Global counterexamples to uniqueness for a Calderón problem with C^k conductivities

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Universität Regensburg, June 5, 2025.

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The Calderón problem,

an inverse problem in geometric analysis

Set-up

- Let Ω be a bounded domain of $\mathbb{R}^n$, $n \geq 3$, with C^∞ boundary.
- Let $\gamma = (\gamma^{ij})$, referred to as a conductivity, be a bounded measurable function from $\overline{\Omega}$ to the set $\mathcal{S}_n$ of positive-definite symmetric matrices, satisfying the uniform ellipticity condition

$$\gamma^{ij}(x)\,\xi_i\,\xi_j \geq c|\xi|^2\,,$$

for a.e. $x \in M$ and for all $\xi \in \mathbb{R}^n$, where $c > 0$ is some positive constant.
- Let λ be a real parameter, referred to as a frequency.

Consider the boundary value problem

$$\begin{cases} L_\gamma u := -\mathrm{div}\,(\gamma \nabla u) = \lambda u, & \text{on } \Omega, \\ u = f & \text{on } \partial\Omega. \end{cases} \tag{1}$$

We have :

Proposition

If $\lambda \notin \sigma_{Dir}(L_\gamma)$ and $f \in H^{1/2}(\partial\Omega)$, then (1) admits a unique solution $u \in H^1(\Omega)$.

The Dirichlet-to-Neumann (DN) map

For γ and f smooth enough, the DN map is defined by

$$\Lambda_{\gamma,\lambda} f = (\gamma \nabla u) \cdot \nu_{\;|\partial\Omega} \,, \tag{2}$$

where $\nu = (\nu^i)$ is the unit outer normal to the boundary.

In general, the DN map $\Lambda_{\gamma,\lambda} : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$ is defined in a weak sense by

$$\langle \Lambda_{\gamma,\lambda} f | g \rangle = \int_\Omega \gamma \nabla u \cdot \nabla v \; dx - \lambda \int_\Omega u \, v \; dx, \text{ for all } f, g \in H^{1/2}(\partial\Omega), \tag{3}$$

where u is the unique solution of (1), v is any element of $H^1(\Omega)$ s.t. $v_{|\partial\Omega} = g$, and $\langle \cdot | \cdot \rangle$ is the standard L^2 duality pairing between $H^{1/2}(\partial\Omega)$ and $H^{-1/2}(\partial\Omega)$.

The DN map $\Lambda_{\gamma,\lambda}$ is an elliptic pseudo-differential operator $\Lambda_{\gamma,\lambda}$ of order 1.

An inverse problem

Question : Does the knowledge of the DN map $\Lambda_{\gamma,\lambda}$ determine uniquely the conductivity γ ?

This is known as the Calderón problem.

Remarks :

- Calderón considered only the case of isotropic conductivities, that is $\gamma^{ij} = c(x)\delta^{ij}$. We are concerned with the general anisotropic case.
- The answer depends significantly on whether $\lambda = 0$ or $\lambda \neq 0$. We shall see that this is due to the differences between the gauge invariances enjoyed by the DN map for these cases.

The Calderón problem at zero frequency

Question : Does the knowledge of the DN map $\Lambda_{\gamma,0}$ at zero frequency $\lambda = 0$ determine uniquely the conductivity γ ?

Due to a natural gauge invariance, the answer to the above question is no.

Indeed :

Proposition

For all $\psi \in \mathrm{Diff}(\overline{\Omega})$ such that $\psi_{|\partial\Omega} = Id$, one has

$$\Lambda_{\psi_*\gamma,0} = \Lambda_{\gamma,0}\,, \tag{4}$$

where

$$\psi_*\gamma := \left(\frac{D\psi \cdot \gamma \cdot (D\psi)^T}{|\det D\psi|}\right) \circ \psi^{-1}\,. \tag{5}$$

This leads to :

Calderón conjecture at zero frequency

Conjecture

Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, be a bounded domain with smooth boundary and let γ_1, γ_2 be bounded measurable conductivities on $\overline{\Omega}$. If

$$\Lambda_{\gamma_1,0} = \Lambda_{\gamma_2,0}$$

then there exists a diffeomorphism $\psi : \overline{\Omega} \to \overline{\Omega}$ such that such that $\psi_{|\partial\Omega} = \mathrm{Id}$ and such that

$$\gamma_2 = \psi_* \gamma_1 \,.$$

In dimension $n \geq 3$, for C^ω conductivities, the Calderón conjecture for $\lambda = 0$ has been proved by Lee-Uhlmann and Lassas-Uhlmann.

For C^∞ rather than C^ω conductivities, the conjecture is still open. There exist counterexamples with $\gamma \in C^\infty(\Omega)$, but only Hölder continuous on a connected component of $\partial\Omega$ (Daudé, K. and Nicoleau). These use operators that fail to satisfy Hörmander's unique continuation principle and are local in that $\mathrm{supp}\, f \subsetneq \partial\Omega$.

Riemannian formulation at zero frequency

The proof of Lee and Uhlmann makes use of an equivalent Riemannian formulation by rewriting

$$L_\gamma u = 0\,, \tag{6}$$

as

$$\Delta_g u = 0\,, \tag{7}$$

where

$$g^{ij} := \det(\gamma^{ij})^{\frac{1}{n-2}}\gamma^{ij}\,.$$

The transformation law (5) for (γ^{ij}) gets converted into the tensorial transformation law for (g^{ij}) :

$$\psi_* g = \left(D\psi \cdot g \cdot (D\psi)^T\right) \circ \psi^{-1}\,.$$

The idea behind the proof of Lee and Uhlmann is to compute the symbol of $\Lambda_{g,0}$ in boundary normal coordinates,

$$g_{ij}\, dx^i dx^j = (dx^n)^2 + \sum_{\alpha,\beta=1}^{n-1} g_{\alpha\beta}(x^\gamma, x^n) dx^\alpha dx^\beta \,,$$

and to show that it determines the Taylor series of g along $\partial\Omega$:

One factorizes

$$-\Delta_g = (i\partial_{x^n} + iE(x^n, x^\gamma) - iA(x^n, x^\gamma)).(i\partial_{x^n} + iA(x^n, x^\gamma)) \mod S^{-\infty} \,,$$

and computes

$$\Lambda_{g,0} f = |g|^{1/2} A\, f \mod S^{-\infty} \,.$$

Gauge invariance in the case $\lambda \neq 0$

When $\lambda \neq 0$, there is a corresponding gauge invariance for the DN map $\Lambda_{\gamma,\lambda}$ which is more subtle than in the case $\lambda = 0$: One has to restrict to the subgroup $\mathrm{SDiff}(\overline{\Omega}) \subset \mathrm{Diff}(\overline{\Omega})$ of diffeomorphisms ψ such that

$$|\det D\psi| = 1 \text{ on } \Omega\,, \quad \psi_{|\partial\Omega} = \mathrm{Id}\,.$$

This is a consequence of :

Lemma

Let $\psi : \overline{\Omega} \to \overline{\Omega}$ be a diffeomorphism and assume that u solves

$$-\mathrm{div}\,((\psi_* \gamma)\nabla u) = \lambda u\,.$$

Then, if we set $\tilde{u} = u \circ \psi$, one has

$$-\mathrm{div}\,(\gamma \nabla \tilde{u}) = \lambda\, |\det D\psi|\ \tilde{u}\,.$$

Then, we obtain immediately :

Proposition

For any $\lambda \notin \sigma_{Dir}(L_\gamma)$ *and* $\psi \in \mathrm{SDiff}(\overline{\Omega})$, *we have*

$$\Lambda_{\psi_*\gamma,\lambda} = \Lambda_{\gamma,\lambda}. \tag{8}$$

In view of the above proposition, we introduce the following definition :

Definition

Let γ_1, γ_2 *be conductivities defined in* $\overline{\Omega}$. *We say that* γ_1 *and* γ_2 *are isometric if there exists* $\psi \in \mathrm{SDiff}(\overline{\Omega})$ *such that* $\gamma_2 = \psi_*\gamma_1$.

Calderón conjecture at non-zero frequency

In the case of non-zero frequency, we are thus led in view of the above discussion to modify the Calderón conjecture as follows :

Conjecture

Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, be a bounded domain with smooth boundary and let γ_1, γ_2 be bounded measurable conductivities on $\overline{\Omega}$. Let $\lambda \neq 0$ be any fixed frequency that does not belong to the Dirichlet spectrum of L_{γ_j}, $j = 1, 2$. If

$$\Lambda_{\gamma_1,\lambda} = \Lambda_{\gamma_2,\lambda}$$

then γ_1 and γ_2 are equal up to isometry.

In what follows , we shall construct C^k counterexamples to this conjecture.

Statement of the main result

Main result

We introduce the following definition :

Definition

Given $k \geq 0$ and $\epsilon > 0$ we say that the conductivities $\gamma_1,\ \gamma_2$ are (ϵ, k)-close if

$$||\gamma_2 - \gamma_1||_{C^k(\overline{\Omega}, \mathcal{S}_n)} \leq \epsilon\,.$$

Our main result is the following :

Theorem

Let $\Omega \subset \mathbb{R}^n,\ n \geq 3$ be a bounded domain with smooth boundary and let γ be a smooth conductivity in $\overline{\Omega}$. Let us consider $\lambda_0 \neq 0$ which does not belong to the Dirichlet spectrum of L_γ. Then, for any $k \geq 1$ and $\epsilon > 0$ there exists a pair of non-isometric conductivities (γ_1, γ_2) on $\overline{\Omega}$ of class C^k, which are (ϵ, k) close to γ and satisfy

$$\Lambda_{\gamma_1, \lambda_0} = \Lambda_{\gamma_2, \lambda_0}\,.$$

Main ideas of the proof

Main ideas of the proof

Our non-uniqueness results are based on the usual conformal invariances for *div-grad* operators and on transformations by suitably chosen diffeomorphisms.

First, we recall the identity :

$$\operatorname{div}(c^2\gamma\nabla v) = c\left[\operatorname{div}(\gamma\nabla(cv)) - \operatorname{div}(\gamma\nabla c)v\right].$$

Thus, if we begin with v satisfying

$$-\operatorname{div}(c^2\gamma\nabla v) = \lambda v, \tag{9}$$

we get immediately

$$-\operatorname{div}(\gamma\nabla(cv)) + \frac{1}{c}\left(\operatorname{div}(\gamma\nabla c) + \lambda(c - \frac{1}{c})\right)(cv) = \lambda(cv).$$

Second, we introduce an auxiliary function $f \in C^\infty(\overline{\Omega})$ which will be chosen in order to use the natural invariance given in Lemma 1 with a suitable diffeomorphism $\psi : \overline{\Omega} \to \overline{\Omega}$ depending on f.

We rewrite (9) as

$$-\mathrm{div}\,(\gamma \nabla (cv)) + \frac{1}{c}\left(\mathrm{div}\,(\gamma \nabla c) + \lambda (c - \frac{1}{c} + cf)\right)(cv) = \lambda (1+f)(cv)\,,$$

If we assume now that the conformal factor c satisfies

$$\mathrm{div}\,(\gamma \nabla c) + \lambda (c - \frac{1}{c} + cf) = 0\,,$$

we get immediately :

$$-\mathrm{div}\,(\gamma \nabla (cv)) = \lambda (1+f)(cv)\,.$$

It remains to choose a suitable function $f \in C^\infty(\overline{\Omega})$.

We do this by choosing $f \in C^\infty(\overline{\Omega})$ which satisfies for some fixed $\alpha \in (0,1)$,

$$\int_\Omega f(x)\, dx = 0 \ , \ ||f||_{C^{0,\alpha}(\overline{\Omega})} \leq \epsilon \ , \tag{10}$$

where $\epsilon > 0$ is small enough.

In particular, we see that $1 + f \geq \frac{1}{2}$ in $\overline{\Omega}$.

Lemma (Dacorogna and Moser)

Under the assumption (10), there exists for all $k \in \mathbb{N}$ a $C^{k+1,\alpha}$ diffeomorphism $\psi : \overline{\Omega} \to \overline{\Omega}$ such that $\psi = Id$ on $\partial\Omega$ and $|\det\, D\psi| = 1 + f$ on Ω. Moreover, we have the following estimate :

$$||\psi - \mathrm{Id}\ ||_{k+1,\alpha} \leq C_k\ ||f||_{k,\alpha},$$

where the constant C_k only depends on k and Ω.

Thus, using the gauge invariance (8), we see that our initial equation (9)

$$-\mathrm{div}\,(c^2\gamma\nabla v) = \lambda v,$$

can be written equivalently in the simpler form :

$$-\mathrm{div}\,(\psi_*\gamma\nabla w) = \lambda w \quad \text{with} \quad w = (cv)\circ\psi^{-1}\,.$$

We therefore immediately get the following result :

Proposition

If c satisfies

$$\mathrm{div}\,(\gamma\nabla c) + \lambda(c - \frac{1}{c} + cf) = 0\,,$$

with

$$c = 1\ ,\ \gamma\nabla c\cdot\nu = 0 \text{ on } \partial\Omega\,,$$

where $f \in C^\infty(\overline{\Omega})$ satisfies for some fixed $\alpha \in (0,1)$ and ϵ small enough,

$$\int_\Omega f(x)\ dx = 0\ ,\ ||f||_{C^{0,\alpha}(\overline{\Omega})} \leq \epsilon\ , \tag{11}$$

then there exists a $C^{k+1,\alpha}$ diffeomorphism $\psi : \overline{\Omega} \to \overline{\Omega}$ such that $\psi = Id$ on $\partial\Omega$ and such that if λ is not a Dirichlet eigenvalue of $L_{c^2\gamma}$, then

$$\Lambda_{c^2\gamma,\lambda} = \Lambda_{\psi_*\gamma,\lambda}.$$

Unique continuation principle

Remark : In the case where $f = 0$ or $\lambda = 0$, the previous Proposition will not lead to counterexamples to uniqueness.

Indeed, the equation for the conformal factor c,

$$\mathrm{div}\,(\gamma \nabla c) + \lambda(c - \frac{1}{c} + cf) = 0\,,$$

can be written in this case as

$$\mathrm{div}\,(\gamma \nabla d) + Vd = 0 \quad \text{on} \quad \Omega,$$

with $d = c - 1$ and

$$V = \lambda \left(\frac{c+1}{c} \right),$$

Then, it follows from the unique continuation principle, that the unique solution is $d = 0$, or equivalently $c = 1$.

In other words, the two conductivities $c^2\gamma$ and $\psi_*\gamma$ are isometric.

Some details of the proof

Numerical range

Let us begin by an abstract result on the numerical range (with constraints) of the operator L_γ.

Lemma

Let $W(L_\gamma)$ the numerical range with constraints of L_γ defined as

$$W(L_\gamma) = \{< L_\gamma u, u > \, ; u \in X\}, \tag{12}$$

where we have set

$$X = \{u \in C_0^\infty(\Omega, \mathbb{R}) \ , \ ||u||_2 = 1 \ , \ \int_\Omega u(x)\ dx = 0\} \, . \tag{13}$$

Then, $W(L_\gamma)$ is an open interval $(m, +\infty)$ with $m := \inf \ W(L_\gamma) > \lambda_1 > 0$, where λ_1 denotes the first Dirichlet eigenvalue of L_γ on Ω.

Conformal factor

Now, let us consider a fixed $\lambda_0 > 0$ (for instance) which does not belong to the Dirichlet spectrum of L_γ.

First, we choose a parameter $\alpha > 0$ such that $\frac{\alpha m}{2\alpha+1} < \lambda_0$. Using the previous Lemma (with $L_\gamma \leftrightarrow \frac{\alpha L_\gamma}{2\alpha+1}$) , we see that

$$\lambda_0 \in (\frac{\alpha m}{2\alpha+1}, +\infty) = W(\frac{\alpha L_\gamma}{2\alpha+1}) .$$

In particular, there exists $u \in X$ such that

$$\lambda_0 = \frac{\alpha}{2\alpha+1} \ < L_\gamma u, u > .$$

For $\epsilon > 0$ small enough, we define the positive conformal factor $c_\epsilon(x)$ on $\overline{\Omega}$ by

$$c_{\epsilon,\alpha}(x) = (1 + \epsilon u(x))^\alpha .$$

This conformal factor satisfies $c_{\epsilon,\alpha}(x) = 1, \ (\gamma \nabla c_{\epsilon,\alpha}(x)) \cdot \nu = 0$ on $\partial\Omega$.

Choice of f

For a suitable frequency $\lambda_{\epsilon,\alpha} > 0$ to be defined later, we set :

$$f_{\epsilon,\alpha} = -\frac{1}{\lambda_{\epsilon,\alpha}\ c_{\epsilon,\alpha}}\ \mathrm{div}(\gamma \nabla c_{\epsilon,\alpha}) + \frac{1}{c_{\epsilon,\alpha}^2} - 1.$$

By construction, our non-linear PDE (with $c \leftrightarrow c_{\epsilon,\alpha},\ f \leftrightarrow f_{\epsilon,\alpha},\ \lambda \leftrightarrow \lambda_{\epsilon,\alpha}$) :

$$\mathrm{div}\,(\gamma \nabla c) + \lambda(c - \frac{1}{c} + cf) = 0\,,$$

is satisfied and we have $f_{\epsilon,\alpha} \in C^\infty(\overline{\Omega})$. Now, we choose $\lambda_{\epsilon,\alpha} > 0$ in order to satisfy

$$\int_\Omega f_{\epsilon,\alpha}(x)\ dx = 0.$$

Using Green's formula, we easily get :

$$\lambda_{\epsilon,\alpha} = \frac{\int_\Omega \frac{\gamma \nabla c_{\epsilon,\alpha} \cdot \nabla c_{\epsilon,\alpha}}{c_{\epsilon,\alpha}^2}\ dx}{\int_\Omega \left(\frac{1}{c_{\epsilon,\alpha}^2} - 1 \right)\ dx},$$

In other words, we have solved our non-linear PDE "backwards" by suitably choosing f.

Some useful asymptotics

We can get easily the asymptotic expansion :

$$\lambda_{\epsilon,\alpha} = \frac{\alpha}{2\alpha+1} < L_\gamma u, u > + O(\epsilon) = \lambda_0 + O(\epsilon).$$

This is why we have considered (a posteriori) $\frac{\alpha L_\gamma}{2\alpha+1}$ instead of L_γ.

For ϵ small enough, we get :

- $\lambda_{\epsilon,\alpha} > 0$.
- for all $k \in \mathbb{N}$ and $\beta \in (0,1)$, $||f_{\epsilon,\alpha}||_{k,\beta} = O(\epsilon)$.
- $c_{\epsilon,\alpha}(x) = 1 + O(\epsilon)$.
- $\lambda_{\epsilon,\alpha}$ is not an eigenvalue $L_{c_{\epsilon,\alpha}\gamma}$.

As a consequence, for all $k \in \mathbb{N}$, there exists a C^{k+1} diffeomorphism $\psi_{\epsilon,\alpha}$ close to the identity such that :

$$\Lambda_{c^2_{\epsilon,\alpha}\gamma, \lambda_{\epsilon,\alpha}} = \Lambda_{(\psi_{\epsilon,\alpha})_* \gamma, \lambda_{\epsilon,\alpha}}.$$

Renormalization at frequency λ_0

Now, if we define the new conductivity :

$$\beta_{\epsilon,\alpha} = \frac{\lambda_0}{\lambda_{\epsilon,\alpha}} \, \gamma,$$

we get obviously :

$$\Lambda_{c^2_{\epsilon,\alpha}\beta_{\epsilon,\alpha},\lambda_0} = \Lambda_{(\psi_{\epsilon,\alpha})_*\beta_{\epsilon,\alpha},\lambda_0} \, .$$

Finally, since volume is an invariant under diffeomorphisms, we can show using the previous asymptotics that for ϵ small enough, the conductivities $c^2_{\epsilon,\alpha}\beta_{\epsilon,\alpha}$ and $(\psi_{\epsilon,\alpha})_*\beta_{\epsilon,\alpha}$ are not isometric.

And the proof is finished.

Thank you very much for your attention !