Causal fermion systems as an approach to
non-smooth Lorentzian geometry

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Causal fermion systems:

- approach to *fundamental physics*
- Limiting cases:
  - classical field theory, general relativity
  - relativistic quantum field theory
- From mathematical perspective:
  - general framework for non-smooth spacetime geometry
  - analytic backbone: causal action principle

- The approach is fully quantum
  - basic object is a family of functions in spacetime (*physical wave functions*, complex-valued or spinorial)
  - geometry is encoded in this family of wave functions

How does one get into the setting of causal fermion systems?
Typical smooth setting:

- Suppose $(\mathscr{M}, g)$ is a *globally Lorentzian spin manifold*
- Let $(\psi_n)_{n=1,\ldots,N}$ be a family of solutions of the *Dirac equation*

$$(\mathscr{D} - m)\psi_n = 0$$

- Similarly: family of *scalar wave* or *Klein-Gordon fields*

What does one really need?

- ψ_n are sections of a vector bundle $S\mathscr{M}$,
  and fibre endowed with inner product

$$\prec . \,|\, . \succ_x : S_x\mathscr{M} \times S_x\mathscr{M} \to \mathbb{C}$$

- simplest case: $S\mathscr{M} = \mathbb{C} \times \mathscr{M}$ and

$$\prec \psi(x)|\phi(x) \succ = \overline{\psi(x)}\phi(x)$$

- (ψ_n) span *Hilbert space* $(\mathcal{H}, \langle . | . \rangle_{\mathcal{H}})$

- This makes it possible to introduce for any $x \in \mathscr{M}$ the local correlation operator $F(x) \in \mathrm{L}(\mathcal{H})$ by

$$\langle \psi | F(x) \phi \rangle_{\mathcal{H}} = - \prec \psi(x) | \phi(x) \succ_x \quad \forall \psi, \phi \in \mathcal{H}$$

- Then $F(x) \in \mathcal{F}$ and thus

$$F : \mathscr{M} \to \mathcal{F}$$

- Next assume a volume measure μ on $\mathscr{M}$ and set

$$\rho := F_* \mu \quad \text{push-forward measure on } \mathcal{F}$$

Definition (causal fermion system)

- Hilbert space $(\mathcal{H}, \langle . | . \rangle_{\mathcal{H}})$
- $n \in \mathbb{N}$ (*spin dimension*)
- $\mathcal{F} \subset \mathrm{L}(\mathcal{H})$ symmetric linear operators of rank at most $2n$,
  with at most n positive and at most n negative eigenvalues
- ρ a measure on $\mathcal{F}$
  Then $(\mathcal{H}, \mathcal{F}, \rho)$ is a causal fermion system

Definition (causal action principle)

$$\mathcal{L} : \mathcal{F} \times \mathcal{F} \to \mathbb{R}_0^+$$
$$\mathcal{S}(\rho) := \int_{\mathcal{F}} d\rho(x) \int_{\mathcal{F}} d\rho(y)\, \mathcal{L}(x,y)$$

Minimize $\mathcal{S}$ varying of ρ under certain constraints.

- $\mathcal{L}(x,y)$ *formed of eigenvalues* of operator product xy.

More details at
www.causal-fermion-system.com

Geometric structures of causal fermion system

- *spacetime* $M := \text{supp}\,\rho$ (topological space, subset of $\mathcal{F}$)

Definition (with A. Grotz, 2012)

$$\begin{aligned} S_x &:= x(\mathcal{H}) && \text{spin space} \\ \prec . | . \succ_x &:= \langle \,.\, | x .\, \rangle_{\mathcal{H}} && \text{spin inner product} \\ \nabla_{x,y} &: S_y \to S_x && \text{spin connection} \end{aligned}$$

Corresponding holonomoy gives curvature

- *Disadvantage:* these structures do not seem to harmonize with the causal action principle

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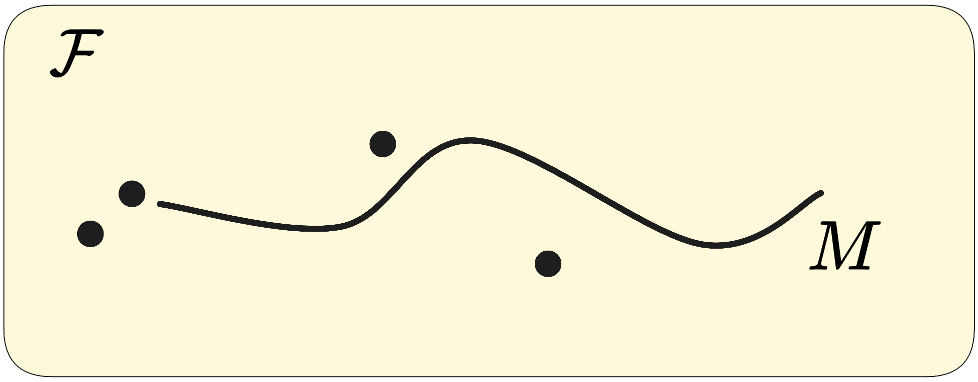
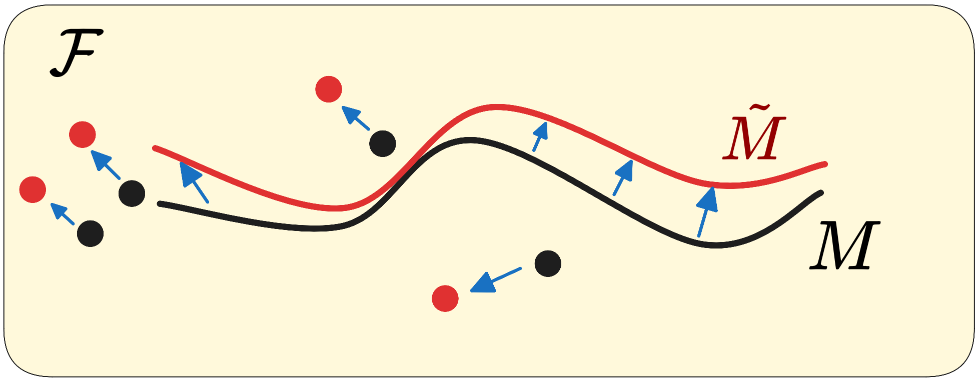
Connection to metric measure spaces

- *Basic structure:*
  - Measure space $(\mathcal{F}, \rho)$

- Lagrangian $\mathcal{L} : \mathcal{F} \times \mathcal{F} \to \mathbb{R}_0^+$
  Resembles *metric measure space*. But there are major differences:
- $M := \text{supp}\, \rho$ typically *singular* or *low-dimensional*



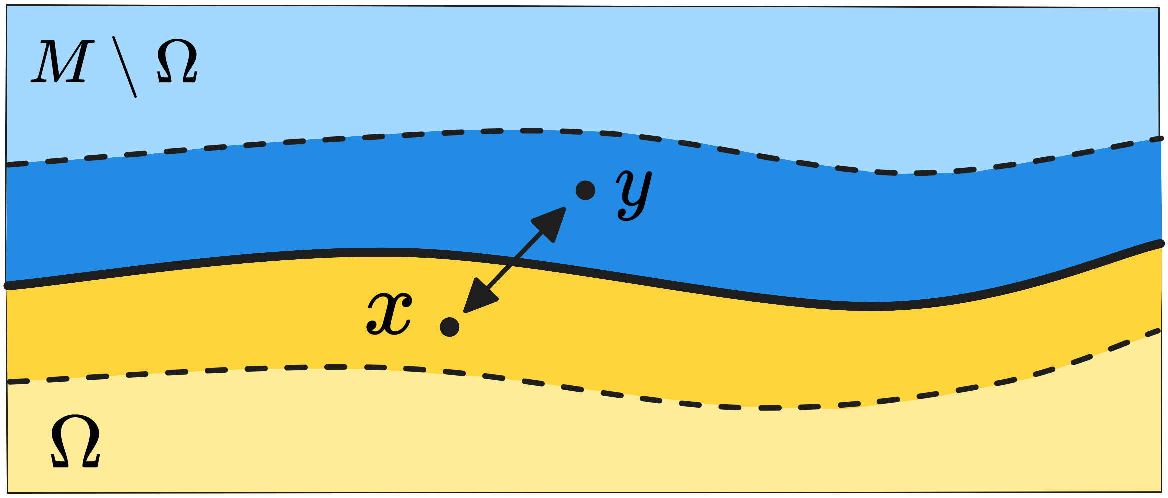
- When varying the measure, also its support changes:



- There are corresponding Euler-Lagrange equations
- There are Noether-like theorems, linearized equations, ...

- Moreover, $\mathcal{L}(x, y)$ is of short range
- Gives rise to structure of *surface layer integral*

$$\int_\Omega d\rho(x) \int_{M \setminus \Omega} d\rho(y)\, (\cdots) \mathcal{L}(x, y)$$

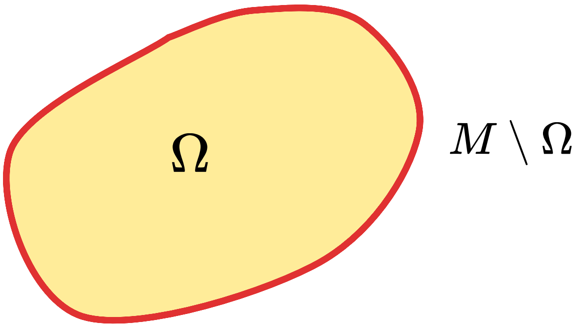


- Finally, it encodes the causal structure

$$x \text{ and } y \text{ are } \begin{cases} \text{timelike separated} & \text{if } \mathcal{L}(x,y) > 0 \\ \text{spacelike separated} & \text{if } \mathcal{L}(x,y) = 0 \end{cases}$$

- All these structures are related to each other via the causal action principle.

A positive quasi-local mass (with N. Kamran)

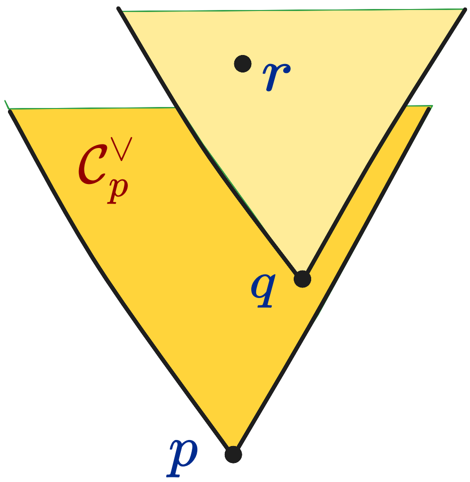


$$\begin{aligned} V &= \rho(\Omega) && \text{volume} \\ A &= \int_\Omega d\rho(x) \int_{N\setminus\Omega} d\rho(y)\, \mathcal{L}(x,y) && \text{area} \end{aligned}$$

- comparing volume and area with flat spacetime gives *positive mass density*
- ongoing project: isoperimetric flow, *scalar curvature*

Cone structures (with M. Kraus)

- use *analytic structures* (causal propagation speed of linearized field equations)



- $\mathcal{C}_p^\vee \subset M$ *future light cone*
- also gives transitive causal structure:

$$q \in \mathcal{C}_p^\vee \;\wedge\; r \in \mathcal{C}_q^\vee \quad \Longrightarrow \quad r \in \mathcal{C}_p^\vee$$

Cone structures II

- so far: all works in *general non-smooth, even discrete spacetimes*
- if M is assumed to have a smooth manifold structure, "localization" gives rise to corresponding local cone structures

$$J_p^\vee \subset T_pM \quad \text{convex cone structure}$$

(similar to Sánchez, Minguzzi, Bernard, Suhr, ...)

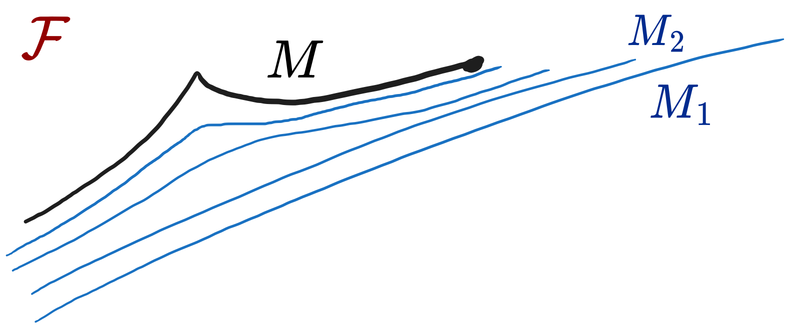
Cone structures III (with C. Paganini, M. van den Beld Serrano)

- go one step further: (M, g) globally hyperbolic spacetime
- $\mathcal{H}$ formed of *regularized Dirac sea configurations*
- Regularization described by *measure* μ_p *on* the boundary of light cone $L_p^\vee$
- ongoing work

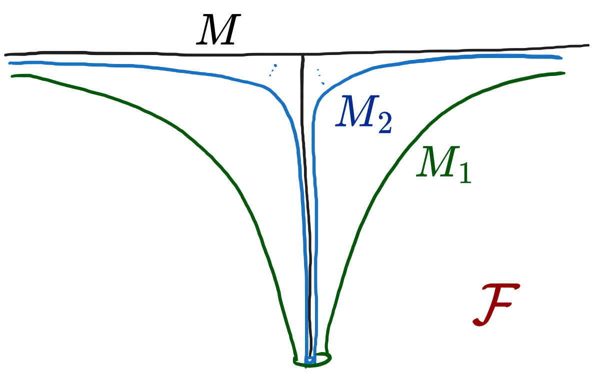
- gravity as statistical theory on the light cone (related to T. Phadmanabhan's *thermodynamical gravity*)
- this should give rise to a *Lorentz-Finsler metric*, ...

Convergence of sequences of spaces of spacetimes
(with O. Müller and N. Kamran)

- *Singular or non-regular limits* of spacetimes



- *Pinching off*: Schwarzschild metric



Convergence of sequences of spaces or spacetimes

- Consider family of causal fermion systems $(\mathcal{H}_n, \mathcal{F}_n, \rho_n)$
- Unitarily identify the Hilbert spaces

$$U_n : \mathcal{H}_n \to \mathcal{H} \quad \text{unitary}$$

- Measure convergence

$$\rho_n\big(U_n^{-1} \,.\, U_n\big) \to \rho$$

Example: Sequences of closed Riemannian manifolds

- (M_n, g_n) family of closed Riemannian manifolds
- $\mathcal{H}_n := L^2(M_n)$
- $\exp(\varepsilon^2 \Delta)$ heat kernel, with $\varepsilon > 0$ *regularization length*
- *local correlation operators*

$$F^\varepsilon(x) := e^{\varepsilon^2 \Delta}\, \delta_x\, e^{\varepsilon^2 \Delta} \;\in\; \mathrm{L}(\mathcal{H}_n)$$

  is symmetric, has rank at most one, thus $F_x \in \mathcal{F}$
- $\rho := F^\varepsilon_* \mu_n$ *push-forward measure*

Thank you for your attention

www.causal-fermion-system.com

Abstract of the talk

The theory of causal fermion systems is an approach to fundamental physics. From the mathematical perspective, it provides a general framework for desribing and analyzing non-smooth geometries.
Its analytic backbone is the causal action principle, a geometric variational principle which generalizes the Einstein-Hilbert action. Afer a few general remarks I want to focus on a few aspects which seem most relevant in the context of non-smooth Lorentzian geometry:

- spin connection and spin curvature of a causal fermion system
- positive mass, quasi-local mass and scalar curvature
- cone structures
- notions of convergence of sequences of spacetimes