Causal Fermion Systems as an Effective Collapse Theory

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Brief summary

- causal fermion systems can describe non-smooth spacetime structures;
  in particular, spacetimes involving fluctuating fields
- causal action principle describes nonlinear dynamics

Main message of this talk:

- gives rise to a effective collapse model,
- has similarities with CSL model.

- Has been worked out in detail in the non-relavitivistic limit with Johannes Kleiner and Claudio Paganini
  - “Causal fermion systems as an effective collapse theory,” arXiv:2405.19254 [math-ph]
- Collapse theory derived from first principles.
- Relativistic model in preparation.

What is a causal fermion system?

- approach to fundamental physics
- novel mathematical model of spacetime
- physical equations are formulated in generalized spacetimes
- Different limiting cases:
  - Continuum limit: Quantized fermionic fields interacting via classical bosonic fields
  - QFT limit: fermionic and bosonic quantum fields (ongoing, more towards the end of the talk)
- For overview, more details (papers, books, videos, online course), physical applications, . . .

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Effective description by nonlocal Dirac equation

- Consider causal fermion system in Minkowski space:
  - Thus Minkowski space, spacetime points $(t, \vec{x})$
  - $\psi_1(t, \vec{x}), \ldots, \psi_f(t, \vec{x})$ family of spinorial wave functions
- causal action principle describes the interaction of all these wave functions
- the linearized interaction can be described effectively by a nonlocal Dirac equation
  - F.F., "Solving the linearized field equations of the causal action principle in Minkowski space," arXiv:2304.00965 [math-ph], to appear in *Adv. Theor. Math. Phys.* (2024)
- There are nonlinear corrections.

Effective description by nonlocal Dirac equation

- Begin in one-particle description (Fock spaces later).
- Describe the dynamics of the causal action principle in terms of a nonlocal Dirac equation

$$
\begin{aligned}
&(i\partial\!\!\!/ + \mathcal{B} - m)\psi = 0 \\
&\qquad (\mathcal{B}\psi)(x) = \int_M \mathcal{B}(x,y)\,\psi(y)\,d^4y \\
&\qquad \mathcal{B}(x,y) = \sum_{a=1}^N \gamma_j\,A_a^j\Big(\frac{x+y}{2}\Big)\,L_a(y-x)
\end{aligned}
$$

Linearized fields in Minkowski space

$$\mathcal{B}(x,y) = \sum_{a=1}^{N} \gamma_j \, A_a^j\Big(\frac{x+y}{2}\Big)\; L_a(y-x)$$

- The kernels $L_a(y-x)$ are nonlocal on the scale $\ell_{\min}$ with

$$\ell_{\text{Planck}} \ll \ell_{\min} \ll \ell_{\text{macro}}$$

(and ℓ_{Planck} denotes the Planck scale)

$$L_a(\xi) = 0 \qquad \text{if } |\xi^0| + |\vec{\xi}| \gtrsim \ell_{\min}$$

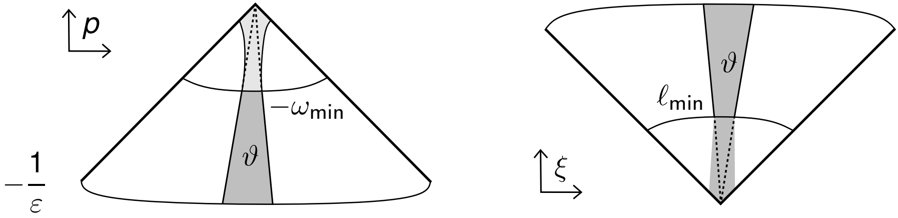
- The number N of fields scales like

$$N \simeq \frac{\ell_{\min}}{\varepsilon}$$

- multitiude of vectorial potentials A_a^j, $a = 1, \ldots, N$, will later be described stochastically
- All potentials satisfy the homogeneous wave equation

$$\Box A_a^j = 0$$

Linearized fields in Minkowski space



$$\vartheta = \frac{1}{\sqrt{\ell_{\min}\,\omega_{\min}}} \qquad \text{with} \qquad \vartheta_{\min} = \sqrt{\frac{\ell_{\text{Planck}}}{\ell_{\min}}}\,.$$

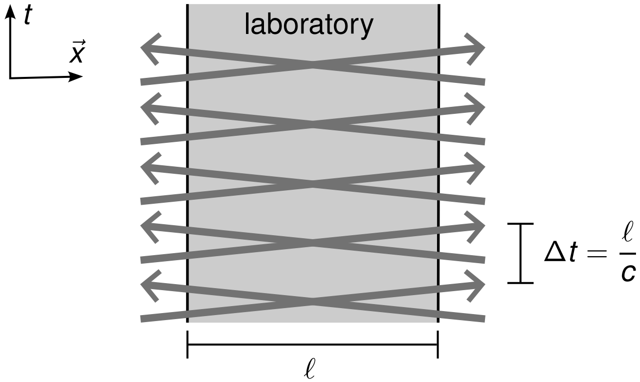
- Different wave functions "feel" different potentials.
- The low-energy wave functions (i.e. $|\omega| \lesssim \ell_{\text{Planck}}^{-1}$) "feel all the potentials at the same time".

Conserved Scalar Product

- Noether-like theorem: conservation law for scalar product

$$
\begin{aligned}
\langle \psi | \phi \rangle_t := \int \prec \psi \,|\, \gamma^0 \, \phi \succ_{(t,\vec{x})} \; d^3x \\
& - i \int_{x^0 < t} d^4x \int_{y^0 > t} d^4y \; \prec \psi(x) \,|\, \mathcal{B}(x,y)\, \phi(y) \succ_x \\
& + i \int_{x^0 > t} d^4x \int_{y^0 < t} d^4y \; \prec \psi(x) \,|\, \mathcal{B}(x,y)\, \phi(y) \succ_x
\end{aligned}
$$

  Has structure of surface layer integral.
- Generalizes probability integral, gives probabilistic interpretation.
- Note: Scalar product depends on stochastic potentials!



Assume potentials are Gaussian and Markovian,

$$\ll A_a^j(x) \gg = 0$$
$$\ll A_a^j(x)\, A_b^k(x) \gg = \delta(x^0 - y^0)\, \delta_{ab}\, C^{jk}(\vec{y} - \vec{x})$$

The non-relativistic limit

- However, the nonlocality of the potential must be taken into account. (Otherwise, no collapse occurs.)

$$\big(i\partial\!\!\!/ + \mathcal{B} - m\big)\psi = 0$$

Hamiltonian formulation:

$$\begin{aligned}
i\partial_t\psi &= \big(H_0 + V\big)\psi \\
H_0 &= -i\gamma^0\vec{\gamma}\vec{\nabla} \\
(V\psi)(t) &= \int_{-\infty}^{\infty} V(t,t')\,\psi(t')\,dt' \\
(V(t,t')\psi)(\vec{x}) &= \int_{\mathbb{R}^3} \Big(-\gamma^0\,\mathcal{B}\big((t,\vec{x}),(t',\vec{y})\big)\Big)\,\psi(\vec{y})\,d^3y
\end{aligned}$$

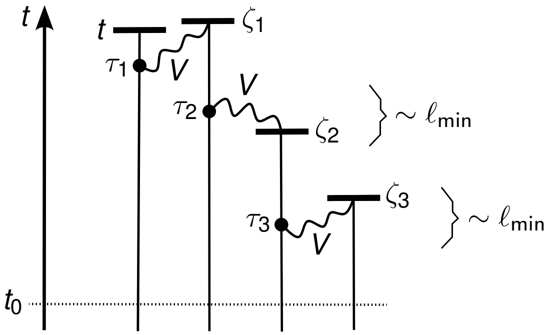
The non-local Dyson series

$$i\partial_t \psi = \big(H_0 + V\big)\psi$$

Can be solved with nonlocal Dyson series

$$\begin{aligned}
\psi(t) &= \psi(t_0) + \int_{t_0}^t \dot{\psi}(\tau)\, d\tau = \psi(t_0) - i \int_{t_0}^t (V\psi)(\tau)\, d\tau \\
&= \cdots = \qquad \text{(apply iteratively)} \\
&= \psi(t_0) + \int_{t_0}^t d\tau \int_{-\infty}^{\infty} d\zeta\, \big(- iV(\tau, \zeta)\big)\, \psi(t_0) \\
&\quad + \int_{t_0}^t d\tau_1 \int_{-\infty}^{\infty} d\zeta_1\, \big(- iV(\tau_1, \zeta_1)\big) \\
&\qquad \times \int_{t_0}^{\zeta_1} d\tau_2 \int_{-\infty}^{\infty} d\zeta_2\, \big(- iV(\tau_2, \zeta_2)\big)\, \psi(t_0) \\
&\quad + \cdots
\end{aligned}$$

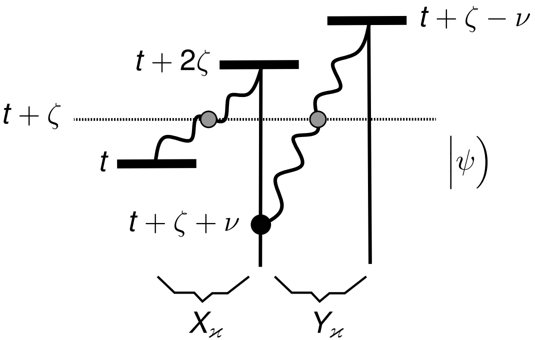
The non-local Dyson series



Gaussian pairings

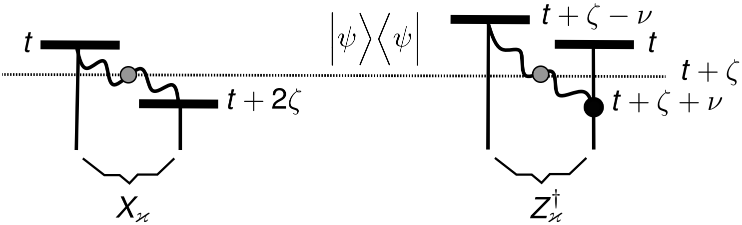
$$\mathcal{B}(x,y) = \sum_{a=1}^{N} \gamma_j \, A^j_a\Big(\frac{x+y}{2}\Big) \, L_a(y-x)$$

$$\ll A^j_a(x)\, A^k_b(x) \gg \; = \delta(x^0 - y^0)\, \delta_{ab}\, C^{jk}(\vec{y} - \vec{x})$$

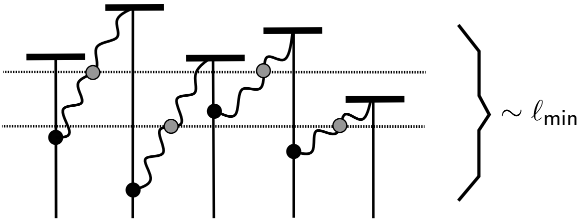


Gaussian pairings

- Take into account pairings between bra and ket



- Example in higher order



Main results of analysis

- Statistical operator σ_t has time evolution of Kossakowski-Lindblad form

$$\frac{d\sigma_t}{dt} = -i[H, \sigma_t] - \frac{1}{2} \sum\!\!\!\!\!\!\int_{\varkappa} \big[K_\varkappa, [K_\varkappa, \sigma_t]\big] \; \Big(1 + \mathcal{O}\big(\ell_{\min} \, \|V\|\big)\Big)$$

- There is dynamical state reduction, in agreement with Born's rule.
- Similar to CSL model, but not the same, due to nonlocality in time.

Derivation of Lindblad dynamics

- Note: Standard scalar product

$$(\psi|\phi)_t := \int \prec\!\psi \,|\, \gamma^0\, \phi\!\succ_{(t,\vec{x})}\; d^3x$$

  is *not* conserved in time.

- Only the modified scalar product is conserved,

$$\begin{aligned}\langle\psi|\phi\rangle_t = (\psi|\phi)_t \\ &- i \int_{x^0<t} d^4x \int_{y^0>t} d^4y\; \prec\!\psi(x)\,|\,\mathcal{B}(x,y)\,\phi(y)\!\succ_x \\ &+ i \int_{x^0>t} d^4x \int_{y^0<t} d^4y\; \prec\!\psi(x)\,|\,\mathcal{B}(x,y)\,\phi(y)\!\succ_x\end{aligned}$$

.

Derivation of Lindblad dynamics

Transform one into the other:

$$\langle \psi | \phi \rangle_{t_0} = \big(\psi \,|\, (\mathbb{1} + \mathcal{S}_{t_0}) \phi \big)_{t_0} \qquad \text{for all } \psi, \phi \in \mathcal{H}_m$$

$$\psi \mapsto \tilde{\psi} := \sqrt{\mathbb{1} + \mathcal{S}_{t_0}}\, \psi$$

$$\langle \psi | \phi \rangle_{t_0} = (\tilde{\psi} | \tilde{\phi})_{t_0}$$

Working with $\tilde{\psi}$, one can use the standard scalar product.

statistical operator $\qquad \sigma_t := \ll |\psi\rangle\langle\psi| \gg \; = \; \ll |\tilde{\psi})(\tilde{\psi}| \gg$

Now compute

$$\frac{d}{dt} \sigma_t = \ll \frac{d}{dt} \Big(|\tilde{\psi})(\tilde{\psi}| \Big) \gg = \cdots$$

to leading order in $\ell_{\min} \|\mathcal{B}\|$.

Reduction of the state vector

Try to use standard assumption:

- Observable $\mathcal{O}$ commutes with Hamiltonian.
- Typical example: Position measurement, use locality of time evolution.

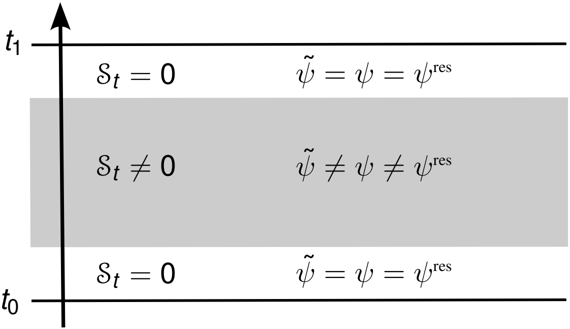
Problem: Operator $\mathcal{S}_t$ is nonlocal! Therefore:

- Work with the original (untilded) wave functions.
- Makes it necessary to also work with the time-dependent scalar product $\langle .|.\rangle_t$.

Reduction of the state vector

Consider situation similar to a scattering process and rescale the wave functions,

$$\psi^{\mathrm{res}}(t) := c(t)\, \psi(t) \qquad \text{with} \qquad c(t) := \frac{1}{\sqrt{\ll\!(\psi(t)|\psi(t))\!\gg}}$$



Reduction of the state vector

$$\frac{d}{dt} \ll\!(\psi^{\text{res}}(t)|\psi^{\text{res}}(t))\!\gg \,= 0$$

$$\frac{d}{dt} \ll\!(\psi^{\text{res}} \,|\, \mathcal{O}\, \psi^{\text{res}})\!\gg \,= 0$$

$$\frac{d}{dt} \ll\!(\psi^{\text{res}} \,|\, \mathcal{O}^2\, \psi^{\text{res}}) - (\psi^{\text{res}} \,|\, \mathcal{O}\, \psi^{\text{res}})^2\!\gg \,\leq 0$$

and strictly negative unless ψ^{res} is an eigenstate.

- Shows collapse
- Proves Born rule

Effective description in Fock spaces

- System can be described at any time t by a

$$\text{Quantum state} \qquad \omega^t : \mathscr{A} \to \mathbb{C}\,,$$

where $\mathscr{A}$ is the algebra of observables.

- can be represented on Fock space $\mathcal{F}$ (fermionic and bosonic)

$$\omega^t(A) = \mathrm{Tr}_{\mathcal{F}}\left(\sigma^t A\right) \overset{\text{if pure state}}{=} <\Psi|A|\Psi>$$

- F.F. and Kamran, N., “Fermionic Fock spaces and quantum states for causal fermion systems,” arXiv:2101.10793 [math-ph], *Ann. Henri Poincaré* **23** (2022) 1359–1398
- F.F., Kamran, N. and Reintjes, M., “Entangled quantum states of causal fermion systems and unitary group integrals,” arXiv:2207.13157 [math-ph], to appear in *Adv. Theor. Math. Phys.* (2024)
- Is ongoing work with C. Dappiaggi, N. Kamran, M. Reintjes.

Effective description in Fock spaces

For our purpose, it suffices to consider Hartree-Fock state

$$\Psi = \psi_1 \wedge \cdots \wedge \psi_q$$

- Dynamics again described by the nonlocal Dirac equation
- Collapse happens as soon as one one-particle wave function collapses.
- Thus collapse phenomena are predominant for mesoscopic and macroscopic systems

Parameters of the model

- The length scale $\ell_{\min}$. Equivalently, the number N of fields related to $\ell_{\min}$ by

$$N \simeq \frac{\ell_{\min}}{\varepsilon}$$

- The strength of the stochastic field as described by the covariance,

$$\ll A_a^j(x)\, A_b^k(x) \gg \,=\, \delta(x^0 - y^0)\, \delta_{ab}\, C^{jk}(\vec{y} - \vec{x})$$

and the nonlocal kernels $L_a(y - x)$

The nonlinear term

Interestingly, it does not need to be specified. Description on various levels possible:

- Take into account nonlinear coupling,

$$-\Box (A_a)^k = e^2 J_a^k$$
$$J_a^k(z) = \int_M \prec \psi(x) | \gamma^k \, L_a(x,y) \, \psi(y) \succ \big|_{x=z-\xi/2,\; y=z+\xi/2} \, d^4\xi$$

- Take the many-particle perspective:

  Hartree-Fock state $\qquad \Psi = \psi_1 \wedge \cdots \wedge \psi_q$

  In the causal fermion system description, the potential $\mathcal{B}$ is encoded in this family of wave functions. Therefore, Dirac equation for Ψ is nonlinear.

The Nature of the Collapse

- It is not the gravitational field which triggers the collapse.
- Instead, it is a multitude of bosonic fields, specific to the causal action principle
- Remark:
  - This multitude of fields can be described effectively by a second-quantized electromagnetic field.
  - Therefore: collapse is closely related to the electromagnetic interaction in QFT
- But: length scale of nonlocality comes into play. Related to Planck scale. Also gives connection to strength of gravitational field.

Outlook: The Relativistic Model

ongoing work also with Simone Murro

- Dirac equation is already relativistic.
- Stochastic background fields break Lorentz invariance.
  - Concept: Stochastic background fields originate from the early universe and/or are generated by the matter on earth and of the surrounding stars and galaxies.
- Replace Markov property by propagation with speed of light. Also gives rise to “smearing in time.”

Summary

- Consider causal fermion systems in Minkowski space
- Described by family of fermionic wave functions, encoded in wave evaluation operator Ψ
- Causal action principle gives rise to plethora of fields
- Coupling of these fields to the Dirac equation is nonlocal on a scale $\ell_{\min} \ll m^{-1}$.
- Similar to CSL model, we obtain a stochastic and a nonlinear term.
- But: has a different mathematical structure, due to nonlocality in time.

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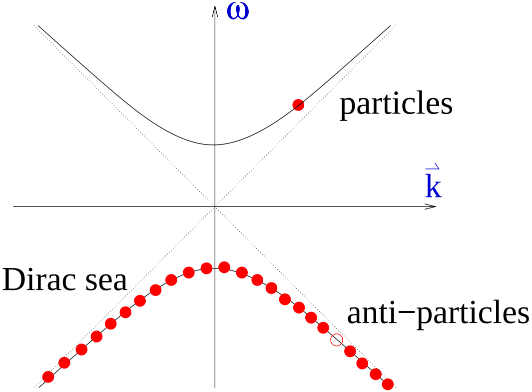
Thank you for your attention!

How causal fermion systems developed (≈ 1989-90)

starting point: Course on relativistic QM and QFT
(following Bjorken-Drell / Itzykson-Zuber)

- Dirac's hole theory (Dirac 1932)



How causal fermion systems developed ($\approx$ 1989-90)

- Problems of the naive Dirac sea picture:
  - infinite charge density
  - infinite negative energy density
- Therefore, we were told in lecture:
  - Dirac sea is not visible due to symmetries (homogeneous, isotropic)
  - Only “deviations” of the sea are observed as particles and anti-particles
  - Forget about the Dirac sea, no longer needed.
- This procedure is implemented in the formalism:
  - Reinterpretation of creation as annihilation operators
  - Wick ordering of field operators in Hamiltonian

I was not convinced by this procedure:

- The interacting Dirac sea should be visible, for example in presence of external potential

$$\big(i\partial\!\!\!/ + A\!\!\!/(x) - m\big)\psi = 0$$

- Pair creation seems an evidence that the Dirac sea is real.

How causal fermion systems developed ($\approx$ 1989-90)

What is the way out?

- Take all the sea states into account.
- In order to avoid the problems of naive Dirac sea, formulate new type of equations, different structure of the physical equations

Goal in general terms:

Formulate a variational principle
directly for the family of wave functions

- Intuitive picture: wave functions "organize themselves" in such a way that the Dirac sea configuration is a minimizer.
- In interacting situation the wave functions organize to solutions of the Dirac equation

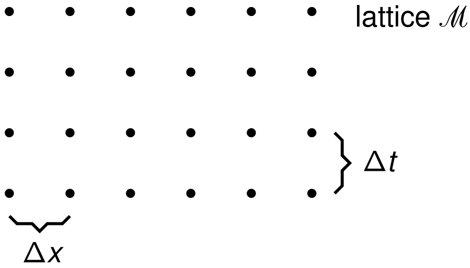
$$(i\gamma^j \partial_j + e\gamma^j A(x) - m)\psi = 0$$

This should serve as the definition of A.

Motivating example: A discrete spacetime

Formulate a variational principle
directly for a family of wave functions

- For simplicity begin with a discrete spacetime, for example 2d-lattice



- Do not make use of nearest neighbor relation and lattice spacing.

Better and simpler: spacetime $\mathscr{M}$ is a discrete set of m points.

Motivating example: A discrete spacetime

- Consider wave functions $\psi_1, \ldots, \psi_f : \mathscr{M} \to \mathbb{C}$ (with $f < \infty$)
- Introduce scalar product; orthonormalize,

$$\langle \psi_k | \psi_l \rangle = \delta_{kl} \,,$$

gives f-dim Hilbert space $(\mathcal{H}, \langle .|. \rangle_{\mathcal{H}})$.

important object: for any lattice point x introduce

local correlation operator $F(x) : \mathcal{H} \to \mathcal{H}$

- define matrix elements by

$$(F(x))^j_k = \overline{\psi_j(x)} \psi_k(x)$$

basis invariant:

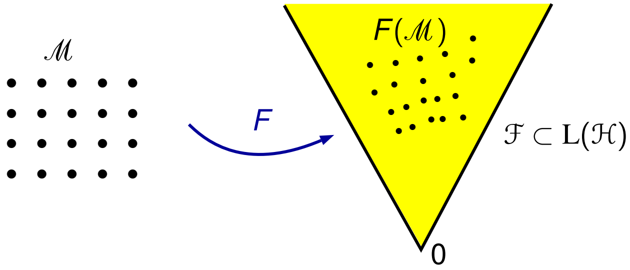
$$\langle \psi, F(x)\, \phi \rangle_{\mathcal{H}} = \overline{\psi(x)} \phi(x) \qquad \text{for all } \psi, \phi \in \mathcal{H}$$

- Hermitian matrix
- Has rank at most one, is positive semi-definite

$$F(x) = e^* e \qquad \text{with} \qquad e : \mathcal{H} \to \mathbb{C} \,, \quad \psi \mapsto \psi(x)$$

Motivating example: A discrete spacetime

$$\mathcal{F} := \big\{ F \text{ Hermitian, rank one, positive semi-definite} \big\}$$



general idea:

- disregard objects on the left
  (nearest neighbors, lattice spacing)
- work instead with the objects on the right
  (only local correlation operators)

Motivating example: A discrete spacetime

How to set up equations in this setting?
Explain idea in simple example:

- local correlation operators $F_1, \ldots, F_m \in \mathcal{F}$
- product $F_i \, F_j$ tells about correlation of wave functions at different space-time points
- $\mathrm{Tr}(F_i F_j)$ is real number
- minimize
$$\mathcal{S} = \sum_{i,j=1}^{m} \mathrm{Tr}(F_i F_j)^2$$
under suitable constraints.

Causal Fermion Systems

[redacted]

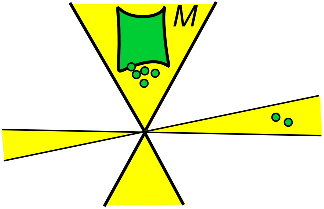
Let $(\mathcal{H}, \langle .|. \rangle_{\mathcal{H}})$ be Hilbert space

Given parameter $n \in \mathbb{N}$ ("spin dimension")

$$\mathcal{F} := \Big\{ x \in \mathrm{L}(\mathcal{H}) \text{ with the properties:}$$

- x is symmetric and has finite rank
- x has at most n positive and at most n negative eigenvalues $\Big\}$

ρ a measure on $\mathcal{F}$



Causal action principle

Let $x, y \in \mathcal{F}$. Then x and y are linear operators.

$x \cdot y \in \mathrm{L}(H)$:

- rank $\leq 2n$
- in general not symmetric: $(x \cdot y)^* = y \cdot x \neq x \cdot y$

thus non-trivial complex eigenvalues $\lambda_1^{xy}, \ldots, \lambda_{2n}^{xy}$

Causal action principle

Nontrivial eigenvalues of xy: $\lambda_1^{xy}, \ldots, \lambda_{2n}^{xy} \in \mathbb{C}$

Lagrangian $$\mathcal{L}(x,y) = \frac{1}{4n} \sum_{i,j=1}^{2n} \Big(|\lambda_i^{xy}| - |\lambda_j^{xy}| \Big)^2 \;\geq\; 0$$

action $$\mathcal{S} = \iint_{\mathcal{F}\times\mathcal{F}} \mathcal{L}(x,y)\, d\rho(x)\, d\rho(y) \;\in\; [0,\infty]$$

Minimize $\mathcal{S}$ under variations of ρ, with constraints

volume constraint: $$\rho(\mathcal{F}) = \text{const}$$

trace constraint: $$\int_{\mathcal{F}} \text{tr}(x)\, d\rho(x) = \text{const}$$

boundedness constraint: $$\iint_{\mathcal{F}\times\mathcal{F}} \sum_{i=1}^{2n} |\lambda_i^{xy}|^2\, d\rho(x)\, d\rho(y) \leq C$$

- F.F., “Causal variational principles on measure spaces,” *J. Reine Angew. Math.* **646** (2010) 141–194

Example: Dirac spinors in Lorentzian space-time

Let $(\mathscr{M}, g)$ be a Lorentzian space-time, for simplicity 4-dimensional, globally hyperbolic, then automatically spin,

$$(S\mathscr{M}, \prec .|.\succ) \qquad \text{spinor bundle}$$

- $S_p\mathscr{M} \simeq \mathbb{C}^4$
- spin scalar product

$$\prec .|.\succ_p \,:\, S_p\mathscr{M} \times S_p\mathscr{M} \to \mathbb{C}$$

is indefinite of signature (2,2)

$$(\mathcal{D} - m)\psi_m = 0 \qquad \text{Dirac equation}$$

Example: Dirac spinors in Lorentzian space-time

- Cauchy problem well-posed, global smooth solutions (for example symmetric hyperbolic systems)
- finite propagation speed

$C^\infty_{\text{sc}}(\mathcal{M}, S\mathcal{M})$ spatially compact solutions

$$(\psi_m|\phi_m)_m := 2\pi \int_{\mathcal{N}} \prec \psi_m | \nu\!\!\!/ \, \phi_m \succ_x \, d\mu_{\mathcal{N}}(x) \quad \text{scalar product}$$

completion gives Hilbert space $(\mathcal{H}_m, (.|.)_m)$

Example: Dirac spinors in Lorentzian space-time

- Choose $\mathcal{H}$ as a subspace of the solution space,

$$\mathcal{H} = \overline{\text{span}(\psi_1, \ldots, \psi_f)}$$

- To $x \in \mathbb{R}^4$ associate a local correlation operator

$$\langle \psi | F(x) \phi \rangle = - \prec \psi(x) | \phi(x) \succ_x \qquad \forall \psi, \phi \in \mathcal{H}$$

  Is symmetric, rank ≤ 4
  at most two positive and at most two negative eigenvalues
- Here ultraviolet regularization may be necessary:

$$\langle \psi | F(x) \phi \rangle = - \prec (\mathfrak{R}_\varepsilon \psi)(x) | (\mathfrak{R}_\varepsilon \phi)(x) \succ_x \qquad \forall \psi, \phi \in \mathcal{H}$$

$$\mathfrak{R}_\varepsilon \;:\; \mathcal{H} \to C^0(\mathcal{M}, S\mathcal{M}) \qquad \text{regularization operators}$$

$$\varepsilon > 0 : \text{regularization scale (Planck length)}$$

- Thus $F(x) \in \mathcal{F}$ where

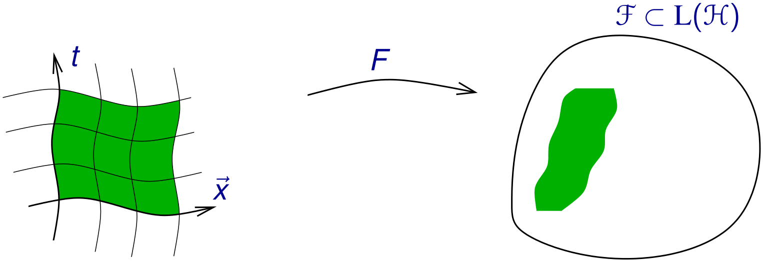
  $$\mathcal{F} := \Big\{ F \in \mathrm{L}(\mathcal{H}) \text{ with the properties:}$$

  ▷ F is symmetric and has rank ≤ 4

  ▷ F has at most 2 positive

  and at most 2 negative eigenvalues $\Big\}$

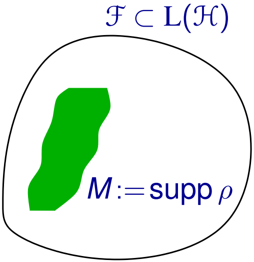
We obtain mapping $\qquad x \mapsto F(x) \in \mathcal{F} \subset \mathrm{L}(\mathcal{H})$



Take push-forward measure

$$\rho := F_*(\mu_{\mathscr{M}}) \qquad \text{(i.e. } \rho(\Omega) := \mu_{\mathscr{M}}\big(F^{-1}(\Omega)\big)\text{)}$$

Example: Dirac spinors in Lorentzian space-time



We thus obtain a causal fermion system of spin dimension two.

A few general remarks

One basic object: measure ρ on set $\mathcal{F}$ of linear operators on $\mathcal{H}$, describes spacetime as well as all objects therein

- Underlying structure: family of fermionic wave functions
- Geometric structures encoded in these wave functions

Matter encodes geometry
Quantum spacetime

- Causal action principle describes spacetime as a whole (similar to Einstein-Hilbert action in GR)
- Causal action principle is a nonlinear variational principle (similar to Einstein-Hilbert action or classical field theory)
- Linear dynamics of quantum theory recovered in limiting case (more details later)