Relative entropy in de Sitter spacetime

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Entanglement entropy in QM and QFT

Entanglement entropy in QM

- Entropy quantifies the amount of information an observer has access to
- For a system in a pure state $|\Psi\rangle$, everything is known $\Leftrightarrow$ entropy is zero
- For a generic state described by density matrix ρ, define von Neumann entropy $S_{\mathrm{vN}}(\rho) = -\operatorname{tr}(\rho \ln \rho) \geq 0$ (since $0 \leq \rho \leq 1$)
- For a system in a pure state $|\Psi\rangle$, but with the observer having access only to degrees of freedom within some region A, define density matrix $\rho_A = \operatorname{tr}_{A^\perp} |\Psi\rangle\langle\Psi|$ (trace over degrees of freedom of the complement region $A^\perp$)
- Entanglement entropy: $S(A) = S_{\mathrm{vN}}(\rho_A) = -\operatorname{tr}(\rho_A \ln \rho_A)$
- Other entropy measures: Tsallis entropy $S^{\mathrm{T}}_q(\rho) = \frac{1}{q-1}(1 - \operatorname{tr}\rho^q)$, Rényi entropy $S^{\mathrm{R}}_\alpha(\rho) = \frac{1}{1-\alpha} \ln \operatorname{tr}\rho^\alpha$, and $\lim_{q\to 1} S^{\mathrm{T}}_q(\rho) = S_{\mathrm{vN}}(\rho) = \lim_{\alpha\to 1} S^{\mathrm{R}}_\alpha(\rho)$
- Thermal density matrix $\rho = \frac{1}{Z} \exp(-\beta H)$ with inverse temperature β, $Z = \exp(-\beta F)$ with free energy F gives $S_{\mathrm{vN}} = \beta(\langle H\rangle - F) = S$ (thermodynamic entropy)

Entanglement entropy in QFT

- In QFT in $d+1$ dimensions, density matrices only exist formally and the trace is infinite, both due to the infinite number of degrees of freedom
- Compute regularised entanglement entropies with UV cutoff ϵ: $S(A) = g_{d-1}[\partial A]\epsilon^{-(d-1)} + \cdots + g_1[\partial A]\epsilon^{-1} + g_0[\partial A]\ln\epsilon + S_0(A) + \mathcal{O}(\epsilon)$, where the g_i are homogeneous functions depending on the boundary ∂A (area law in 4D)
- Source of divergences: high-energy vacuum fluctuations of the fields
- $\Rightarrow$ differences in entropies between different states are finite
- Relative entropy (Kullback–Leibler divergence): $S(\rho\|\sigma) = \operatorname{tr}(\rho\ln\rho - \rho\ln\sigma)$
- Relative Rényi entropy: $S^{\mathrm{R}}_\alpha(\rho\|\sigma) = \frac{1}{\alpha-1}\ln\operatorname{tr}\left(\rho^\alpha\sigma^{1-\alpha}\right)$
- α-z-Rényi entropy (generalized quantum Rényi div.): $S^{\mathrm{RG}}_{\alpha,z}(\rho\|\sigma) = \frac{z}{\alpha-1}\ln\operatorname{tr}\left(\rho^{\frac{\alpha}{z}}\sigma^{\frac{1-\alpha}{z}}\right)$ and $\lim_{\alpha\to1} S^{\mathrm{R}}_\alpha(\rho\|\sigma) = S(\rho\|\sigma) = \lim_{\alpha,z\to1} S^{\mathrm{RG}}_{\alpha,z}(\rho\|\sigma)$

Tomita–Takesaki theory

Tomita–Takesaki theory

- Mathematically, difference between QM and QFT is the type of (factors of) von Neumann algebra of operators (I, II, III)
- Tomita–Takesaki theory gives information on structure of vN algebra $\mathfrak{A} \subset \mathcal{B}(\mathcal{H})$ acting on Hilbert space $\mathcal{H}$, given a cyclic and separating vector $\Omega \in \mathcal{H}$
- Tomita operator S is the closure of the map $S_0 \colon a\Omega \to a^\dagger\Omega$ for $a \in \mathfrak{A}$
- Polar decomposition $S = J\Delta^{\frac{1}{2}}$ gives positive modular operator $\Delta = S^\dagger S \geq 0$ and antilinear modular conjugation J
- Modular flow $\sigma_s(a) = \Delta^{\mathrm{i}s} a\, \Delta^{-\mathrm{i}s} \in \mathfrak{A}$ for $a \in \mathfrak{A}$
- State ω defined by Ω is a thermal (KMS) state: $\omega(\sigma_s(a)b) = (\Omega, \sigma_s(a)b\Omega)$ satisfies $\omega(\sigma_{s-\mathrm{i}}(a)b) = \omega(b\, \sigma_s(a))$, with inverse temperature normalised to $\beta = 1$
- Both J and $\Delta^{\frac{1}{2}}$ map $\mathfrak{A}$ to commutant $\mathfrak{A}'$

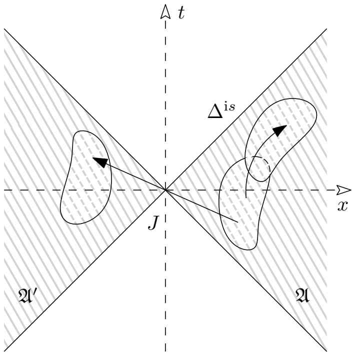
Tomita–Takesaki theory

- Relative Tomita operator $S_{\Phi|\Psi}$ is closure of map $a\Phi \mapsto a^\dagger\Psi$ for $a \in \mathfrak{A}$ and cyclic and separating vectors Φ, $\Psi \in \mathcal{H}$, relative modular operator $\Delta_{\Phi|\Psi}$ and relative modular conjugation $J_{\Phi|\Psi}$ defined by polar decomposition $S_{\Phi|\Psi} = J_{\Phi|\Psi} \Delta_{\Phi|\Psi}^{1/2}$
- Araki formula relates relative modular Hamiltonian $\ln \Delta_{\Phi|\Psi}$ to relative entropy: $S(\Phi\|\Psi) = -\left(\Phi, \ln \Delta_{\Phi|\Psi}\Phi\right)$ (well-defined and finite)
- Important case: $\Phi = uu'\Omega$ and $\Psi = vv'\Omega$ for unitary operators $u, v \in \mathfrak{A}$ and $u', v' \in \mathfrak{A}'$ commuting with u and v
- $\Rightarrow \Delta_{\Phi|\Psi} = u'v\Delta_\Omega v^\dagger(u')^\dagger$ and $S(\Phi\|\Psi) = -\left(v^\dagger u\Omega, \ln \Delta_\Omega v^\dagger u\Omega\right)$
- Relative entropy between two "excited" states relative to a "vacuum" state Ω can be computed using only the modular Hamiltonian $\ln \Delta_\Omega$ of the "vacuum" state, e.g., for coherent state with $u = u' = v' = \mathbb{1}$ and $v = \exp[\mathrm{i}\phi(f)]$

Modular Hamiltonians

Modular Hamiltonians

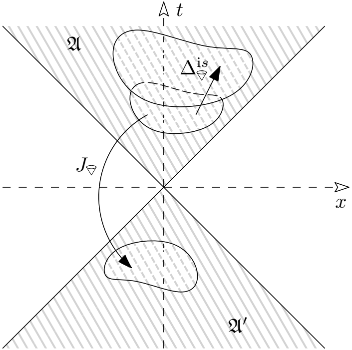
- (Relative) modular Hamiltonian $\ln \Delta_\Omega$ only known in special cases
- Minkowski vacuum state Ω and algebra $\mathfrak{A}$ generated by fields restricted to (right) Minkowski wedge $W_1 = \{x^1 \geq |x^0|\}$: $\ln \Delta_\Omega = \mathrm{i} M_{01}$, the generator of boosts
- Modular conjugation maps fields between left and right wedge
- Result for arbitrary (Wightman) quantum fields, including interacting ones



(Bisognano/Wichmann, On the duality condition for a Hermitian scalar field 1975, On the duality condition for quantum fields 1976)

Modular Hamiltonians

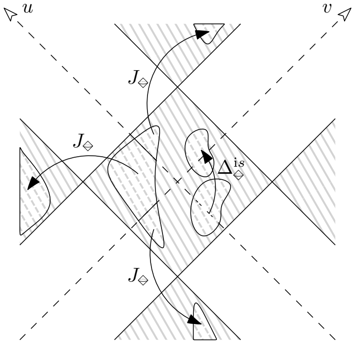
- Minkowski vacuum state Ω and algebra $\mathfrak{A}$ generated by free massless scalar fields restricted to future lightcone with tip $(\tau, \mathbf{0})$: $\ln \Delta_\Omega = 2\pi(D - \tau H)$, a linear combination of time translations and dilations
- Modular conjugation maps to past lightcone



(Buchholz, On the structure of local quantum fields with non-trivial interaction 1977, for $\tau = 0$)

Modular Hamiltonians

- Minkowski vacuum state Ω and algebra $\mathfrak{A}$ generated by free massless scalar fields restricted to diamond of size ℓ with center $(\tau, \mathbf{0})$: $\ln \Delta_\Omega = \frac{\pi}{\ell}\left[(\ell^2 - \tau^2)H + 2\tau D + K\right]$, a linear combination of time translations, dilations and special conformal transformations
- Modular conjugation maps in future/past lightcone and spacelike separated region



(Hislop/Longo, Modular structure of the local algebras associated with the free massless scalar field theory 1982,
Hislop, Conformal Covariance, Modular Structure, and Duality for Local Algebras in Free Massless Quantum Field Theories 1988, for $\tau = 0$)

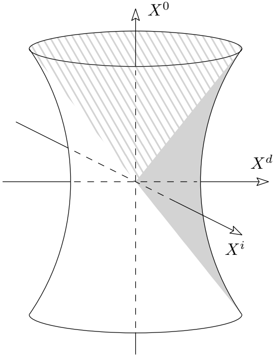
Modular Hamiltonians

- Many more examples for free, massless fermions and CFTs in 1+1 dimensions (Casini/Huerta/Rehren/Hollands/Tonni/Peschel/...), Schwarzschild (Kay/Sewell/Wald), see arXiv:2308.14797 for list
- Massive fields (even free) much more complicated
- Recent result (Cadamuro/Fröb/Minz arXiv:2312.04629): algebra $\mathfrak{A}$ of free fermions of small mass in 1+1 dimensions inside diamond of size ℓ, Minkowski vacuum state Ω
- $\ln \Delta_\Omega = \sum_{a,b=1}^2 \iint \psi_a(x) H_{ab}(x,y) \psi_b(y) \,\mathrm{d}x\,\mathrm{d}y$ (on Cauchy hypersurface $t = 0$)
- $H_{11}(x,y) = -H_{22}(x,y) = \mathrm{i}\pi \frac{\ell^2 - xy}{\ell} \delta'(x-y) + \mathcal{O}(m^2 \ln m)$
- $H_{12}(x,y) = -H_{21}(x,y) = 2\pi \mathrm{i} m \ell K_{12}(x,y) + \mathcal{O}(m^2 \ln m)$
- $K_{12}(x,y) = \ln\left(m\ell \frac{\ell^2 - x^2}{2\ell} \mu \right) \frac{\ell^2 - x^2}{2\ell^2} \delta(x+y) + \frac{1}{8\ell^2} \lvert x-y \rvert - \frac{\ell^2 - x^2}{2\ell^2} \delta(x-y) - \frac{2\ell^2 - x^2 - y^2}{8\ell^2} Pf_\mu \frac{1}{\lvert x+y \rvert}$
- Generically a non-local operator, contrary to the wedge or massless fields

De Sitter spacetime and relative entropy

De Sitter spacetime

- d-dimensional de Sitter is embedded in $(d+1)$-dimensional Minkowski space as hyperboloid $\eta_{AB} X^A X^B = H^{-2}$
- Expanding half of dS (Poincaré patch) with metric $\mathrm{d}s^2 = \eta_{AB}\, \mathrm{d}X^A\, \mathrm{d}X^B = -\,\mathrm{d}t^2 + \mathrm{e}^{2Ht}\, \mathrm{d}\mathbf{x}^2$ describes primordial inflation and current accelerated expansion of our universe
- Maximally symmetric solution of Einstein's equations with cosmological constant $\Lambda = (d-1)H^2$
- Generator of boosts is tangent to hyperboloid: $M_{0j} = X_0 \partial_{X^j} - X_j \partial_{X^0} = -\frac{1}{2H}\left(H^2\mathbf{x}^2 - \mathrm{e}^{-2Ht} + 1\right)\partial_j - \mathbf{x}_j \partial_t + H \mathbf{x}_j \mathbf{x}^i \partial_i$
- Modular Hamiltonian is known for dS vacuum state Ω and algebra $\mathfrak{A}$ generated by fields restricted to intersection of hyperboloid and wedge $W_1 = \{X^A : X^1 \geq |X^0|\}$: $\ln \Delta_\Omega = \mathrm{i} M_{01}$



(Borchers/Buchholz, Global properties of vacuum states in de Sitter space 1999)

Relative entropy in de Sitter spacetime

- Compute relative entropy $S(\Omega\|\Psi_f)$ for coherent state $\Psi_f = \mathrm{e}^{\mathrm{i}\phi(f)}\Omega$ via Araki formula (= relative entanglement entropy, with entanglement region A the intersection of hyperboloid and wedge $A = \{(t,\mathbf{x})\colon 2H\mathbf{x}^1 \geq \left|1 - \mathrm{e}^{-2Ht} + H^2\mathbf{x}^2\right|\}$, and $\operatorname{supp} f \subset A$)
- $S(\Omega\|\Psi_f) = \mathrm{i}\pi\langle\Delta(M_{01}f),\Delta f\rangle$ with commutator function $\Delta(x,y) = \mathrm{i}[\phi(x),\phi(y)]$ and symplectic product $\langle f,g\rangle = \mathrm{i}\int\left[f^*(t,\mathbf{x})\dot{g}(t,\mathbf{x}) - g(t,\mathbf{x})\dot{f}^*(t,\mathbf{x})\right]_{t=0}\mathrm{d}^{d-1}\mathbf{x}$
- Further manipulations: $S(\Omega\|\Psi_f) = 2\pi\int\left[\mathbf{x}^1\mathcal{H}(\hat{f},x) + \frac{1}{2}H\mathbf{x}^2\partial_1\hat{f}\partial_t\hat{f} - H\mathbf{x}^1\mathbf{x}^i\partial_i\hat{f}\partial_t\hat{f}\right]_{t=0}\mathrm{d}^{d-1}\mathbf{x}$ with $\hat{f}(x) = \int\Delta(x,y)f(y)\sqrt{-g}\,\mathrm{d}^d y$ and $\mathcal{H}(g,x) = \frac{1}{2}\left(\dot{g}^2 + \mathrm{e}^{-2Ht}\partial_i g\partial^i g + m^2 g^2\right)$ the Hamiltonian density $\Rightarrow$ not manifestly positive!
- However, correct flat-space limit $H \to 0$

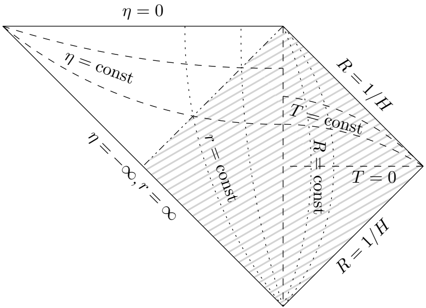
Relative entropy in de Sitter spacetime

- Solution: evaluate symplectic product on different Cauchy surface $\Sigma = \{(t, \mathbf{x})\colon 2Ht + \ln(1 + H^2\mathbf{x}^2) = 0\}$ instead of just $t = 0$
- $S(\Omega \| \Psi_f) = 2\pi \int \mathcal{Q}(\hat{f}) \left(1 + H^2\mathbf{x}^2\right)^{-\frac{d}{2}} \mathrm{d}^{d-1}\mathbf{x}$ with $\mathcal{Q}(h) = \frac{\mathbf{x}^1}{2\sqrt{1+H^2\mathbf{x}^2}} \left[\partial_n h \partial_n h + \left[(1 + H^2\mathbf{x}^2)\delta^{kl} - H^2\mathbf{x}^k\mathbf{x}^l\right] \hat{\partial}_k h \hat{\partial}_l h + m^2 h^2\right]_\Sigma \geq 0$
- $\mathcal{Q}(h) = -n_\mu \xi^{(1)}_\nu T^{\mu\nu}(h)$ with n_μ normal to Σ, $\xi^{(1)}_\nu$ Killing vector associated to boosts: $M_{01} = \xi^\mu_{(1)} \partial_\mu$, and $T_{\mu\nu}$ canonical stress tensor
- $\mathcal{Q}$ is a Noether charge density, compare Wald (Black hole entropy is the Noether charge 1993) and Floerchinger (Lectures on quantum fields and information theory)
- Relative entropy is also convex: $\lambda S(\Omega \| \Psi_f) + (1-\lambda) S(\Omega \| \Psi_g) \geq S(\Omega \| \Psi_{\lambda g + (1-\lambda) h})$ for $\operatorname{supp} f, g \subset A$ and $\lambda \in [0, 1]$

Observers in de Sitter spacetime

De Sitter spacetime, static patch

- However, not all of Poincaré patch is accessible to a single observer
- Relevant patch: static patch with metric $\mathrm{d}s^2 = -(1 - H^2R^2)\,\mathrm{d}T^2 + (1 - H^2R^2)^{-1}\,\mathrm{d}R^2 + R^2\,\mathrm{d}\Omega^2_{d-2}$
- Cosmological horizon at $R = 1/H$ or $r = -\eta$
- Modular Hamiltonian is known for dS vacuum state Ω and algebra $\mathfrak{A}$ generated by fields restricted to the static patch: $\mathrm{i}[\ln \Delta_\Omega, \phi(x)] = 2\pi H^{-1}\partial_T \phi(x)$

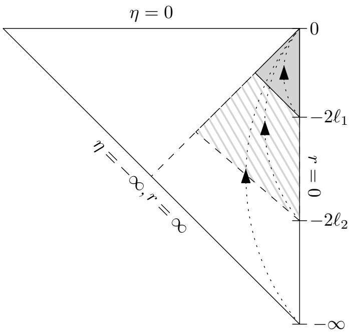


(Figari/Höegh-Krohn/Nappi, Interacting Relativistic Boson Fields in the De Sitter Universe with Two Space-Time Dimensions 1975, Gibbons/Hawking, Cosmological event horizons, thermodynamics, and particle creation 1977, Chandrasekaran/Longo/Penington/Witten, An algebra of observables for de Sitter space 2023)

De Sitter spacetime, static patch

- Recent result: Modular Hamiltonian dS vacuum state Ω and algebra $\mathfrak{A}$ generated by massless, conformally coupled fields of dimension Δ restricted to diamonds of size ℓ inside the static patch
- $\ln \Delta_\Omega = -2\pi D + \pi/\ell K$, a linear combination of dilations and special conformal transformations
- $i[\ln \Delta_\Omega, \hat{\mathcal{O}}(x)] = 2\pi H^{-1} \partial_T \hat{\mathcal{O}}(x) - \frac{\pi}{H^2 \ell} \frac{e^{-HT}}{\sqrt{1-H^2R^2}} \left[\partial_T - H(1 - H^2 R^2)(R\partial_R + \Delta) \right] \hat{\mathcal{O}}(x)$

(Fröb, arXiv:2308.14797)



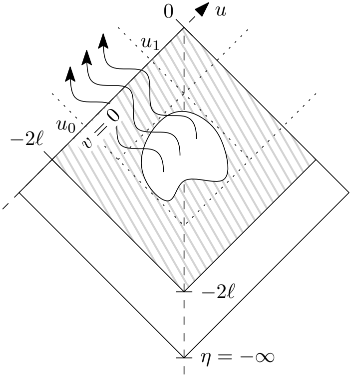
Relative entropy in de Sitter diamonds

- Compute relative entropy $S(\Omega \| \Psi_f)$ for coherent state $\Psi_f = \mathrm{e}^{\mathrm{i}\phi(f)}\Omega$, now for diamonds
- Computation easier in conformally flat coordinates $(\eta, \mathbf{x})$ of expanding Poincaré patch
- $\mathcal{S}\left(\Omega \| \mathrm{e}^{\mathrm{i}\Phi(f)}\Omega\right) = \frac{\pi}{2\ell} \int_{\eta=\chi} \left[(\ell^2 - \mathbf{x}^2)(\partial_\eta(\Delta f_\omega)\partial_\eta(\Delta f_\omega) + \partial_i(\Delta f_\omega)\partial^i(\Delta f_\omega)) + 2(\Delta f_\omega)^2 \right] \mathrm{d}^3\mathbf{x}$ with $f_\omega(x) \equiv (-H\eta)^{\Delta-4} f(x)$
- Manifestly positive $\mathcal{S}\left(\Omega \| \mathrm{e}^{\mathrm{i}\Phi(f)}\Omega\right) \geq 0$
- Convex under shrinking of the size of the diamond: $\partial_\ell^2 \mathcal{S}\left(\Omega \| \mathrm{e}^{\mathrm{i}\Phi(f)}\Omega\right) \geq 0$ (D'Angelo/Fröb/Galanda/Meda/Much/Papadopoulos, arXiv:2311.13990)
- Example of half-sided modular inclusion of algebras: $\mathrm{e}^{\mathrm{i}\tau \ln \Delta} \mathfrak{A}_{\text{sub}} \mathrm{e}^{-\mathrm{i}\tau \ln \Delta} \subset \mathfrak{A}_{\text{sub}}$ for ≥ 0, where $\mathfrak{A}_{\text{sub}}$ is the subalgebra of fields with support inside the smaller diamond and $\ln \Delta$ the modular Hamiltonian of the bigger one (Borchers, On revolutionizing quantum field theory with Tomita's modular theory 2000, Ciolli/Longo/Ranallo/Ruzzi, Relative entropy and curved spacetimes 2022)

Entropy–area law

- Consider backreaction of quantum fields on spacetime via semiclassical Einstein equations
- Evaluate relative entropy on horizon (characteristic surface) instead of Cauchy surface: $\mathcal{S}\left(\Omega \| e^{i\Phi(f)}\Omega\right) = -\frac{2\pi}{\ell} \int_{-2\ell}^{0} \int T_{uu}\big|_{v=0} u(u+2\ell)\, d\Omega\, du$
- Raychaudhuri equation relates change of area of the null surface with stress tensor: $\frac{d\delta\Theta}{du}\Big|_{v=0} = -32\pi G_N T_{uu}$, where $\delta\Theta$ is the geodesic expansion of the horizon
- Result: $\mathcal{S}\left(\Omega \| e^{i\Phi(f)}\Omega\right) = \frac{1}{4G_N} \frac{1}{2\ell} \int_{-2\ell}^{0} \delta A_H(u)\, du$ with horizon cross-sectional area $A_H(u) \equiv \int \sqrt{\gamma(u)}\, d\theta\, d\phi$

(D'Angelo/Fröb/Galanda/Meda/Much/Papadopoulos, arXiv:2311.13990)



Local temperature

- Consider observer with proper time t, four-velocity $v^\mu = -g^{\mu\nu} \partial_\nu t$, crossing Cauchy surface Σ normally such that $v_\mu \big|_\Sigma = n_\mu$ with normal vector n_μ
- Observer measures relative entropy $\mathcal{S}\left(\Omega \| \mathrm{e}^{\mathrm{i}\Phi(f)}\Omega\right) = \int_\Sigma T_{\mu\nu}(\Delta f_\omega)\xi^\mu n^\nu \, \mathrm{d}^3\Sigma = \int_\Sigma s \, \mathrm{d}^3\Sigma$ with entropy density s, and energy density $e \equiv v^\mu v^\nu T_{\mu\nu}$
- First law of thermodynamics: $\delta s = \beta \delta e$ with inverse temperature $\beta \Rightarrow \beta = \partial t / \partial(-\tau)$ with τ the parameter of the modular flow, $\xi^\mu \partial_\mu = \partial_\tau$
- Agrees with thermal time hypothesis (Connes/Rovelli/Martinetti/Longo/Rehren/...)
- Result: $\beta = \frac{2\pi}{H}\left[1 - \mathrm{e}^{-H(T_\tau - T_{\min})}\right]$ with T_τ static time of observer, $T_{\min} = -\frac{1}{H}\ln(2H\ell)$ static time of lower tip of the diamond (D'Angelo et al., arXiv:2311.13990)
- For large diamonds $\ell \to \infty$ we have $T_{\min} \to -\infty$ and recover the (known) temperature of the static patch $\beta_{\mathrm{dS}} = 2\pi/H$, for finite-size diamonds subleading corrections that decay exponentially fast

Thank you for your attention

Questions?

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References: arXiv:2308.14797, arXiv:2310.12185, arXiv:2311.13990, and arXiv:2312.04629