Non-regular spacetime geometry via metric geometry and optimal transport

Working Seminar "Mathematical Physics"
University of Regensburg

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Lorentzian geometry (1/2)

Definition

(M, g) *Lorentzian manifold*, M smooth, connected manifold, g Lorentzian metric, i.e., a symmetric, non-degenerate $(0, 2)$ tensor with signature $(-, +, +, +, \dots)$, usually g smooth

Definition

(M, g) Lorentzian manifold, $v \in T_pM$ is

$$\begin{cases} \textit{timelike} \\ \textit{null} \\ \textit{causal} \\ \textit{spacelike} \end{cases} \quad \text{if} \quad g_p(v,v) \quad \begin{cases} < 0 \\ = 0 \text{ and } v \neq 0 \\ \leq 0 \text{ and } v \neq 0 \\ > 0 \text{ or } v = 0 \end{cases}$$

analogously for curves into M of sufficient regularity

length of a curve γ: $L_g(\gamma) := \int_a^b \sqrt{|g_{\gamma(s)}(\dot\gamma(s), \dot\gamma(s))|}\, \mathrm{d}s$

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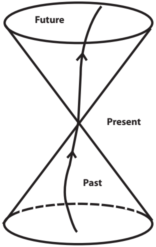

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(M,g) *spacetime*: (M,g) Lorentzian manifold, *time-oriented*, i.e., $\exists$ timelike vector field T

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v causal is *future directed* if $g_p(v, T(p)) < 0$

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Causal relations: $p \ll q :\Leftrightarrow \exists$ f.d. timelike curve from p to q, $I^+(p) := \{q \in M : p \ll q\}$
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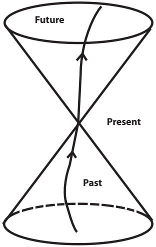

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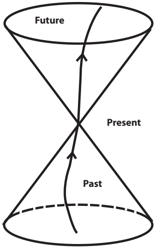

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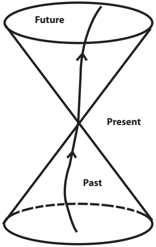

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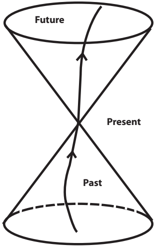

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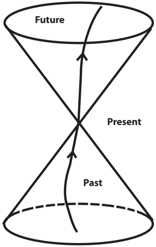

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Motivation (1/3) - Curvature

Einstein equations relate curvature to matter

$$\mathrm{Ric} - \frac{1}{2} R\, g = \frac{8\pi G}{c^4} T$$

Classical *curvature* as (2nd) derivative of the metric (Riemann tensor)
$\rightsquigarrow$
what if the metric is *non-smooth* (non-regular)? as e.g.

- *PDE* point-of-view, physically relevant *models* (matched spacetimes, shock waves, impulsive gravitational waves, etc.)
- approaches to *Quantum Gravity* (no metric)
- *singularities* vs *inextendibility* vs *curvature blow-up* — *cosmic censorship hypothesis* of Penrose

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Motivation (2/3) - How to detect curvature?

Triangle comparison:

Theorem (Toponogov)

(smooth) Riemannian manifold has $Sec(g) \geq K$ ($\leq$) if $\forall \triangle abc$ (small enough), p, q on the sides of $\triangle abc$

$$d(p,q) \geq \bar{d}(\bar{p},\bar{q}) \qquad (d(p,q) \leq \bar{d}(\bar{p},\bar{q}))$$

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(smooth) semi-Riemannian manifold has $Sec(g) \geq K$ ($\leq$) if *spacelike* sectional curvatures $\geq K$ ($\leq$) and *timelike* sectional curvatures $\leq K$ ($\geq$)

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Riemannian manifolds	$\subsetneq$	metric spaces
Lorentzian manifolds / spacetimes	$\subsetneq$	?

analog of metric space in the *Lorentzian setting*?

$\rightsquigarrow$ *Lorentzian (pre-)length spaces* (Kunzinger-S. '18)

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Lorentzian (pre-)length spaces

X (metrizable) topological space, $\ell\colon X \times X \to \{-\infty\} \cup [0, \infty]$ such that $\ell(x, x) \geq 0\ \forall x \in X$, $\tau := \max(\ell, 0)$, $\ll := \ell^{-1}((0, \infty))$, $\leq := \ell^{-1}([0, \infty))$

Definition

(X, ℓ) is a *Lorentzian pre-length space* if

$$\tau(x, z) \geq \tau(x, y) + \tau(y, z) \qquad (x \leq y \leq z)\,,$$

and τ is l.s.c.
ℓ and τ called *(extended) time separation function*

- *smooth spacetimes* (M, g) with usual time separation function $\ell(p, q) := \sup\{L_g(\gamma) : \gamma$ f.d. causal from p to $q\}$
- *Lorentz-Finsler* spacetimes, spacetimes of *low regularity*
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- Lorentz cylinder $S^1_1 \times \mathbb{R}$: every non-constant locally Lipschitz curve is timelike and causal $\rightsquigarrow$ need causality conditions
- Minkowski spacetime $\mathbb{R}^3_1$: $t \mapsto (t, \cos(t), \sin(t))$ has null tangent but is timelike

Proposition (Kunzinger-S '18)

continuous, strongly causal spacetimes: *different notions* of causal curves *agree*

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Length and maximal causal curves

Definition

$\gamma\colon [a,b] \to X$ f.d. causal, *τ-length* defined by

$$L_\tau(\gamma) := \inf\{\sum_{i=0}^{N-1} \tau(\gamma(t_i), \gamma(t_{i+1})) : a = t_0 < t_1 < \ldots < t_N = b\}$$

- $t \mapsto (t, \cos(t), \sin(t))$ timelike but *τ-length zero*
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- γ is *maximal* if $L_\tau(\gamma) = \tau(\gamma(a), \gamma(b)$

Proposition (Kunzinger-S '18)

$(M, \ll, \leq, \tau)$ the Lorentzian pre-length space induced by a *smooth* and *strongly causal* spacetime (M, g), then $L_\tau(\gamma) = L_g(\gamma)$

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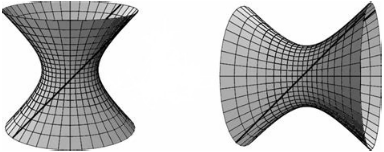
Lorentzian model spaces of constant curvature $K \in \mathbb{R}$

$$M_K = \begin{cases} \tilde{S}^2_1(r) & K = \frac{1}{r^2} \\ \mathbb{R}^2_1 & K = 0 \\ \tilde{H}^2_1(r) & K = -\frac{1}{r^2} \end{cases}$$



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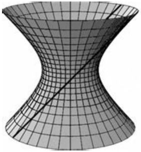

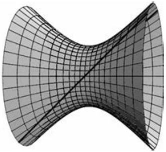

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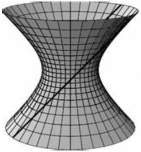

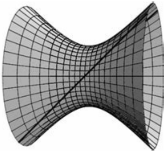

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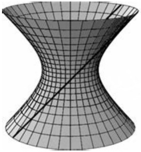

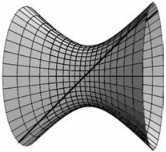

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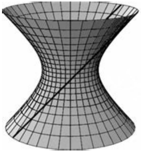

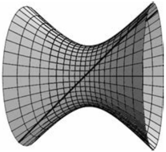

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Timelike curvature via triangle comparison

Definition

Lorentzian pre-length space X has *timelike curvature bounded below (above)* by $K \in \mathbb{R}$ if all points in X have nhd. U s.t.:

1. $\tau|_{U\times U}$ *finite* and *continuous*
2. $x, y \in U$ with $x \ll y \Rightarrow \exists$ f.d. *maximal causal curve* in U from x to y
3. Δxyz *small timelike geodesic triangle* in U, $\Delta\bar{x}\bar{y}\bar{z}$ *comparison triangle* of Δxyz in M_K, then for p, q points on the sides of Δxyz and $\bar{p}, \bar{q}$ *corresponding points* in $\Delta\bar{x}\bar{y}\bar{z}$:

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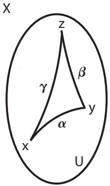

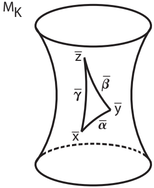

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Beran, Kunzinger, Ohanyan, Rott '24: *smooth Lorentzian manifold* with *timelike sectional curvature bounds*
TLCBB implies TL *non-branching*

Optimal transport and (Ricci) curvature

- *Optimal Transport*: Monge, Kantorovich, move matter in the cheapest / optimal way from A to B
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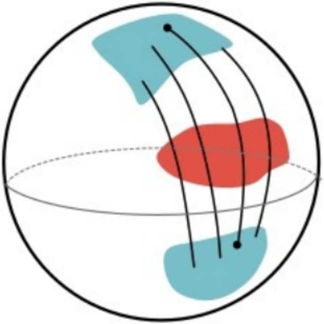
- What is *optimal*? depends on *distances* and *geometry*!
- Turn this on its head to define *curvature* by requiring that optimal transport behaves as in model spaces

Transporting *clouds* of points on the sphere

in the *Lorentzian* case

cost $c = \tau$ *time separation*



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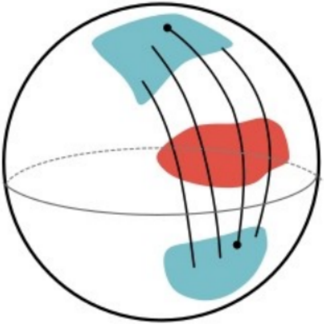

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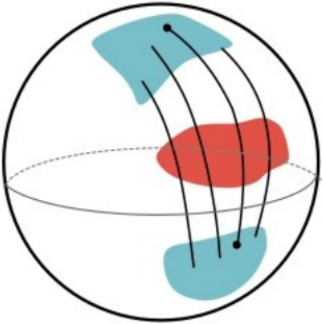

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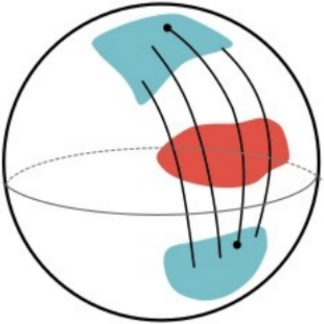

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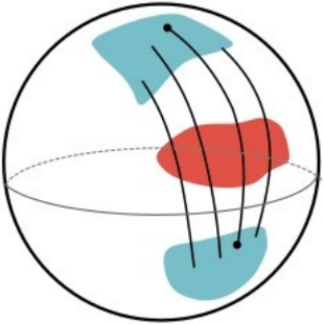

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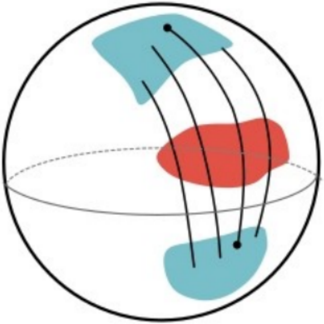

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2. *Generalized cones*: Lorentzian warped products of metric spaces with 1-dim base and singularity thms (Alexander-Graf-Kunzinger-S. '23)
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preprint 2408.15968 w/ Beran, Braun, Calisti, Gigli, McCann, Ohanyan, Rott

Motivation:

in smooth spacetimes: ∇f f.d. causal, γ f.d. causal curve, by reverse Cauchy-Schwarz

$$f(\gamma(1)) - f(\gamma(0)) = \int_0^1 df(\dot\gamma(t))dt \geq \int_0^1 |\nabla f|_g(\gamma(t))|\dot\gamma(t)|dt.$$

$\rightsquigarrow$ need to know that what $|\nabla f|$ and $|\dot\gamma|$ is

in the metric case: *minimal weak upper gradient* and *metric speed*

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develop such a differential calculus in the *synthetic Lorentzian* setting

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Differentiating the reverse triangle inequality

Proposition (Diff. the reverse triangle inequality on the halfsquare)

$T \colon \{(s,t) \in [0,1]^2,\ s \le t\} \to [0,\infty)$ such that

$$T(r,s) + T(s,t) \le T(r,t) \qquad \forall 0 \le r \le s \le t \le 1$$

then exists *maximal measure* μ among Borel measures ν on $[0,1]$ sat.

$$\nu((a,b)) \le T(a,b) \qquad \forall a,b \in [0,1],\ a \le b.$$

μ is the *weak limit* as $h \downarrow 0$ of μ_h having densities $\mathrm{d}\mu_h(t) := \frac{T(t,t+h)}{h}\,\mathrm{d}t$, writing $\mu = \rho\mathcal{L}^1 + \mu^\perp$ with $\mu^\perp \perp \mathcal{L}^1$, $0 \neq p < 1$, then

$$\frac{1}{p}\int_0^1 \rho^p = \inf \sum_i \frac{T(t_i,t_{i+1})^p}{p(t_{i+1}-t_i)^{p-1}}$$

infimum taken over all finite partitions $0 = t_0 < \ldots < t_N = 1$ of $[0,1]$

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Causal speed and energy

Definition (causal speed)

$\gamma\colon [0,1] \to X$ causal path, apply Prop. to $T(s,t) := \tau(\gamma(s),\gamma(t)) \rightsquigarrow$ maximal measure $|\dot\gamma| := \mu$ on $[0,1]$ is the *causal speed* of γ

Lebesgue decomposition is $|\dot\gamma| = |\dot\gamma|\mathcal{L}^1 + |\dot\gamma|^\perp$

γ need not be continuous! $\rightsquigarrow$ *left-continuous causal paths* $LCC([0,1],X)$

Definition (p-energy of left-continuous causal paths)

$0 \neq p < 1$

$$\mathcal{A}_p(\gamma) := \frac{1}{p}\int_0^1 |\dot\gamma|(t)^p \,\mathrm{d}t$$

γ *timelike geodesic*, i.e., $\tau(\gamma(s),\gamma(t)) = (t-s)\tau(\gamma(0),\gamma(1)) > 0$, if and only if γ *maximizes* the p-energy

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Left-continuous causal paths of probability measures

Lifting the causal structure to $\mathcal{P}(X)$: $\mu, \nu \in \mathcal{P}(X)$, μ *causally precedes* ν, i.e., $\mu \preceq \nu$, if $\exists \pi \in \Pi_{\leq}(\mu, \nu)$ (π coupling of μ, ν with $\pi(\leq) = 1$) $\rightsquigarrow$ *left-continuous causal paths of probability measures* $LCC([0,1], \mathcal{P}(X))$

Definition (p-Lorentz-Wasserstein distance, Eckstein-Miller '17)

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$$\ell_p(\mu, \nu) := \sup_{\pi \in \Pi_{\leq}(\mu,\nu)} \Big(\int_{X^2} \tau^p \, \mathrm{d}\pi \Big)^{\frac{1}{p}}$$

Prop. $\Rightarrow$ causal paths of probability measures have causal speed

Definition (p-energy of left-continuous causal paths of prob. meas.)

$0 \neq p < 1$, $\mu \in LCC([0,1], \mathcal{P}(X))$

$$\mathcal{A}_p(\mu) := \frac{1}{p} \int_0^1 |\dot{\mu}|_p(t)^p \, \mathrm{d}t$$

μ *timelike ℓ_p-geodesic*, i.e., $\ell_p(\mu(s), \mu(t)) = (t-s)\ell_p(\mu(0), \mu(1)) > 0$, if and only if μ *maximizes* the p-energy

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Calculus for causal functions (1/2)

Definition (Causal (or monotone) function)

$f\colon X \to [-\infty, \infty]$ is a *causal function* if it is *monotone* (or causally order-preserving), i.e., $\forall x, y \in X$, $x \leq y \Rightarrow f(x) \leq f(y)$

From now on: X with polish topology, $\mathfrak{m}$ Radon measure on X

Definition (Test plan)

$\pi \in \mathcal{P}(LCC([0,1], X))$ *test plan* if $\exists C > 0$ s.t. $(\mathrm{eval}_t)_\sharp \pi \leq C\,\mathfrak{m}\ \forall t \in [0,1]$

Definition (Weak subslope)

f Borel causal function, $G\colon X \to [0,\infty]$ is a *weak subslope* of f if every test plan π satisfies

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f Borel causal function, then $\exists$ $\mathfrak{m}$-a.e. unique *maximal weak subslope* $G\colon X \to [0,\infty]$ of f and $|\mathrm{d}f| := G$

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Vertical right-differentiation

Definition (Perturbations of causal functions)

$f\colon X \to [-\infty,\infty]$ causal, $\mathrm{Pert}(f)$ *admissible perturbations* of f:

$$\mathrm{Pert}(f) := \{g\colon X \to [-\infty,\infty] : \exists \varepsilon > 0 \text{ s.t. } f + \varepsilon g \text{ is causal}\}$$

Definition (Vertical right-differentiation)

$f\colon X \to [-\infty,\infty]$ causal, $g \in \mathrm{Pert}(f)$, $p < 0$, define

$$\mathrm{d}^+ g(\nabla f)|\,\mathrm{d}f|^{p-2} := \begin{cases} 0, & \text{on } \{f = \pm\infty\} \\ \lim\limits_{\varepsilon\downarrow 0} \dfrac{|\,\mathrm{d}(f+\varepsilon g)|^p - |\,\mathrm{d}f|^p}{p\,\varepsilon}, & \text{on } f^{-1}(\mathbb{R}) \end{cases}$$

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Relating vertical with horizontal derivative

calculus rules for perturbations: *chain rule*, *Leibniz rule* $\rightsquigarrow$

Theorem (Vertical vs horizontal derivative)

$(M, \ell, \mathfrak{m})$ *infinitesimally Minkowskian*, $f \colon M \to \bar{\mathbb{R}}$ causal, $g \colon M \to \bar{\mathbb{R}}$ s.t. $\pm g \in \mathrm{Pert}(f)$; then

$$\lim_{t \downarrow 0} \int \frac{g(\gamma_t) - g(\gamma_0)}{t} \, \mathrm{d}\pi(\gamma) = \int \mathrm{d}^+ g(\nabla f) \, |\mathrm{d}f|^{p-2}(\gamma_0) \, \mathrm{d}\pi(\gamma)$$

where π *represents the p-gradient of f*, i.e., for $p^{-1} + q^{-1} = 1$ with $0 \neq q < 1$, $|\mathrm{d}f|^p \in L^1(\mathfrak{m})$ and $\limsup_{t \to 0} t^{-1} \iint_0^t |\dot{\gamma}_r|^q \, \mathrm{d}r \, \mathrm{d}\pi(\gamma) < +\infty$

$$\limsup_{t \to 0} \int \frac{f(\gamma_t) - f(\gamma_0)}{t} \, \mathrm{d}\pi(\gamma)$$

$$\leq \frac{1}{p} \int |\mathrm{d}f|^p(\gamma_0) \, \mathrm{d}\pi(\gamma) + \liminf_{t \to 0} \frac{1}{tq} \iint_0^t |\dot{\gamma}_r|^q \, \mathrm{d}r \, \mathrm{d}\pi(\gamma)$$

Smooth (q-)d'Alembertian comparison (1/2)

Definition (q-d'Alembertian)

$q < 0$, $f \in C^2(M)$ with timelike gradient, define

$$\Box_q(f) := \operatorname{div}\left(\nabla f |\nabla f|_g^{q-2}\right).$$

Theorem (Smooth q-d'Alembertian comparison)

(M, g) globally hyperbolic spacetime of dimension n s.t. Ric$(v, v) \geq (n-1)K$ $\forall v \in TM$ unit timelike, $K \in \mathbb{R}$, then $\forall y \in M$ $(\tau_y := \tau(., y))$, on $I^-(y) \setminus C_T^-(y)$:

$$\Box_q \frac{\tau_y^p}{p} \leq \begin{cases} 1 + (n-1)\sqrt{K}\tau_y \cot(\sqrt{K}\tau_y), & K > 0, \\ n, & K = 0, \\ 1 + (n-1)\sqrt{-K}\tau_y \coth(\sqrt{-K}\tau_y), & K < 0. \end{cases}$$

Smooth (q-)d'Alembertian comparison (2/2)

- since $|\nabla\tau_y|_g = 1$, $\Box_q \tau_y = \Box\tau_y$ for any $q < 0$
- Expectation: q-d'Alembertian comparison holds for any Kantorovich potential, not just $p^{-1}\tau_y^p$
- $\Box_q \frac{\tau_y^p}{p} = \Box \frac{\tau_y^2}{2} \rightsquigarrow$ classical result:

Theorem (D'Alembertian comparison)

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$$\Box\tau_y \leq \begin{cases} (n-1)\sqrt{K}\cot(\sqrt{K}\tau_y), & K > 0, \\ \frac{n-1}{\tau_y}, & K = 0, \\ (n-1)\sqrt{-K}\coth(\sqrt{-K}\tau_y), & K < 0. \end{cases}$$

Synthetic timelike non-negative Ricci curvature

- [McCann, '18], [Mondino, Suhr, '18], [Cavalletti, Mondino, '20], [Braun, '23].
- $\mu, \nu \in \mathcal{P}(X)$ *timelike p-dualizable* if $\exists$ optimal $\pi \in \Pi(\mu, \nu)$ with $\pi(\ll) = 1$
- *Entropy:* $\mathrm{Ent}(\mu, \mathfrak{m}) := \int_X \rho \log(\rho) \, \mathrm{d}\mathfrak{m}$ if $\mu = \rho \, \mathfrak{m}$ and $(\rho \log(\rho))_+$ integrable, otherwise $\mathrm{Ent}(\mu, \mathfrak{m}) = \infty$
- $p \in (0, 1)$, $N \in (0, \infty)$: X is a $\mathrm{TCD}_p(0, N)$ space if $\forall (\mu_0, \mu_1) \in \mathcal{P}_{ac}(X)^2$ timelike p-dualizable $\exists$ timelike ℓ_p-geodesic $\mu \colon [0, 1] \to \mathcal{P}_{ac}(X)$ from μ_0 to μ_1 s.t. $t \mapsto e(t) := \mathrm{Ent}(\mu(t), \mathfrak{m})$ is *semi-convex*, i.e., $e'' - \frac{(e')^2}{N} \geq 0$ in the distributional sense

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X w/ *synthetic TL Ricci curvature* ≥ 0 and *dim* $\leq N$, $N > 1$, τ continuous, $0 \neq q < 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\varphi = \frac{-\ell(.,o)^q}{q}$ (or τ^q/q*-concave*), then $\forall 0 \leq g \in \mathrm{Pert}(\varphi)$ bounded, compact support:

$$\int \mathrm{d}^+ g(\nabla\varphi)|\,\mathrm{d}\varphi|^{p-2}\,\mathrm{d}\mathfrak{m} \leq N \int g\,\mathrm{d}\mathfrak{m}$$

- also for *synthetic TL Ricci curvature* $\geq K \in \mathbb{R}$
- if $\mathrm{Pert}(\varphi)$ rich enough conclusion holds in *distributional sense*
- Eschenburg (1988) proved such estimates where $\tau(\cdot, z)$ is smooth; we extend across the *timelike cut locus* for the first time
- thus even on *smooth globally hyperbolic spacetimes* new results
- $d^+g(\nabla\varphi)|\,\mathrm{d}\varphi|^{p-2}$ is a *measure*, but non unique unless infinitesimal Minkowskianity holds and $\mathrm{Pert}(\varphi)$ dense

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preprint 2410.12632 w/ Braun, Gigli, McCann, Ohanyan

- using the *p-D'Alembertian* $\Box_p(f) := \operatorname{div}\left(\nabla f |\nabla f|_g^{p-2}\right)$ instead of the D'Alembertian gives ellipticity (while losing linearity), $p < 0$
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- other techniques from *optimal transport* and the *synthetic calculus* on Lorentzian length spaces

Theorem (Lorentzian splitting, Newman; Eschenburg; Galloway)

(M, g) globally hyperbolic or timelike geodesically complete, Ric$(v, v) \geq 0$ for timelike $v \in TM$, containing a complete timelike line, then $(M, g) \cong (\mathbb{R} \times S, -dt^2 + h)$

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Outlook

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- *Coordinates* for Lorentzian length spaces with timelike curvature bounded below (w.i.p. w/ Beran, Harvey, Rott)
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