Causal fermion systems as an approach to non-smooth geometry

Felix Finster

Fakultät für Mathematik
Universität Regensburg

Johannes-Kepler-Forschungszentrum
für Mathematik, Regensburg

Talk at BIRS Workshop Granada, May 2025

What is a causal fermion system?

- approach to fundamental physics
- novel mathematical model of spacetime

  "quantum spacetime," "quantum geometry"
- physical equations are formulated in generalized spacetimes

  causal action principle, causal variational principles

How to get into the setting of causal fermion systems?

Let us begin with following setup:

- $S\mathscr{M}$ vector bundle over a smooth manifold $\mathscr{M}$ (spinor bundle, tangent bundle, complex line bundle, ...)
- Assume that each fiber $S_x\mathscr{M}$ is endowed with an inner product

$$\prec .|. \succ_x \;:\; S_x\mathscr{M} \times S_x\mathscr{M} \to \mathbb{C}\,.$$

- Consider a family (ψ_n) of sections (for example wave functions, vector fields, ...
- Assume that sections form a Hilbert space $\mathcal{H}$, endowed with scalar product $\langle .|. \rangle_{\mathcal{H}}$

$$\langle .|. \rangle_{\mathcal{H}} \;:\; \mathcal{H} \times \mathcal{H} \to \mathbb{C}\,.$$

How to get into the setting of causal fermion systems?

$$\prec . | . \succ_x \;:\; S_x\mathscr{M} \times S_x\mathscr{M} \to \mathbb{C}$$
$$\langle . | . \rangle_{\mathcal{H}} \;:\; \mathcal{H} \times \mathcal{H} \to \mathbb{C}$$

- For any $x \in \mathscr{M}$ introduce the local correlation operator $F(x)$ by

$$\langle \psi | F(x)\, \phi \rangle_{\mathcal{H}} := \prec \psi(x) \,|\, \phi(x) \succ \quad \forall\, \psi, \phi \in \mathcal{H}\,.$$

- This gives rise to a mapping

$$F : \mathscr{M} \to \mathcal{F} \subset \mathrm{L}(\mathcal{H})\,.$$

- Assume a volume measure $\mu_{\mathscr{M}}$ on $\mathscr{M}$. Introduce the push-forward measure,

$$\rho := F_*(\mu_{\mathscr{M}}) \qquad \text{(i.e. } \rho(\Omega) := \mu_{\mathscr{M}}\big(F^{-1}(\Omega)\big)\text{)}$$

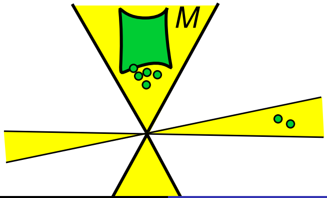
Definition. Let $(\mathcal{H}, \langle .|. \rangle_{\mathcal{H}})$ be Hilbert space
Given parameter $n \in \mathbb{N}$ ("spin dimension")
$\mathcal{F} := \Big\{ x \in \mathrm{L}(\mathcal{H})$ with the following properties:

- x is symmetric and has finite rank
- x has at most n positive
  and at most n negative eigenvalues $\Big\}$

ρ a measure on $\mathcal{F}$

$(\rho, \mathcal{F}, \mathcal{H})$ is a causal fermion system.

$M := \mathrm{supp}\, \rho$ is the spacetime of the causal fermion system.



Example: Dirac spinors in Lorentzian spacetime

Let $(\mathscr{M}, g)$ be a Lorentzian spacetime,
for simplicity globally hyperbolic,
4-dimensional, signature $(+,-,-,-)$,
then automatically spin,

$$(S\mathscr{M}, \prec.|.\succ) \qquad \text{spinor bundle}$$

- $S_p\mathscr{M} \simeq \mathbb{C}^4$
- spin inner product

$$\prec.|.\succ_p \;:\; S_p\mathscr{M} \times S_p\mathscr{M} \to \mathbb{C}$$

is indefinite of signature (2,2)

$$(\mathcal{D} - m)\psi_m = 0 \qquad \text{Dirac equation}$$

Example: Dirac spinors in Lorentzian spacetime

- Cauchy problem well-posed, global smooth solutions (for example symmetric hyperbolic systems)
- finite propagation speed

$C^\infty_{\text{sc}}(\mathcal{M}, S\mathcal{M})$ spatially compact solutions

$$(\psi_m | \phi_m)_m := \int_{\mathcal{N}} \prec \psi_m | \slashed{\nu} \phi_m \succ_x \, d\mu_{\mathcal{N}}(x) \quad \text{scalar product}$$

completion gives Hilbert space $(\mathcal{H}_m, (.|.)_m)$

Example: Dirac spinors in Lorentzian spacetime

- Choose $\mathcal{H}$ as a subspace of the solution space,

$$\mathcal{H} = \overline{\text{span}(\psi_1, \ldots, \psi_f)}$$

- To $x \in \mathbb{R}^4$ associate a local correlation operator

$$\langle \psi | F(x) \phi \rangle = - \prec \psi(x) | \phi(x) \succ_x \qquad \forall \psi, \phi \in \mathcal{H}$$

  Is symmetric, rank ≤ 4
  at most two positive and at most two negative eigenvalues

- Here ultraviolet regularization may be necessary:

$$\langle \psi | F(x) \phi \rangle = - \prec (\mathfrak{R}_\varepsilon \psi)(x) | (\mathfrak{R}_\varepsilon \phi)(x) \succ_x \qquad \forall \psi, \phi \in \mathcal{H}$$

$$\mathfrak{R}_\varepsilon \;:\; \mathcal{H} \to C^0(\mathscr{M}, S\mathscr{M}) \qquad \text{regularization operators}$$

$$\varepsilon > 0 : \text{regularization scale (Planck length)}$$

Example: Dirac spinors in Lorentzian spacetime

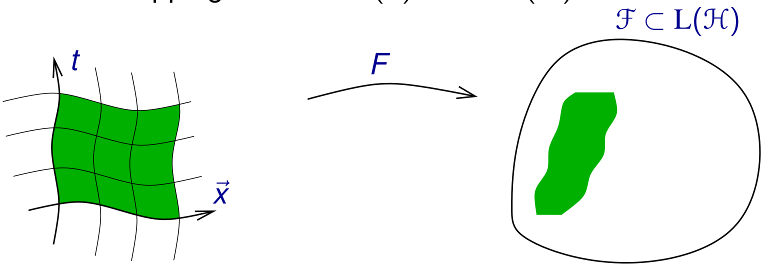
- Thus $F(x) \in \mathcal{F}$ where

  $$\mathcal{F} := \Big\{ F \in \mathrm{L}(\mathcal{H}) \text{ with the properties:}$$

  - ▷ F is symmetric and has rank ≤ 4
  - ▷ F has at most 2 positive and at most 2 negative eigenvalues $\Big\}$

Example: Dirac spinors in Lorentzian spacetime

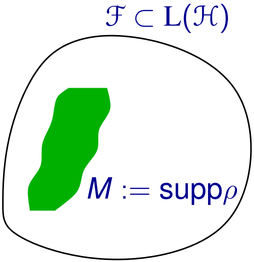
We obtain mapping $\quad x \mapsto F(x) \in \mathcal{F} \subset \mathrm{L}(\mathcal{H})$



Take push-forward measure

$$\rho := F_*(\mu_{\mathcal{M}}) \qquad \text{(i.e. } \rho(\Omega) := \mu_{\mathcal{M}}\big(F^{-1}(\Omega)\big)\text{)}$$

Example: Dirac spinors in Lorentzian spacetime



We thus obtain a causal fermion system of spin dimension two.

Causal action principle

Let $x, y \in \mathcal{F}$. Then x and y are linear operators.

$x \cdot y \in \mathrm{L}(H)$:

- rank $\leq 2n$
- in general not self-adjoint: $(x \cdot y)^* = y \cdot x \neq x \cdot y$

thus non-trivial complex eigenvalues $\lambda_1^{xy}, \ldots, \lambda_{2n}^{xy}$

Causal action principle

Nontrivial eigenvalues of xy: $\lambda_1^{xy}, \ldots, \lambda_{2n}^{xy} \in \mathbb{C}$

Lagrangian $\quad \mathcal{L}(x,y) = \frac{1}{4n} \sum_{i,j=1}^{2n} \left(|\lambda_i^{xy}| - |\lambda_j^{xy}| \right)^2 \;\geq\; 0$

action $\quad \mathcal{S} = \iint_{\mathcal{F}\times\mathcal{F}} \mathcal{L}(x,y)\, d\rho(x)\, d\rho(y) \;\in\; [0,\infty]$

Minimize $\mathcal{S}$ under variations of ρ, with constraints

volume constraint: $\quad \rho(\mathcal{F}) = \text{const}$

trace constraint: $\quad \int_{\mathcal{F}} \text{tr}(x)\, d\rho(x) = \text{const}$

boundedness constraint: $\quad \iint_{\mathcal{F}\times\mathcal{F}} \sum_{i=1}^{2n} |\lambda_i^{xy}|^2\, d\rho(x)\, d\rho(y) \leq C$

- F.F., "Causal variational principles on measure spaces," *J. Reine Angew. Math.* **646** (2010) 141–194

A few general remarks

One basic object: measure ρ on set $\mathcal{F}$ of linear operators on $\mathcal{H}$, describes spacetime as well as all objects therein

- Underlying structure: family of fermionic wave functions
- Geometric structures encoded in these wave functions

Matter encodes geometry
"Quantum spacetime"

The setting allows for the description of both continuum and discrete spacetimes.

- Causal action principle describes spacetime as a whole (similar to Einstein-Hilbert action in GR)
- Causal action principle is a nonlinear variational principle (similar to Einstein-Hilbert action or classical field theory)
- Linear dynamics of quantum theory recovered in limiting case (more details later)

Results of the theory

Continuum limit
(classical fields coupled to second-quantized Dirac field):

- interactions of the standard model (electroweak + strong)
- general relativity
- quantum mechanics

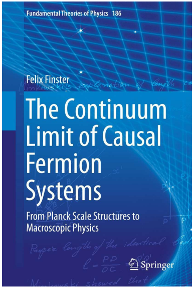
Other limiting case (more recently, with C. Dappiaggi, N. Kamran and M. Reintjes)

- quantum field theory
  (second-quantized fermionic and bosonic fields)

Analysis in the Continuum Limit

The above example of a Lorentzian spacetime is the starting point for continuum limit analysis:

- Consider Dirac systems in a classical bosonic field
  Are measures critical points in the limit $\varepsilon \searrow 0$?



Fundamental Theories of Physics **186**
Springer, 2016
548+xi pages

arXiv:1605.04742 [math-ph]

classical fields coupled to second-quantized Dirac field:

- interactions of the standard model (electroweak + strong)
- general relativity

The limiting case of classical GR

- Space-time goes over to a Lorentzian manifold
- The EL equations of the causal action principle give rise to the Einstein equations,

$$R_{jk} - \frac{1}{2}\,R\,g_{jk} + \Lambda\,g_{jk} = \kappa\,T_{jk} + \mathcal{O}\big(\ell_{\text{Planck}}^4\,\text{Riem}^2\big)$$

- $\kappa \sim \ell_{\text{Planck}}^2$ is determined by the length scale of the microscopic space-time structure.

How to go beyond classical GR?

- $M := \operatorname{supp} \rho$ no longer has a manifold structure
- no tensor equations
- Instead: Work directly with structures of causal fermion system
- In particular: EL equations of causal action principle still well-defined

Inherent structures of a causal fermion system

Let $(\rho, \mathcal{F}, \mathcal{H})$ be a causal fermion system of spin dimension n, base space $M := \text{supp}\,\rho$.

points in M are linear operators on $\mathcal{H}$

- For $x \in M$, consider eigenspaces of x.
- For $x, y \in M$,
  - consider operator products xy
  - project eigenspaces of x to eigenspaces of y

Gives rise to:

- vector bundles, sections therein
- geometric structures (connection, curvature)
- analytic structures

A Lorentzian quantum geometry

- A causal fermion systems has inherent geometric structures:
  - spinor space $(S_x, \prec.|.\succ_x)$,

$$S_x := x(\mathcal{H}) \subset \mathcal{H} \qquad \text{“spin space”}, \ \dim S_x \leq 2n$$

$$\prec u|v \succ_x : S_x \times S_x \to \mathbb{C}\,, \qquad \prec u|v \succ_x := -\langle u \,|\, x\, v \rangle_{\mathcal{H}}$$

  - Physical wave functions
    Let $u \in \mathcal{H}$ and $\pi_x : \mathcal{H} \to S_x$ orthogonal projection

$$\psi^u(x) := \pi_x\, u$$

  - kernel of fermionic projector

$$P(x,y) = \pi_x\, y|_{S_y} \ : \ S_x \to S_y$$

“gives relations between space-time points”

Inherent geometric structures

$P(x,y)\ :\ S_yM \to S_xM$ gives relations between spin spaces

idea: a polar decomposition gives

$$D_{x,y}\ :\ S_yM \to S_xM \text{ unitary} \qquad \text{spin connection}$$

holonomy of connection gives curvature

$$\mathfrak{R}(x,y,z) = \mathcal{D}_{x,y}\,\mathcal{D}_{y,z}\,\mathcal{D}_{z,x}\ :\ S_xM \to S_xM$$

Additional structures:

- tangent space T_xM, carries Lorentzian metric,

$$\nabla_{x,y}\ :\ T_yM \to T_xM \qquad \text{corresponding metric connection}$$

- spin and metric connections are compatible

→ F.F., A. Grotz, "A Lorentzian Quantum Geometry," arXiv:1107.2026 [math-ph], *Adv. Theor. Math. Phys.* **16** (2012) 1197-1290

Correspondence to Lorentzian spin geometry

Let $(\mathscr{M}, g)$ be a globally hyperbolic Lorentzian manifold.

Choose $P^\varepsilon(x,y)$ as *regularized Dirac sea structure*:

- ε is regularization scale
- regularization can be removed:

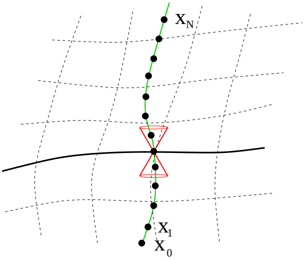
$$P^\varepsilon(x,y) \xrightarrow{\varepsilon \searrow 0} P(x,y)$$

where $P(x,y)$ is two-point distribution of Hadamard form

Correspondence to Lorentzian spin geometry

Theorem. In the limit $\varepsilon \searrow 0$:

- $D_{x,y}$ goes over to the metric spin connection.
  Curvatures gives the Riemann curvature tensor.



$$\lim_{N\to\infty} \lim_{\varepsilon\searrow 0} D_{x_N,x_{N-1}} D_{x_{N-1},x_{N-2}} \cdots D_{x_1,x_0}$$

$$= D^{\text{LC}}_{x,y} + \mathcal{O}\left(L(\gamma)\, \frac{\|\nabla R\|}{m^2} \right) \left(1 + \mathcal{O}\left(\frac{\mathfrak{s}}{m^2}\right)\right)$$

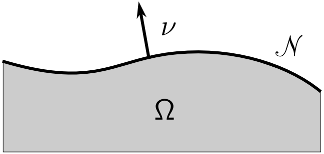
$\rightarrow$ F.F., A. Grotz, “A Lorentzian Quantum Geometry,” arXiv:1107.2026 [math-ph], *Adv. Theor. Math. Phys.* **16** (2012) 1197-1290

Synthetic Notions of curvature

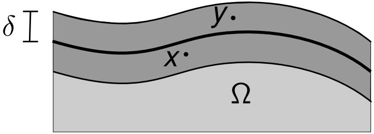
Instead of recovering notions of differential geometry, alternative approach: introduce synthetic notions of curvature:

- Compare volumes and areas (isoperimetric inequalities, positive mass, . . .)
- Compare interacting spacetimes with vacuum spacetime

Surface layer integrals



$$\int_{\mathcal{N}} \cdots \, d\mu_{\mathcal{N}}$$



$$\int_{\Omega} d\rho(x) \int_{M \setminus \Omega} d\rho(y) \, \cdots \, \mathcal{L}(x,y)$$

Here $(\cdots)$ stands for a suitable differential operator.

$\rightarrow$ F.F., J. Kleiner, "Noether-like theorems for causal variational principles," arXiv:1506.09076 [math-ph], *Calc. Var. Partial Differential Equations* **55:35** (2016)

The total mass of a static causal fermion system

Synthetic notions have been studied mainly in the static setting

- positive mass (generalizes ADM mass)
- positive quasi-local mass
- synthetic scalar curvature
- right now, we are studying isoperimetric flows using minimizing movements, ..., ...

→ F.F., A. Platzer, "A positive mass theorem for static causal fermion systems," arXiv:1912.12995 [math-ph], Adv. Theor. Math. Phys. **25** (2021) 1735–1818

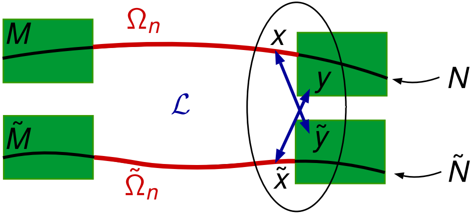
→ F.F., N. Kamran, N., "A positive quasilocal mass for causal variational principles," arXiv:2310.07544 [math-ph], Calc. Var. **64** (2025) 91pp

The total mass abstractly

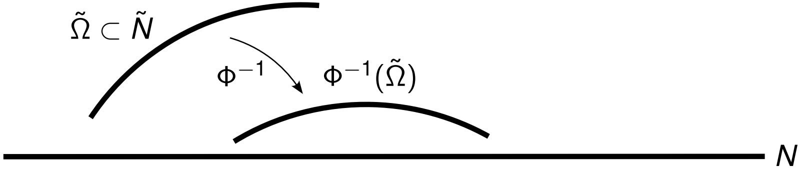
- Let $(\Omega_n)_{n \in \mathbb{N}}$ be exhaustion of N by compact sets, $(\tilde{\Omega}_n)_{n \in \mathbb{N}}$ exhaustion of $\tilde{N}$ with

$$\mu(\Omega_n) = \tilde{\mu}(\tilde{\Omega}_n) \quad \forall n$$

$$\mathfrak{M} := \lim_{n \to \infty} \bigg(\int_{\tilde{\Omega}_n} d\tilde{\mu}(\tilde{x}) \int_{N \setminus \Omega_n} d\mu(y)\, \mathcal{L}(\tilde{x}, y) - \int_{\Omega_n} d\mu(x) \int_{\tilde{N} \setminus \tilde{\Omega}_n} d\tilde{\mu}(\tilde{y})\, \mathcal{L}(x, \tilde{y}) \bigg)$$



The quasilocal mass



Φ an isometry of the Lagrangian in the sense that

$$\mathcal{L}\big(\Phi(x), \Phi(y)\big) = \mathcal{L}(x,y) \qquad \text{for all } x,y \in \mathcal{F} \,.$$

$$\mathfrak{M}(\tilde{\Omega}) := \inf \Big\{ \mathfrak{M}_{\tilde{\mu}, \Phi_* \mu}\big(\tilde{\Omega}, \Omega\big) \;\Big|\; \Phi \in \mathcal{G}, \Omega \subset \Phi(N) \text{ with } \Omega \text{ and } \tilde{\Omega} \text{ have the same volume}, \ldots \Big\}$$

Applications

Physical applications:

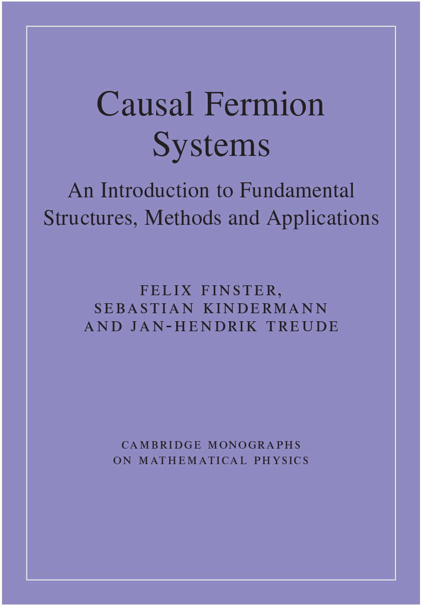
- baryogenesis (see poster by Marco van den Beld Serrano)
- wave function collapse, reduction of the wave functions
- corrections to classical field equations and quantum field theory, ...

Mathematical applications:

- singular limits of manifolds (with Niky Kamran and Olaf Müller)

$$\rho_n \to \rho \qquad \text{as measures on } \mathcal{F}$$

An introductory book



to appear in October in

Cambridge Monographs on Mathematical Physics
Cambridge University Press
483+xii pages

arXiv:2411.06450 [math-ph]

- Conference dedicated to causal fermion systems (both math and physics)
- introductory “summer school” at the beginning
- You can register at

www.causal-fermion-system.com/conference2025