A New Proof of the Classical Minkowski Inequality via a Divergence Identity

We present a new proof of the classical Minkowski inequality for a bounded, convex domain $\Omega \subset \mathbb{R}^n$ with $n \geq 3$ and smooth boundary using Robinson's method. Our main result is a parametric geometric inequality involving the p-capacitary potential for the domain Ω, which is derived from a divergence identity using nonlinear potential theory. From it, we obtain the L^p-Minkowski inequality, which yields the classical Minkowski inequality when taking the limit $p \to 1^+$. Additionally, this parametric geometric inequality allows us to also derive two new inequalities, which we dub weighted L^p-Minkowski and quantitative L^p-Willmore type inequality. Finally, we compare our approach using a divergence identity with that of Fogagnolo, Mazzieri, and Pinamonti where monotonicity formulas are employed.