R}^n \backslash \overline{\Omega} \\ u = 1 & \text{on} \quad \partial\Omega \\ u(x) \to 0 & \text{as} \quad |x| \to \infty, \end{cases} \tag{1}$$

with $1 < p < n$.

p-Capacity

Definition (Normalized p-capacity, see e.g. Fogagnolo–Mazzieri–Pinamonti (FMP) '19)

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with a smooth boundary. Then

$$C_p(\Omega) = \inf \left\{ \left(\frac{p-1}{n-p} \right)^{p-1} \frac{1}{|\mathbb{S}^{n-1}|} \int_{\mathbb{R}^n} |Dv|^p d\mu \;\middle|\; v \in C_0^\infty(\mathbb{R}^n), v \geq 1 \text{ on } \Omega \right\}$$

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Lemma (see e.g. FMP '19)

If u is a p-potential of Ω, then

$$C_p(\Omega) = \left(\frac{p-1}{n-p} \right)^{p-1} \frac{1}{|\mathbb{S}^{n-1}|} \int_{\partial\Omega} |Du|^{p-1} d\sigma.$$

Existence, uniqueness and regularity of p-potentials

Theorem (Existence & regularity of p-potentials, Lewis '77)

Let $n \geq 3$ and $1 < p < n$. Let $\Omega \subset \mathbb{R}^n$ be a convex bounded domain with smooth boundary $\partial\Omega$. Then the following statements hold

1. *There exists a* **unique** *weak solution $u \in C^\infty(\mathbb{R}^n \backslash \overline{\Omega}) \cap C(\mathbb{R}^n \backslash \Omega)$ to (1).*
2. $0 < u < 1$ *and* $|Du| \neq 0 \in \mathbb{R}^n \backslash \overline{\Omega}$.
3. *Let $C_p(\Omega)$ be the normalized p-capacity of Ω. Then*

$$C_p(\Omega) = \frac{1}{\left(\frac{n-p}{p-1}\right)^{p-1} |\mathbb{S}^{n-1}|} \int_{\mathbb{R}^n \backslash \overline{\Omega}} |Du|^p d\mu.$$

Orthogonal Decomposition

Let u be a p-harmonic function and f an arbitrary smooth function, then

$$D^{\perp} f = \left\langle Df, \frac{Du}{|Du|} \right\rangle \frac{Du}{|Du|} \quad \text{and} \quad D^{\top} f = Df - D^{\perp} f$$

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$$\big|D|Df|\big|^2 = \big|D^{\perp}|Df|\big|^2 + \big|D^{\top}|Df|\big|^2.$$

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$$\big|D|Df|\big|^2 = \big|D^{\perp}|Df|\big|^2 + \big|D^{\top}|Df|\big|^2.$$

Moreover, we define

$$\Delta^{\top} f = \Delta f - D^2 f(E_n, E_n)$$

with $\{E_1, \dots, E_{n-1}, E_n := Du/|Du|\}$ an orthonormal frame.

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Main Result - Parametric Geometric Inequality

Theorem (Parametric geometric inequality)

Let $n \geq 3$ and $\Omega \subset \mathbb{R}^n$ be a convex bounded domain with smooth boundary $\partial\Omega$. Let u be the p-potential associated to Ω and consider parameters $c, d \in \mathbb{R}$ satisfying $c + d \geq 0$ and $d \geq 0$. Consider a parameter $\beta \in \mathbb{R}$ that satisfies $\beta \geq (p-1)\frac{n-2}{n-1}$. Then,

$$\begin{aligned} &\mathbf{d}\frac{p-1}{\beta-p+2}\left(\frac{n-p}{p-1}\right)^{\beta+1} C_p(\Omega)^{\frac{n-\beta-2}{n-p}}|\mathbb{S}^{n-1}| \\ &\qquad \leq \mathbf{(c+d)}\int_{\partial\Omega}|Du|^{\beta}H d\sigma \\ &\qquad\qquad + (p-1)\left[\frac{\mathbf{d}}{\beta-p+2} - \mathbf{(c+d)}\frac{n-1}{n-p}\right]\int_{\partial\Omega}|Du|^{\beta+1}d\sigma \qquad (2) \end{aligned}$$

holds. Finally, equality holds in (2) if and only if Ω is a round ball (unless $c = d = 0$).

Corollary I

1. In the parametric geometric inequality set $(c, d) = (-1, 1)$ and $(c, d) = (1, 0)$ to get two distinct inequalities.

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2. Apply Hölder's inequality to the second inequality.
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Corollary (see e.g. FMP '19 with $q = \frac{\beta+1}{p-1}$)

Let $\Omega \subset \mathbb{R}^n$ with $n \geq 3$ be a convex bounded domain with smooth boundary $\partial\Omega$. Let $\beta \geq (p-1)\frac{n-2}{n-1}$ with $1 < p < n$, then

$$C_p(\Omega)^{\frac{n-\beta-2}{n-p}} |\mathbb{S}^{n-1}| \leq \int_{\partial\Omega} \left(\frac{H}{n-1}\right)^{\beta+1} d\sigma \tag{3}$$

holds. Moreover, we have equality in (2) if and only if Ω is a round ball.

Corollary II

Setting $\beta = p - 1$ we obtain the L^p-*Minkowski inequality*:

Corollary (L^p-Minkowski inequality, Agostiniani–Fogagnolo–Mazzieri (AFM) '22)

Let $n \geq 3$ and let $\Omega \subset \mathbb{R}^n$ be a convex bounded domain with smooth boundary $\partial\Omega$. Then, for every $1 < p < n$ the following inequality

$$C_p(\Omega)^{\frac{n-p-1}{n-p}} \leq \frac{1}{|\mathbb{S}^{n-1}|} \int_{\partial\Omega} \left(\frac{H}{n-1} \right)^p d\sigma \tag{4}$$

holds. Equality holds in (4) if and only if Ω is a ball.

Corollary III

Taking the limit $p \to 1^+$ we obtain the *classical Minkowski inequality*:
Using that

$$\lim_{p \to 1^+} C_p(\Omega) = \frac{|\partial\Omega|}{|\mathbb{S}^{n-1}|}$$

Theorem (Classical Minkowski inequality, Minkowski 1903)

If $\Omega \subset \mathbb{R}^n$ with $n \geq 3$ is a convex bounded domain with smooth boundary $\partial\Omega$, then

$$\left(\frac{|\partial\Omega|}{|\mathbb{S}^{n-1}|}\right)^{\frac{n-2}{n-1}} \leq \frac{1}{|\mathbb{S}^{n-1}|} \int_{\partial\Omega} \frac{H}{n-1} d\sigma, \tag{5}$$

holds. Here, H is the mean curvature of $\partial\Omega$ computed with respect to the outward pointing unit normal. Equality holds in (5) if and only if Ω is a ball.

Corollary VI

Corollary (Quantitative L^p-Willmore-type inequality)

Let $\Omega \subset \mathbb{R}^n$ with $n \geq 3$ be a convex bounded domain with smooth boundary $\partial\Omega$ and u the p-potential associated to Ω, where $1 < p < n$. Then

$$\int_{\partial\Omega} |Du|^{(p-1)\frac{n-2}{n-1}} \left[H - \left(\frac{n-1}{n-p}\right) |Du| \right] d\sigma \leq (n-1) \left(\frac{n-p}{p-1}\right)^{(p-1)\frac{n-2}{n-1}} |\mathbb{S}^{n-1}|^{\frac{n-2}{n-1}} C_p(\Omega)^{\frac{n-2}{n-1}} \times \left[\left(\int_{\partial\Omega} \left(\frac{H}{n-1}\right)^{n-1} d\sigma \right)^{\frac{1}{n-1}} - \frac{|\mathbb{S}^{n-1}|^{\frac{1}{n-1}}}{p-1} \right]. \tag{6}$$

Moreover, we have equality in (6) if and only if Ω is a round ball.

Corollary V

Corollary (Weighted L^p-Minkowski inequality)

Let $\Omega \subset \mathbb{R}^n$ with $n \geq 3$ be a convex bounded domain with smooth boundary $\partial\Omega$. Let u be the p-potential associated with Ω, where $1 < p < n$. Then

$$\int_{\partial\Omega} |Du|^{(p-1)\frac{n-2}{n-1}} \left[H - \left(\frac{n-1}{n-p}\right)|Du| \right] d\sigma \leq (n-1)\left(\frac{n-p}{p-1}\right)^{(p-1)\frac{n-2}{n-1}} |\mathbb{S}^{n-1}|^{\frac{n-2}{n-1}} \left(\frac{C_p(\Omega)}{|\partial\Omega|}\right)^{\frac{n-2}{n-1}} \times \left[\int_{\partial\Omega} \frac{H}{n-1} d\bar{\sigma} - \frac{|\partial\Omega|^{\frac{n-2}{n-1}} |\mathbb{S}^{n-1}|^{\frac{1}{n-1}}}{p-1} \right], \tag{7}$$

where $d\bar{\sigma} = \left(\frac{|Du|^{p-1}}{\frac{1}{|\partial\Omega|}\int_{\partial\Omega}|Du|^{p-1}d\sigma}\right)^{\frac{n-2}{n-1}} d\sigma$ is a weighted area measure on $\partial\Omega$. Moreover, equality holds in (7) if and only if Ω is a ball.

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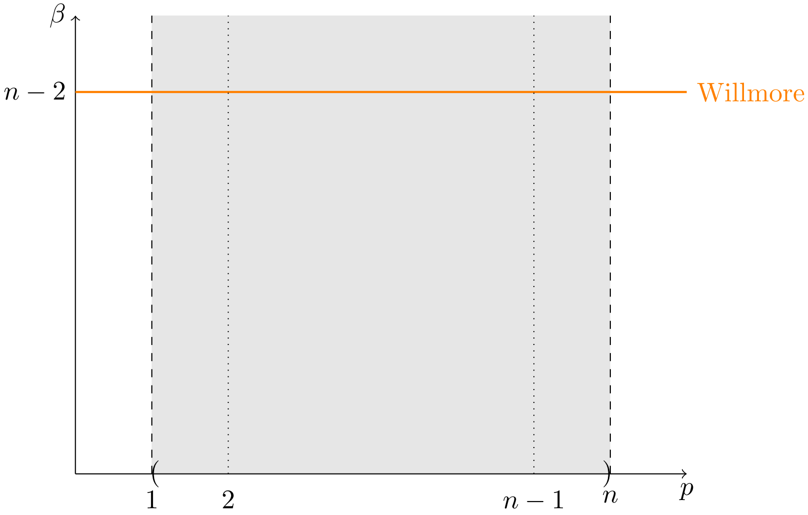
Overview of the different inequalities

Recall $1 < p < n$ and $\beta \geq (p-1)\frac{n-2}{n-1}$.



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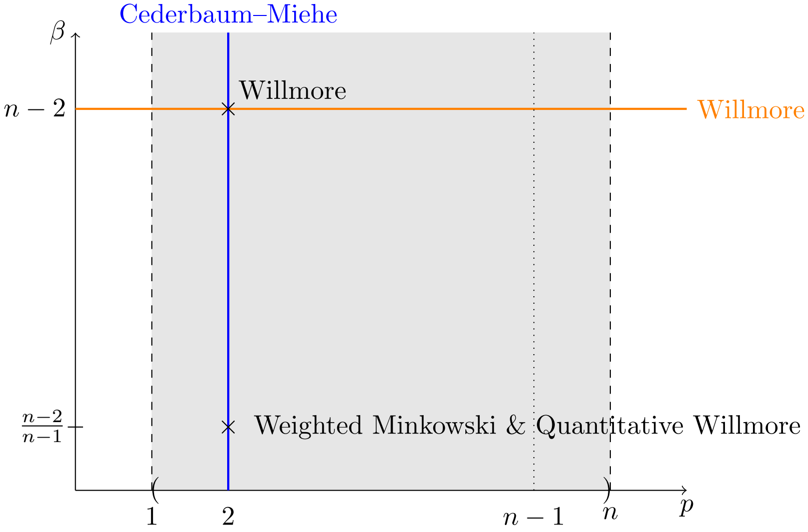

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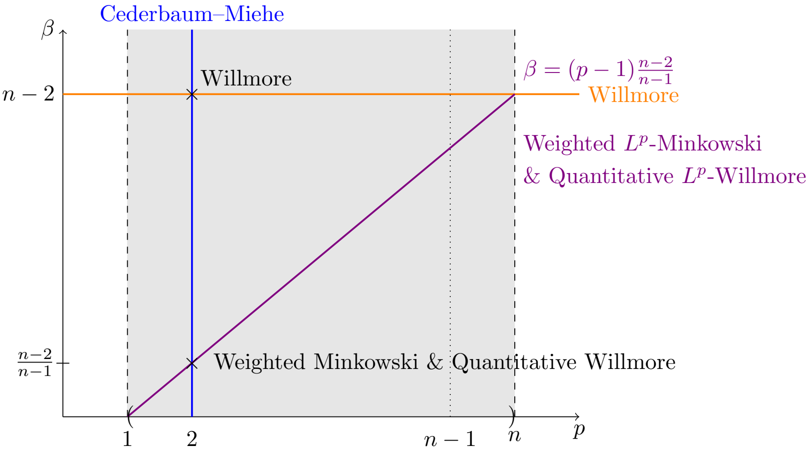

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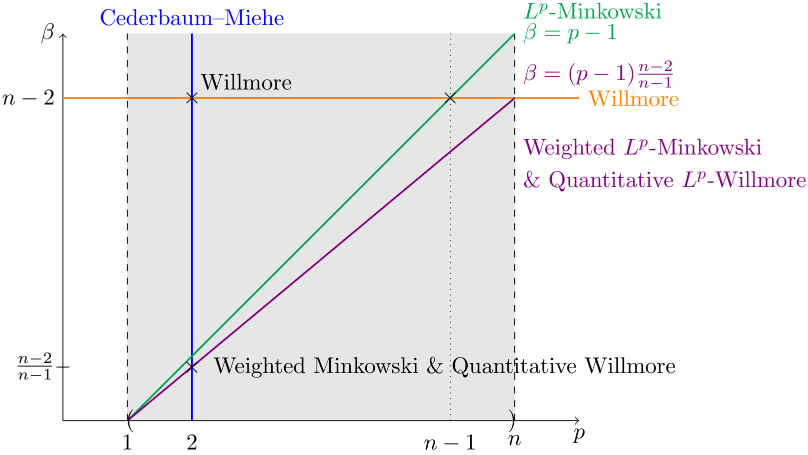

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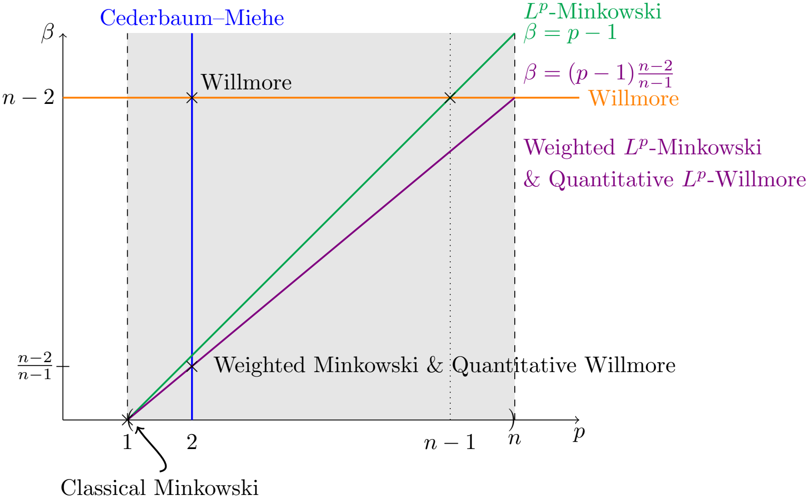

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6. All these proofs use solutions to the linear Laplace equation. This is the first time solutions to the non-linear p-Laplace equation are used.

Divergence Identity

Theorem (Divergence identity)

Let u be the unique p-potential. Let $\beta \geq 0$ and set $a_{\beta,p} = \frac{\beta(p-1)}{4}\left[\beta - (p-1)\frac{n-2}{n-1}\right]$. Then, the divergence identity

$$\operatorname{div}\left(F(u)(D|Du|^{\beta} + (p-2)D^{\perp}|Du|^{\beta}) + G(u)|Du|^{\beta}Du\right)$$
$$= F(u)|Du|^{\beta-4}\left\{a_{\beta,p}\left|D^{\perp}|Du|^2 - \frac{2(n-1)|Du|^2}{(n-p)u}Du\right|^2 \right.$$
$$\left. + \beta\left|(D^{\top})^2u - \frac{\Delta^{\top}u}{n-1}\delta^{\top}\right|^2 + \frac{\beta^2}{4}\left|D^{\top}|Du|^2\right|^2\right\}$$

holds on $\mathbb{R}^n \backslash \overline{\Omega}$ for $c, d > 0$, $\beta \neq p-2$ and smooth functions

$$F(u) = (cu+d)u^{-(\beta-p+2)\frac{n-1}{n-p}+1},$$
$$G(u) = -(p-1)\frac{(n-1)\beta}{(n-p)u}F(u) + \frac{\beta(p-1)}{\beta-p+2}du^{-(\beta-p+2)\frac{n-1}{n-p}}.$$

Proof of divergence identity

1. Compute the divergence of the ansatz

$$W := F(u)(D|Du|^{\beta} + (p-2)D^{\perp}|Du|^{\beta}) + G(u)|Du|^{\beta}Du.$$

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2. Apply p-Bochner formula, which for the flat case reduces to

$$\begin{aligned}\Delta|Du|^\beta = \beta(\beta-2)|Du|^{\beta-2}\big|D|Du|\big|^2 \\ + \beta|Du|^{\beta-2}\big\{|D^2u|^2 + \langle Du, D\Delta u\rangle\big\}.\end{aligned}$$

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3. Compute $\langle Du, D\Delta u\rangle$.
4. Use the Kato-type identity

$$\begin{aligned}|D^2u|^2 &= |Du|^2\left|(D^\top)^2u - \frac{\Delta^\top u}{n-1}\delta^\top\right|^2 \\ &\quad + \frac{1}{4}\left(1 + \frac{(p-1)^2}{n-1}\right)|Du|^{-2}\big|D^\perp|Du|^2\big|^2 + \frac{1}{2}|Du|^{-2}\big|D^\top|Du|^2\big|^2.\end{aligned}$$

Proof of divergence identity

5. Compute

$$\left| D^{\perp} |Du|^2 - \frac{2(n-1)|Du|^2}{(n-p)u} Du \right|^2$$

and formally compare with the coefficients in the divergence to get two coupled ODEs for F and G.

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and formally compare with the coefficients in the divergence to get two coupled ODEs for F and G.
8. Solve the ODEs, making use of the fact that the problem can be transformed into a second-order Cauchy-Euler equation for F, from which then G can be obtained.

Sign of the divergence

Theorem

Let u be the unique p-potential. If $\beta \geq (p-1)\frac{n-2}{n-1}$, then

$$|Du|^4 \operatorname{div}\left(F(u)(D|Du|^\beta + (p-2)D^\perp|Du|^\beta) + G(u)|Du|^\beta Du\right) \geq 0,$$

holds on $\mathbb{R}^n \backslash \overline{\Omega}$ for any $c, d \in \mathbb{R}$ satisfying $c + d \geq 0$ and $d \geq 0$. Equality holds above if and only if Ω is a round ball (unless $c = d = 0$).

Here, c and d are the constants appearing in F, G:

$$F(u) = (cu + d)u^{-(\beta - p + 2)\frac{n-1}{n-p} + 1},$$
$$G(u) = -(p-1)\frac{(n-1)\beta}{(n-p)u}F(u) + \frac{\beta(p-1)}{\beta - p + 2}du^{-(\beta - p + 2)\frac{n-1}{n-p}}.$$

Proof of the rigidity case

By the choice of c, d and β it is clear that F and $a_{\beta,p}$ are non-negative. It thus suffices to show the rigidity case.

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"$\Leftarrow$": If Ω is a round ball, then one can compute that the divergence vanishes.

"$\Rightarrow$":

1. Assume equality.
2. Umbilicity of the level sets follows from the divergence identity.
3. As $\partial\Omega$ is a regular level set of u, it is a sphere, and thus Ω is a round ball.

Parametric Geometric Inequality

Theorem (Parametric Geometric Inequality)

Let u be the unique p-potential. Let $c+d \geq 0$ and $d \geq 0$ such that $F(u) \geq 0$. Let $\beta \geq (p-1)\frac{n-2}{n-1}$, then

$$
\begin{aligned}
&d\frac{p-1}{\beta-p+2}\left(\frac{n-p}{p-1}\right)^{\beta+1} C_p(\Omega)^{\frac{n-\beta-2}{n-p}}|\mathbb{S}^{n-1}| \\
&\quad \leq (c+d)\int_{\partial\Omega}|Du|^{\beta} H d\sigma \\
&\qquad + (p-1)\left[\frac{d}{\beta-p+2}-(c+d)\frac{n-1}{n-p}\right]\int_{\partial\Omega}|Du|^{\beta+1}d\sigma
\end{aligned} \tag{8}
$$

holds. Equality holds if and only if Ω is a round ball (unless $c=d=0$).

Proof of the parametric geometric inequality

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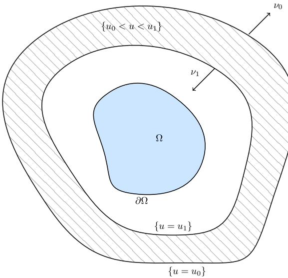

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3. Use the asymptotics of u given by

$$u(x) = \frac{C_p(\Omega)^{\frac{1}{p-1}}}{|x|^{\frac{n-p}{p-1}}} + o\left(|x|^{-\frac{n-p}{p-1}}\right) \quad \text{as} \quad |x| \to \infty$$

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10. Evaluate the first term at the boundary, that is, $u = 1$.
11. Rearrange to find the final result.

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- The proof of FMP relies on discovering *monotonicity formulas* for newly constructed functionals V_q^p, holding along the level set flow of the p-potential u associated with Ω.

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$$V_q^p(t) = \left(\frac{C_p(\Omega)}{t^{p-1}} \right)^{\frac{(n-1)(q-1)}{(n-p)}} \int_{\{u=t\}} |Du|^{q(p-1)} d\sigma.$$

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- To show monotonicity, FMP need to change into a conformal picture $\Rightarrow$ **Tedious computations and lots of subtle analysis!**

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Relation to the Monotonicity Formulas

Note that the two inequalities, one gets from the parametric geometric inequality,

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(-1,1): &\quad \left(\frac{n-p}{p-1}\right)^{\beta+1} C_p(\Omega)^{\frac{n-\beta-2}{n-p}} |\mathbb{S}^{n-1}| \leq \int_{\partial\Omega} |Du|^{\beta+1} d\sigma, \\
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coincide with the monotonicity formulas

$$(V^p_{q(\beta)})'(1) \geq 0 \qquad \text{and} \qquad \lim_{t\to 0^+} V^p_{q(\beta)}(t) \leq V^p_{q(\beta)}(1)$$

for $q(\beta) = \frac{\beta+1}{p-1}$.

Relation to the Functionals $V^p_{q(\beta)}$

For $\beta \geq (p-1)\frac{n-2}{n-1}$ and $c+d \geq 0$, $d \geq 0$ we can introduce new functionals $\mathcal{H}^{c,d}_{\beta,p} : (0,1] \to \mathbb{R}$ defined as

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These functionals are related to the $V^p_{q(\beta)}$ with $q(\beta) = \frac{\beta+1}{p-1}$ as follows:

$$\mathcal{H}^{c,d}_{\beta,p}(t) = \frac{\beta(p-1)}{\beta-p+2} C_p(\Omega)^{-\frac{(\beta-p+2)(n-1)}{(p-1)(n-p)}} \left[(ct+d)t(V^p_{q(\beta)})'(t) + dV^p_{q(\beta)}(t)\right]$$

$$(V^p_{q(\beta)})'(t) = \frac{\beta-p+2}{\beta(p-1)} C_p(\Omega)^{\frac{(\beta-p+2)(n-1)}{(p-1)(n-p)}} \frac{1}{t^2}\mathcal{H}^{1,0}_{\beta,p}(t),$$

$$V^p_{q(\beta)}(t) = \frac{\beta-p+2}{\beta(p-1)} C_p(\Omega)^{\frac{(\beta-p+2)(n-1)}{(p-1)(n-p)}} \left[\mathcal{H}^{-1,1}_{\beta,p}(t) + \left(1-\frac{1}{t}\right)\mathcal{H}^{1,0}_{\beta,p}(t)\right].$$

In particular, for $d = 0$ and $c = 1$ one can deduce from the integral identity that $\mathcal{H}^{1,0}_{\beta,p}(t) \geq 0$ and thus

$$0 \leq \frac{\beta - p + 2}{\beta(p-1)} C_p(\Omega)^{-\frac{(\beta-p+2)(n-1)}{(p-1)(n-p)}} \frac{\mathcal{H}^{1,0}_{\beta,p}(t)}{t^2} = (V^p_{q(\beta)})'(t).$$

Integration over $(0, 1]$ yields the second monotonicity formula $\lim_{t \to 0^+} V^p_{q(\beta)}(t) \leq V^p_{q(\beta)}(1)$.

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- **We get new monotone quantities.**

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- From our approach, we recover the monotonicity of their constructed functionals $V^p_{q(\beta)}$.
- In the future, one could be looking into generalizing these results to more general domains with smooth boundary by analyzing the sets of critical points of the p-potentials.

The End

Thank you for your attention!