qmetric: A tool to describe the small-scale structure of spacetime

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outline

consider spacetime endowed with existence of a minimum length
(i.e., with quadratic intervals -> finite limit at coincidence)
[minimum-length metric or quantum metric or qmetric]
Kothawala 1307.5618; Kothawala, Padmanabhan 1405.4967; JaffinoStargen, Kothawala 1503.03793

allow for this description to include null intervals
AP 1812.01275

apply it to black hole horizons
Krishnendu N V, S. Chakraborty, A. Perri, AP (ongoing work)

minimum-length metric

Kothawala 1307.5618; Kothawala, Padmanabhan 1405.4967; JaffinoStargen, Kothawala 1503.03793

existence of a minimum length L affects geometry itself in the small scale (i.e., not regarded as L-blurring of sources in an ordinary spacetime)

modification introduced in the quadratic interval $\sigma^2(x, x')$ (before g_{ab}):
$\sigma^2(x, x') \mapsto S(\sigma^2)$ with $S(\sigma^2) \to \epsilon L^2$ finite in the coincidence limit $x \to x'$
(with $S(\sigma^2) \approx \sigma^2$ when $|\sigma^2| \gg L^2$, i.e., when x is far apart from x)

for it, one needs a metric singular everywhere: how to deal with this?

we face the unavoidable nonlocality accompanying gravity in the smallest scales

convenience of nonlocal objects to describe this: use bitensors (just like $\sigma^2(x, x')$, which is a biscalar)

to require

$\sigma^2(x, x') \mapsto S(\sigma^2)$ with $S(\sigma^2) \to \epsilon L^2$ finite in the coincidence limit $x \to x'$

along the connecting geodesic, which such remains (with a same character) also in the new metric

implies

$$g_{ab}(x) \mapsto q_{ab}(x, x') = A\, g_{ab}(x) + \epsilon\, (1/\alpha - A)\, t_a(x)\, t_b(x)$$

$t_a =$ tangent vector

$\alpha = \alpha(\sigma^2), A = A(\sigma^2)$
biscalars

$\epsilon = g^{ab}\, t_a t_b = \pm 1$

q_{ab} turns out to be completely fixed if a condition is additionally posed on the 2-point function $G(x, x')$ of any field (namely, this is about causality):

one requires that, when spacetime is maximally symmetric,

$$G(\sigma^2) \mapsto \widetilde{G}(\sigma^2) = G(S(\sigma^2))$$

where

G and $\widetilde{G}$ are Green functions of $\Box$ and ${}_x\widetilde{\Box}_{x'}$ resp., and ${}_x\widetilde{\Box}_{x'}$ is the d'Alembertian associated to $q_{ab}(x, x')$

one gets:

Kothawala 1307.5618; Kothawala, Padmanabhan 1405.4967; JaffinoStargen, Kothawala 1503.03793

$$q_{ab}(x,x') = A\, g_{ab} + \epsilon\,(1/\alpha - A)\, t_a t_b$$

t_a unit tangent to connect. geod.

ϵ = -/+ 1 for time/space sep.

with

$$A = \frac{S}{\sigma^2}\left(\frac{\Delta}{\Delta_S}\right)^{\frac{2}{D-1}} \qquad \alpha = \frac{S}{\sigma^2\, S'^2}$$

(D-dim spt.)

$S' \equiv dS/d(\sigma^2)$

$$\Delta(x,x') = -\frac{1}{\sqrt{g(x)g(x')}}\det\left[-\nabla_a^{(x)}\nabla_b^{(x')}\frac{1}{2}\sigma^2(x,x')\right]$$

van Vleck determinant

$\Delta_S = \Delta(\tilde{x}, x')$ with $\tilde{x}$ such that $\sigma^2(\tilde{x},x') = S$ on the connecting geodesic

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q_{ab} is singular everywhere in the $x \to x'$ limit, and $q_{ab} \approx g_{ab}$ for x, x' far apart

null separations

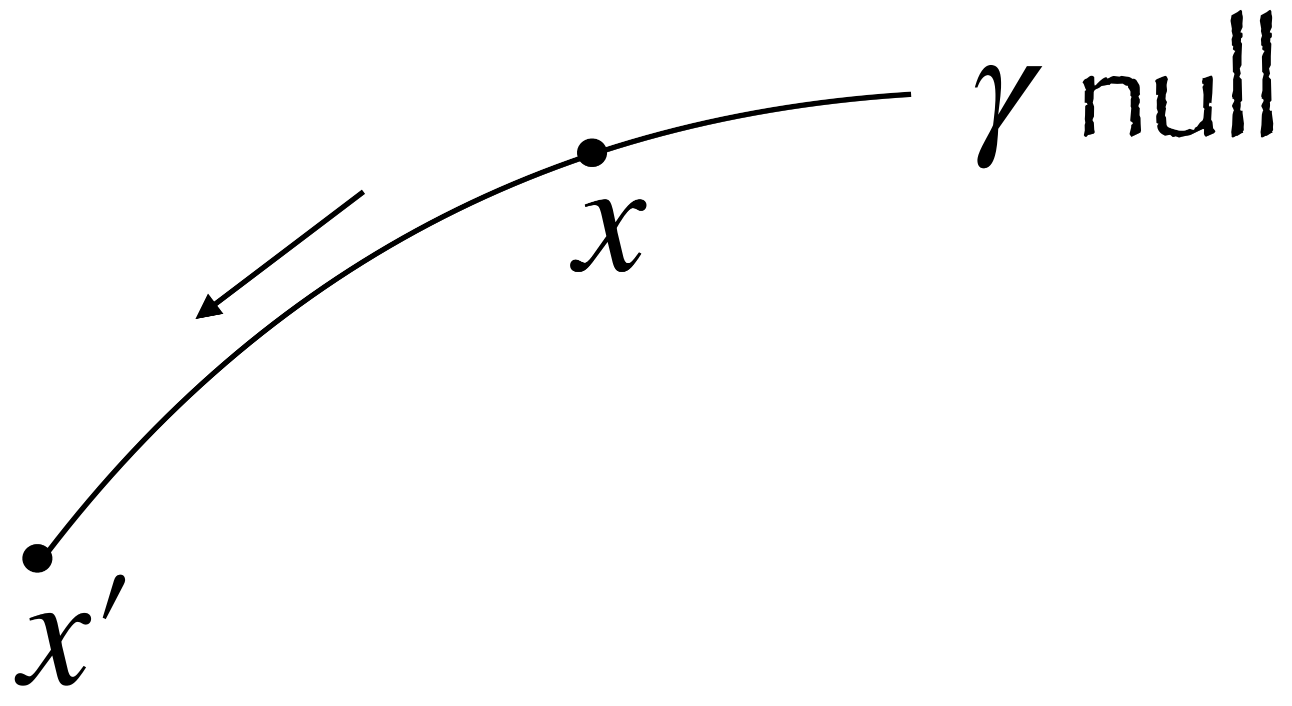
AP 1812.01275, 2207.12155

what's the meaning of a finite distance limit in this case?

key: affine λ = measure of distance by the canonical observer

qmetric:
this observer at x' will find a finite lower bound L to $\lambda - \lambda_{x'}$

take $\lambda_{x'} = 0$,
$\lambda \mapsto \tilde{\lambda}(\lambda)$, with $\tilde{\lambda} \to L$ when $\lambda \to 0$
(with $\tilde{\lambda}(\lambda) \approx \lambda$ when $\lambda \gg L$)



we seek $q_{ab}^{(\gamma)}$ of the form

$$q_{ab}^{(\gamma)}(x,x') = A_{(\gamma)}\, g_{ab}(x) + (A_{(\gamma)} - 1/\alpha_{(\gamma)})\,(l_a(x)n_b(x) + n_a(x)l_b(x))$$

$$A_{(\gamma)} = A_{(\gamma)}(\lambda)$$
$$\alpha_{(\gamma)} = \alpha_{(\gamma)}(\lambda)$$

n_a null with $l^a n_a = -1$

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from

$$\tilde{l}^b\, \widetilde{\nabla}_b\, \tilde{l}_a = 0$$

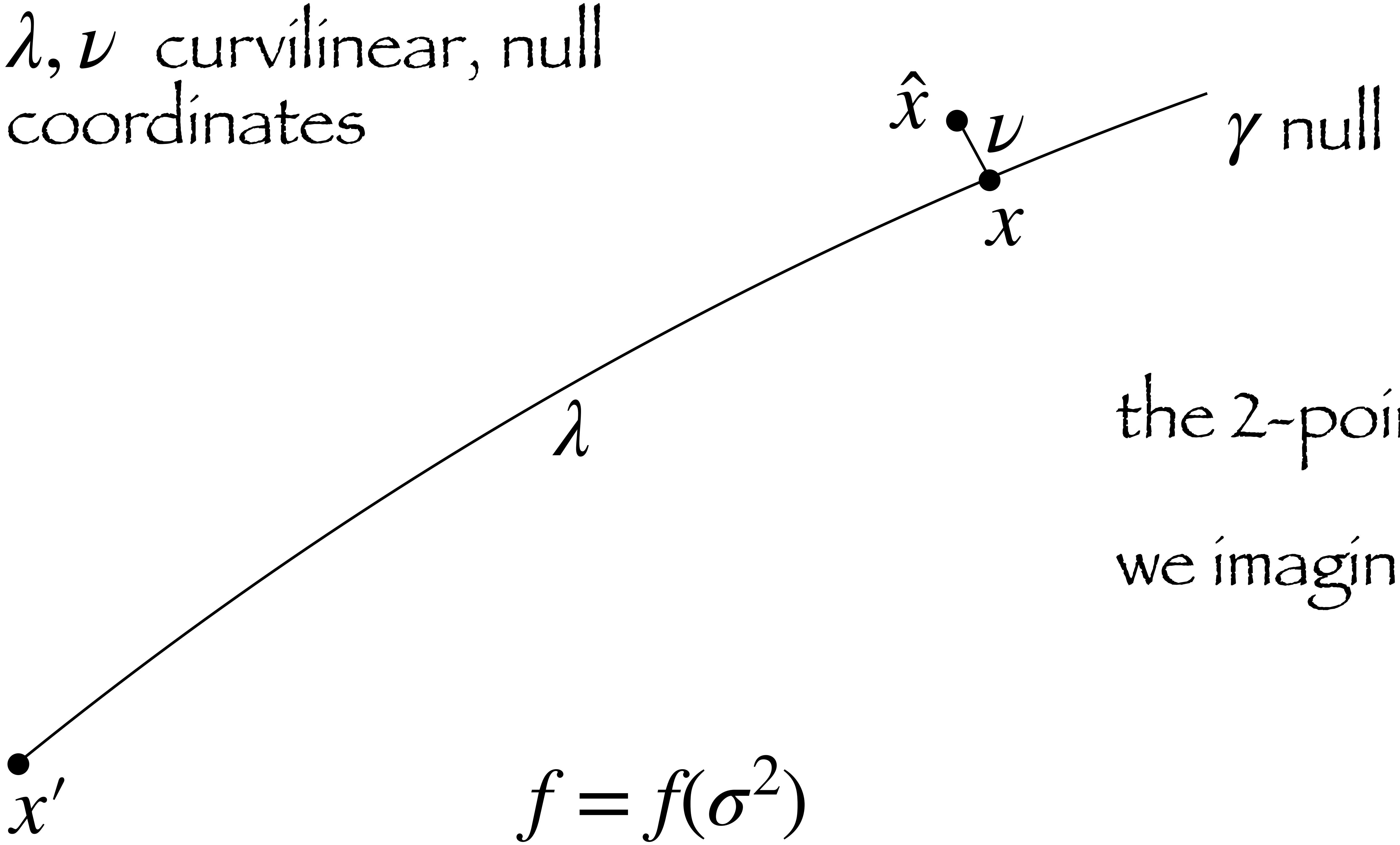
with $\tilde{l}^a = \dfrac{dx^a}{d\tilde{\lambda}} = l^a \dfrac{d\lambda}{d\tilde{\lambda}}$,

and $\widetilde{\nabla}_b \tilde{v}_a = \nabla_b \tilde{v}_a - \dfrac{1}{2} q^{cd}(-\nabla_d q_{ba} + 2\nabla_{(b} q_{a)d})\,\tilde{v}_c$

we obtain

$$\alpha_{(\gamma)} = \frac{C}{d\tilde{\lambda}/d\lambda},$$

with C real const.



the 2-point function $G(x, x')$ diverges on γ

we imagine to be slightly off γ

$$f = f(\sigma^2)$$

$$\Box f = (4 + 2\lambda \nabla_a l^a) \frac{df}{d\sigma^2} \quad \text{at } x \in \gamma \qquad l^a = \frac{dx^a}{d\lambda}$$

we implement then the d'Alembertian condition this way:

$\widetilde{G}(\sigma^2) = \widetilde{G}(S(\sigma^2))$ is solution of

$$(4 + 2\tilde{\lambda}\,\widetilde{\nabla}_a \tilde{l}^a)\,\frac{d\tilde{G}}{dS}_{|\tilde{\lambda}} = (4 + 2\tilde{\lambda}\,\widetilde{\nabla}_a \tilde{l}^a)\left(\frac{d\tilde{G}}{d\sigma^2}\right)_{|\lambda=\tilde{\lambda}} = 0 \qquad (1)$$

when

$G(\sigma^2)$ is solution of

$$(4 + 2\lambda\,\nabla_a l^a)\,\frac{dG}{d\sigma^2}_{|\lambda} = 0 \qquad (2)$$

using $\widetilde{\nabla}_b \tilde{l}_a$ and the expression for $\alpha_{(\gamma)}$ we already have, eq. (1) is

$$4 + 2\tilde{\lambda}\frac{d\lambda}{d\tilde{\lambda}}\nabla_a l^a{}_{|\lambda} + \tilde{\lambda}(D-2)\frac{d\lambda}{d\tilde{\lambda}}\frac{d}{d\lambda}\ln A_{(\gamma)} = 0$$

$D =$ spacetime dim.

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from (2) at $\tilde{\lambda}$, i.e., $4 + 2\tilde{\lambda}\nabla_a l^a{}_{|\tilde{\lambda}} = 0$,

and $\nabla_a l^a{}_{|\lambda} = \dfrac{D-2}{\lambda} - \dfrac{d}{d\lambda}\ln\Delta, \quad \nabla_a l^a{}_{|\tilde{\lambda}} = \dfrac{D-2}{\tilde{\lambda}} - \dfrac{d}{d\tilde{\lambda}}\ln\Delta_{\tilde{\lambda}},$

we obtain

$$\frac{d}{d\lambda}\ln\Big[\frac{\lambda^2}{\tilde{\lambda}^2}\Big(\frac{\Delta_{\tilde{\lambda}}}{\Delta}\Big)^{\frac{2}{D-2}}A_{(\gamma)}\Big] = 0$$

which is

$$A_{(\gamma)} = C' \, \frac{\tilde{\lambda}^2}{\lambda^2} \left(\frac{\Delta}{\Delta_{\tilde{\lambda}}} \right)^{\frac{2}{D-2}}, \qquad C' > 0 \ \text{const.}$$

———————

from

$$q_{ab}^{(\gamma)} = A_{(\gamma)}\, g_{ab} + (A_{(\gamma)} - 1/\alpha_{(\gamma)})\,(l_a n_b + n_a l_b) \ \approx \ g_{ab} \qquad \text{when } \lambda \gg L\,,$$

we get $C' = 1 = C$

———————

then, final expression is

$$q_{ab}^{(\gamma)} = A_{(\gamma)}\, g_{ab} + (A_{(\gamma)} - 1/\alpha_{(\gamma)})\,(l_a n_b + n_a l_b)$$

with

$$\alpha_{(\gamma)} = \frac{1}{d\tilde{\lambda}/d\lambda}, \qquad A_{(\gamma)} = \frac{\tilde{\lambda}^2}{\lambda^2} \left(\frac{\Delta}{\Delta_{\tilde{\lambda}}} \right)^{\frac{2}{D-2}}.$$

q_{ab} singular everywhere
when $x \to x'$

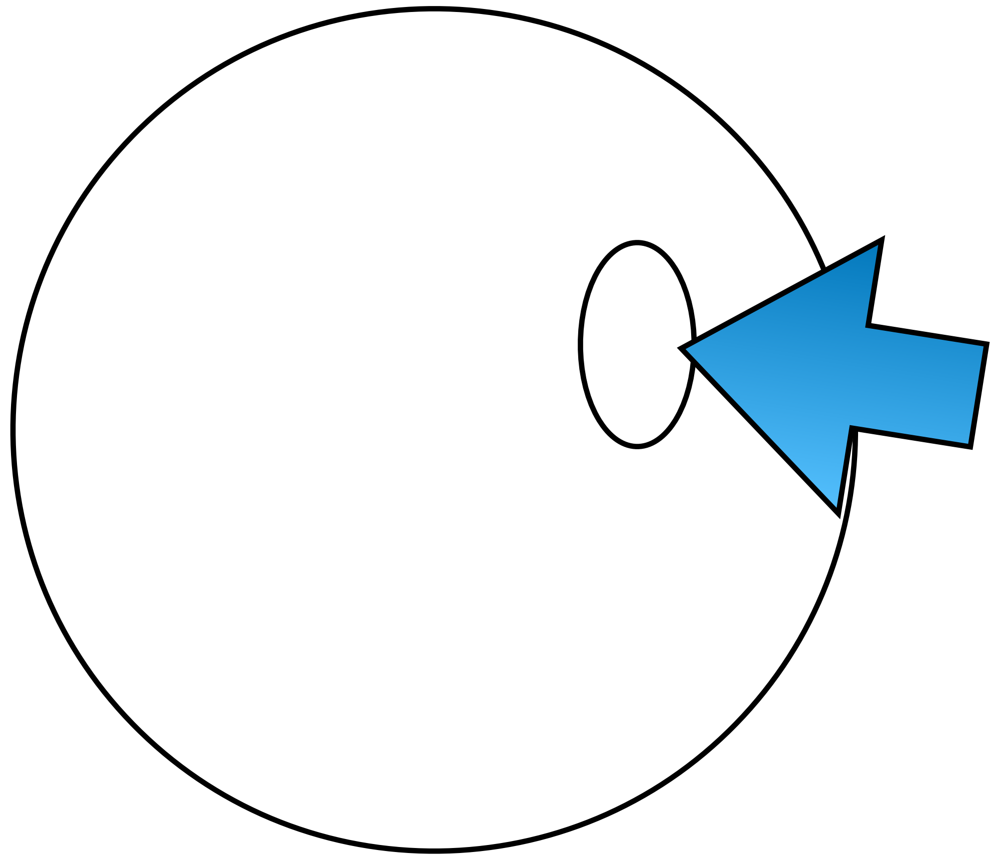
Ricci scalar

$$\lim_{x \to x'} \tilde{R}(x, x') = \epsilon \, D \, R_{ab} t^a t^b \; + \; O(L)$$ time/space sep.

Kothawala, Padmanabhan 1405.4967; JaffinoStargen, Kothawala 1503.03793

$$\lim_{x \to x'} \tilde{R}_{(\gamma)}(x, x') = (D - 1) \, R_{ab} l^a l^b \; + \; O(L)$$ null sep. AP 1911.04135

$\delta Q =$ heat flow through horizon



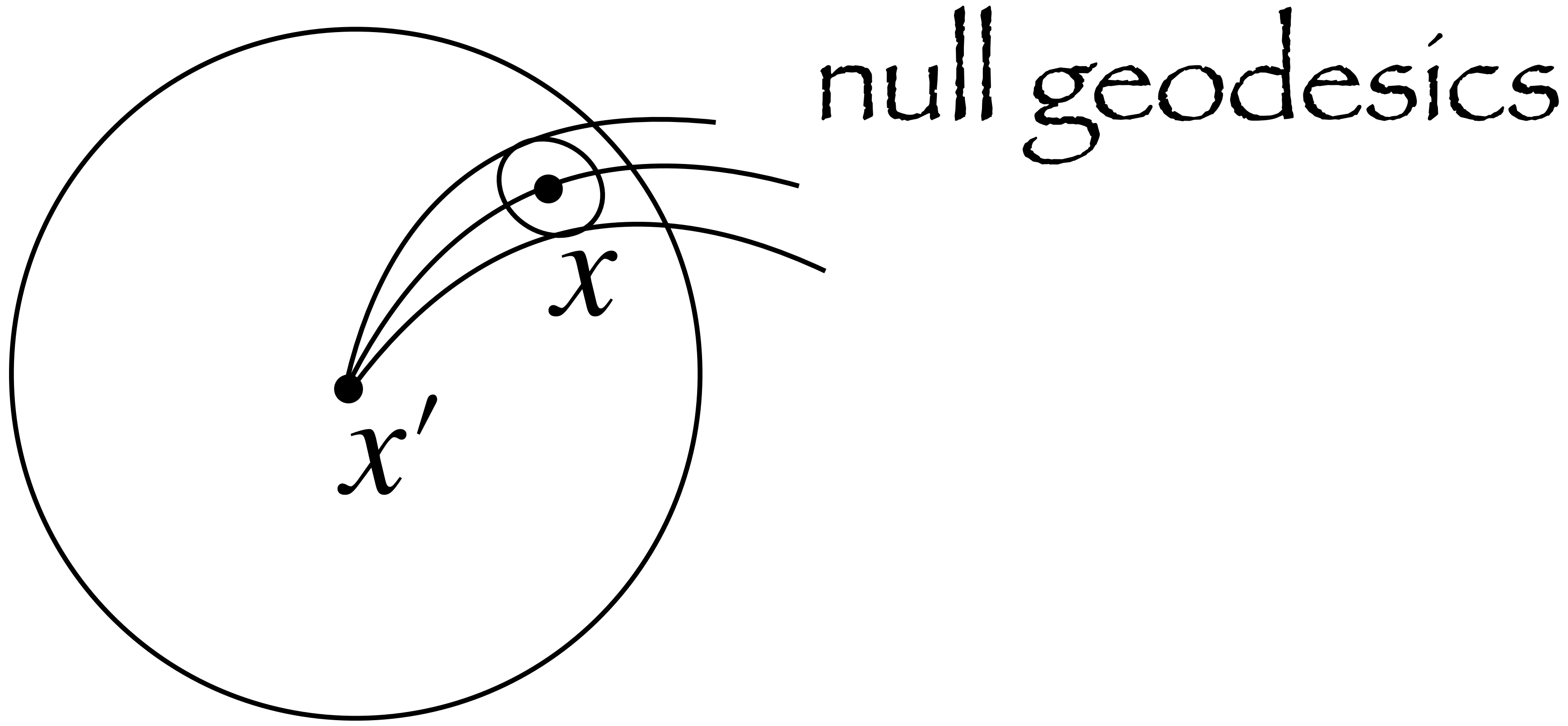
$$\delta Q = \lim_{x \to x'} \tilde{R}$$

qmetric introduces gravitational, local dofs
(geometric)

areas shrink to finite values

transverse metric:

$$\tilde{h}_{ab} = A_{(\gamma)} h_{ab}$$



$$d^{D-2}\tilde{a}(x) =$$

$$= \sqrt{\det \tilde{h}_{ab}(x) / \det h_{ab}(x)}\; d^{D-2}a(x) = \sqrt{\det \tilde{h}_{ab}(x) / \det h_{ab}(x)}\; \lambda^{D-2} d\Omega_{(D-2)} =$$

$$= \tilde{\lambda}^{D-2} \frac{\Delta}{\Delta_{\tilde{\lambda}}} d\Omega_{(D-2)} \;\to\; L^{D-2} \frac{1}{\Delta_{\tilde{\lambda}=L}} d\Omega_{(D-2)} \approx L^{D-2}\, d\Omega_{(D-2)}$$

for $x \to x'$

$$\Delta_{\tilde{\lambda}=L} = 1 + \frac{1}{6} L^2 R_{ab} l^a l^b + \ldots$$

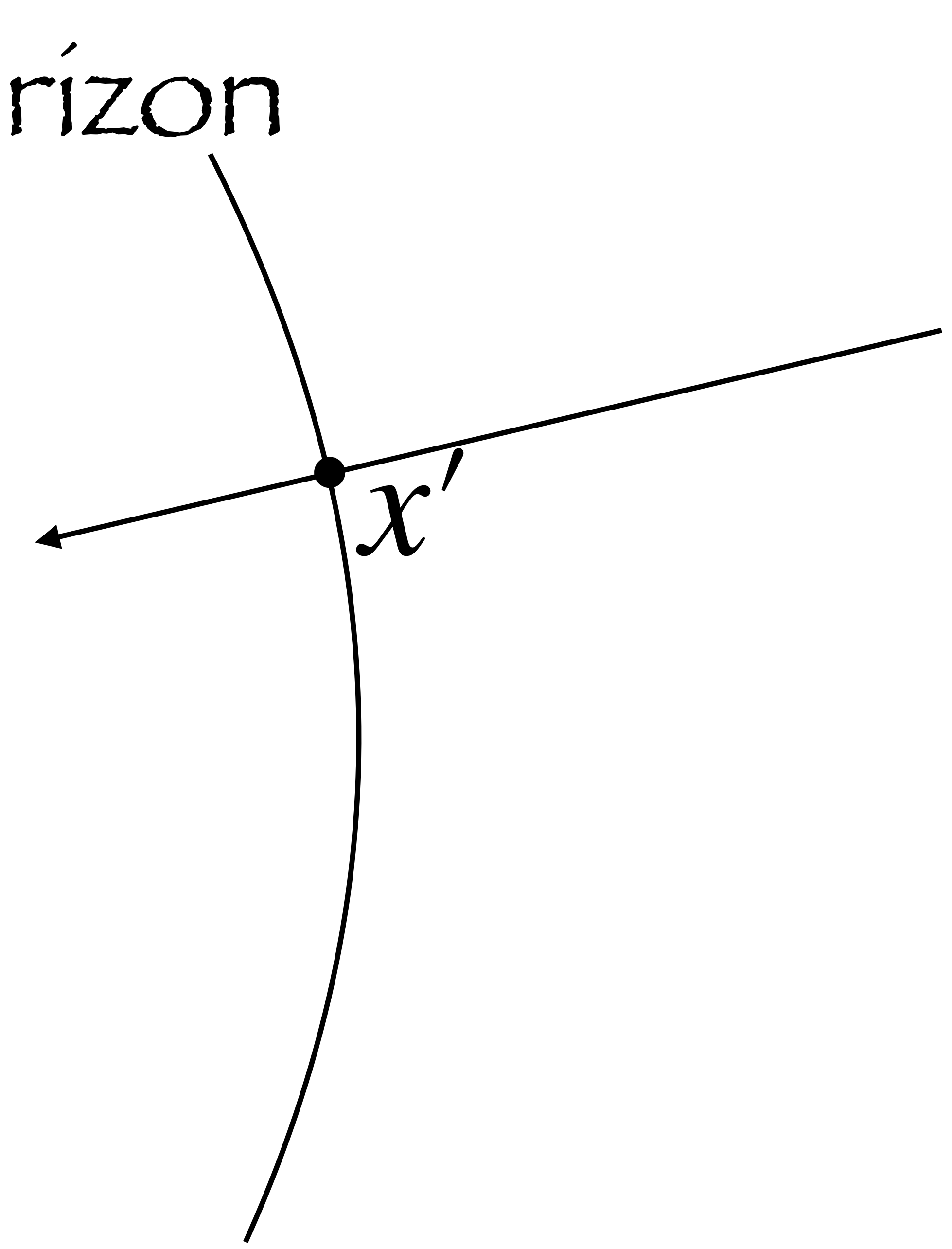
which is finite for a given $d\Omega_{(D-2)}$

Kothawala 1406.2672 ; Padmanabhan 1508.06286; AP 1812.01275

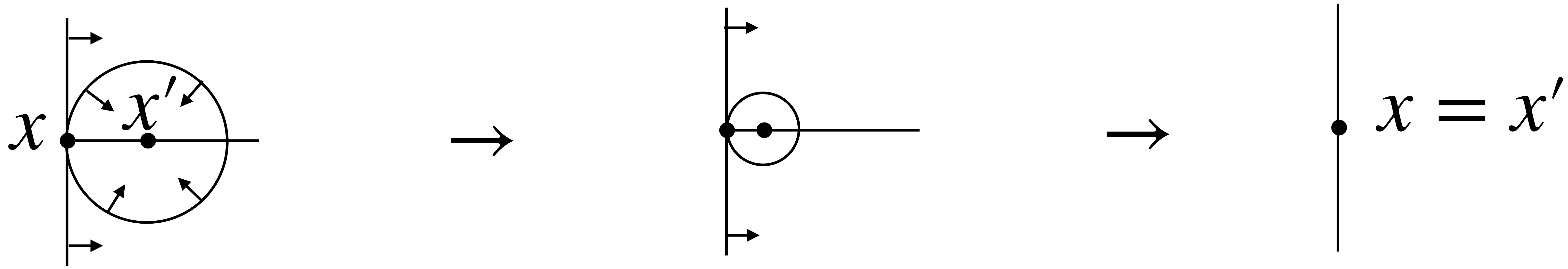
use on horizons

(Krishnendu N V (ICTS, Bengaluru), S. Chakraborty (IACS, Kolkata), A. Perri (Bologna), AP)

x' = event of crossing of the horizon by some chunk of energy



we describe the coincidence event in local Gullstrand-Painleve' at x'



horizon irreducible area
$T \ll -L$
x'
X
c
$\lambda \approx \tilde{\lambda}(\lambda) \gg L$
horizon at $\tilde{\lambda}\,(> L)$
horizon at $\lambda\,(\approx L)$
$T \approx -L$
$\tilde{\lambda}(\lambda) > L$
x'
X
$\lambda \approx L$
horizon at $\tilde{\lambda}\,(\gtrsim L)$
horizon at $\lambda\,(\ll L)$
$T < 0, |T| \ll L$
X
$\tilde{\lambda} \gtrsim L \gg \lambda$
$\lambda \ll L$
COINCIDENCE
horizon at $\tilde{\lambda} = L$
horizon at $\lambda = 0$
$T = 0$
irreducible area, $\tilde{\lambda} = L$
hor. irreducible area
X
$x' = x$, $\lambda = 0$
RELAXATION
$T = T_1 > 0$
$T = T_2 > T_1$

this means

$$A \to A' = A + 4\pi L^2$$

minimum step: $\delta A_{\min} = 4\pi L^2 = 4\pi\beta^2 l_p^2 = 4\pi\beta^2\hbar$ $\beta \equiv L/l_p$

energy conservation $\Rightarrow$ threshold energy E_0 to have absorption;
for energies $E < E_0$, no absorption

induced reflectivity $\mathscr{R} \neq 0$:

$$\mathscr{R}(\omega) = 1 \qquad \omega < \omega_0$$

$$\mathscr{R}(\omega) = 0 \qquad \omega \geq \omega_0$$

$$\omega_0 = E_0/\hbar$$

from

$$\delta M = \frac{1}{8\pi}\kappa\left(\delta A + 4\pi\,\frac{\delta(J^2)}{\sqrt{M^4 - J^2}}\right)$$

$$\kappa = \frac{r_+ - M}{r_+^2 + J^2/M^2}$$ surf. grav.

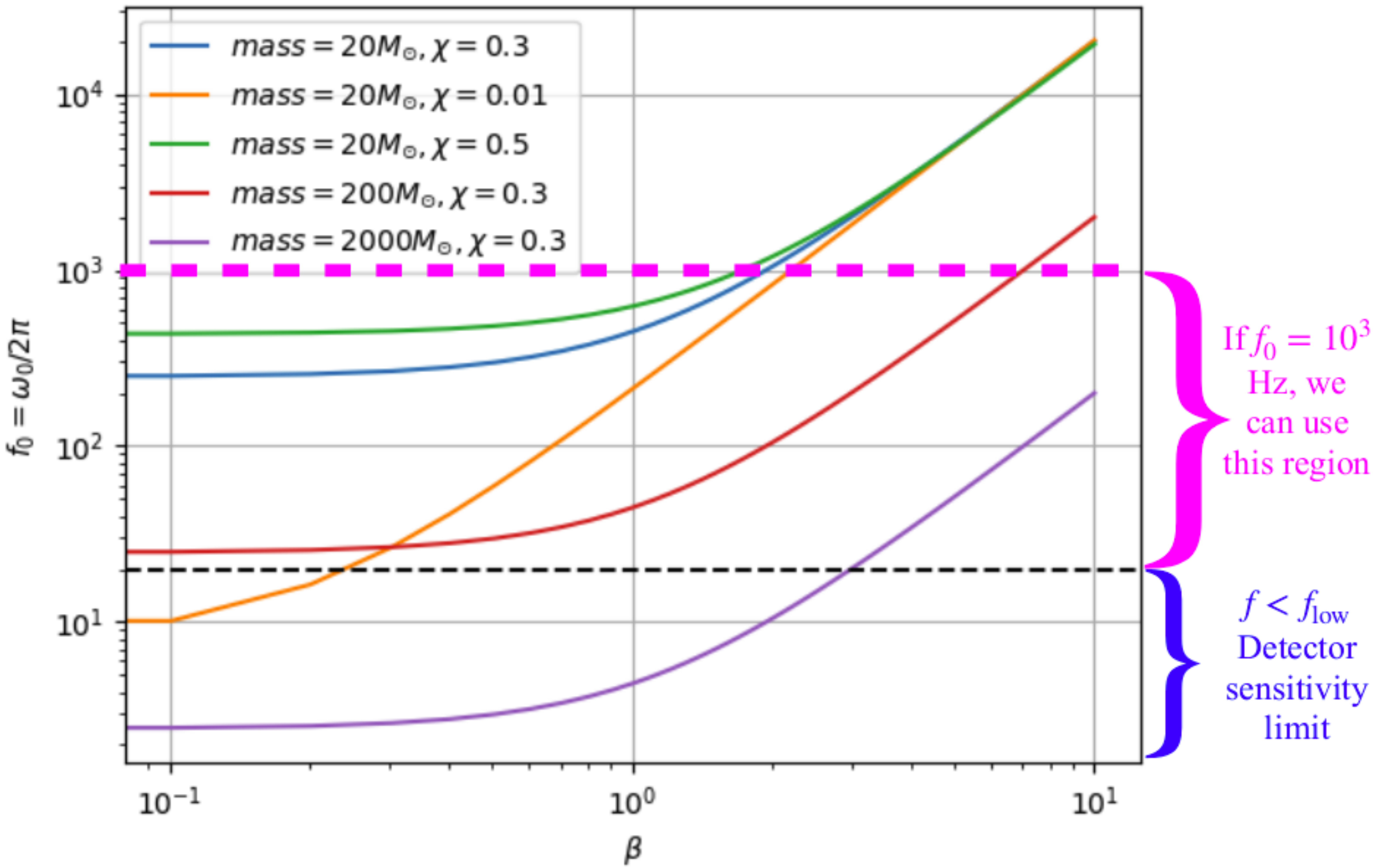
$$r_+ = M + \sqrt{M^2 - J^2/M^2}$$ outer hor.

we get $(\delta A = \delta A_{\min}, \delta J_{(\min)} = 2\hbar, E_0 = \delta M_{\min})$

$$\omega_0 = \frac{\kappa}{2}\beta^2 + 2\,\Omega = \kappa\left(\frac{\beta^2}{2} + \frac{2}{\sqrt{(M^2/J)^2 - 1}}\right)$$

$$\Omega = \frac{J/M}{2\,M\,r_+}$$ ang. vel. at the hor.

Region of parameter space that we can constrain: just from the frequency estimations

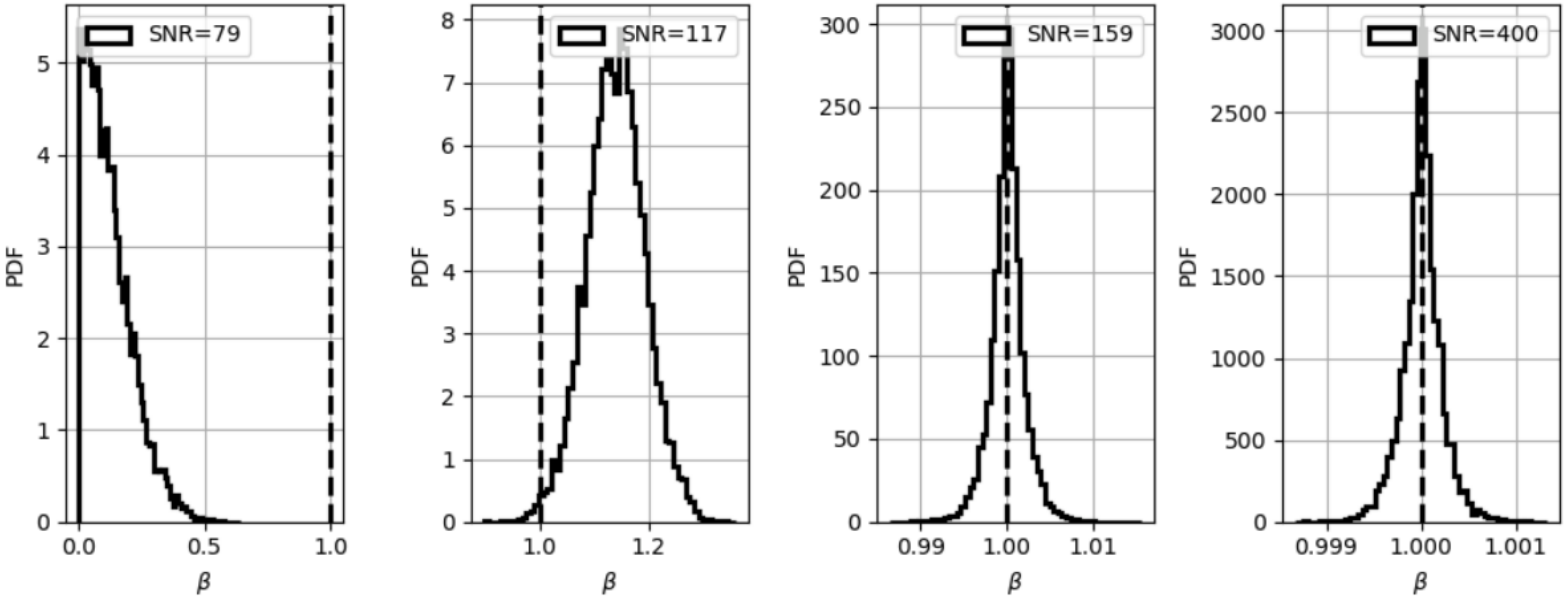


$$\chi = \frac{J}{M^2}$$

<u>If the BH mass is in the stellar mass range, highly spinning cases will have more constraining power</u>

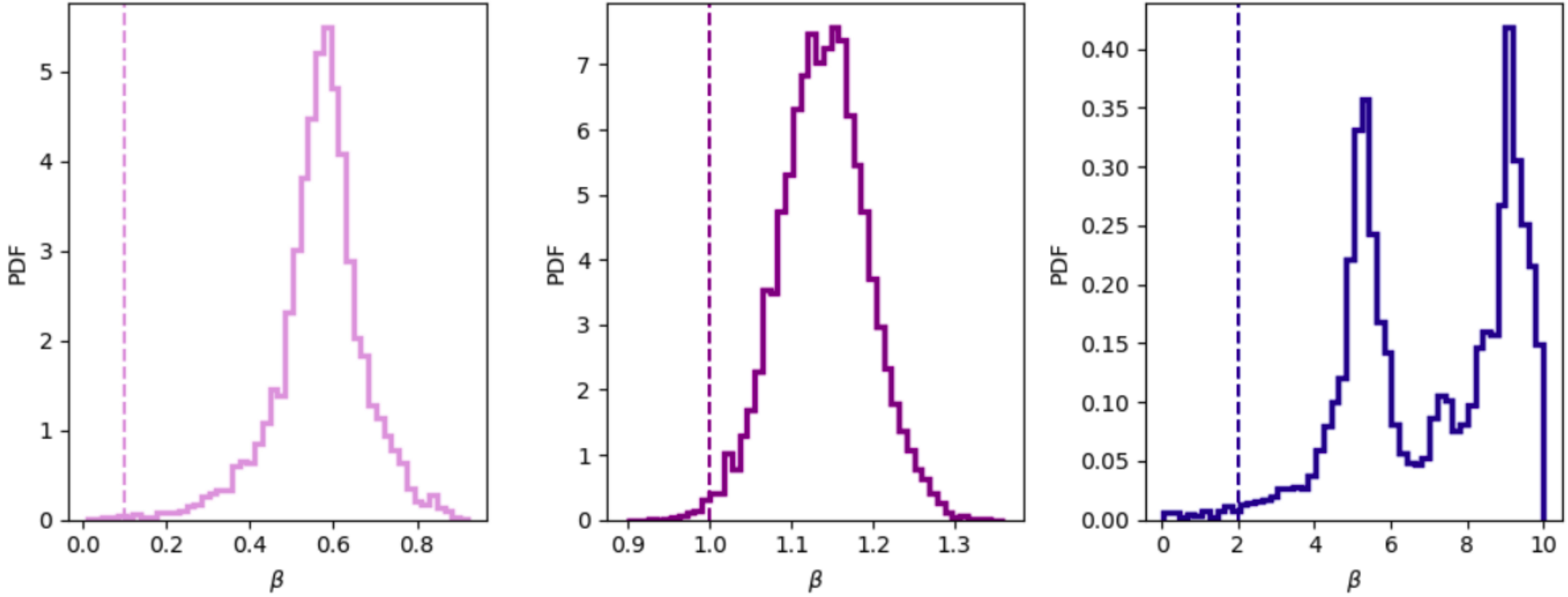
(from Krishnendu)

Posteriors: Effect of SNR $\{\beta, m_i, \chi_i\}$



(from Krishnendu)

Different injected values $\{\beta, m_i, \chi_i\}$, SNR = 119



(from Krishnendu)

in conclusion

-qmetric: tool to investigate imprints of quantum gravity from limit length

-it can be usefully considered also for null separated events

-when applied to horizons, it shows existence of a limit step in area increase

-this induces $\mathscr{R} \neq 0$ below a given threshold energy E_0

-$\omega_0 = E_0/\hbar$ is in the sensitivity range of ground-based GW detectors
for fast spinning black holes