The fearful symmetry of quantum billiards

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Tyger, tyger, burning bright,
In the forests of the night;
What immortal hand or eye,
Could frame thy fearful symmetry?

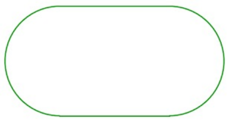
W. Blake

Eigenfunctions of a planar domain

Given a smooth bounded domain $\Omega \subset \mathbb{R}^2$, consider its *Neumann eigenvalues* μ_n and *eigenfunctions* $u_n : \overline{\Omega} \to \mathbb{R}$ (not identically 0):

$$\Delta u_n + \mu_n u_n = 0 \quad \text{in } \Omega, \qquad \nu \cdot \nabla u_n = 0 \quad \text{on } \partial\Omega\,.$$

There are infinitely many eigenvalues $0 = \mu_0 < \mu_1 \leq \mu_2 \leq \cdots \to \infty$ (the *spectrum* of Ω). All the eigenfunctions are smooth.



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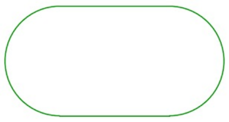
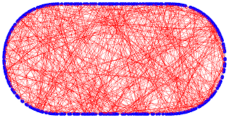

Mechanical interpretation: quantum billiards

- Eigenfunctions are the stationary states of the *free Schrödinger equation* on Ω . The dynamics is written off explicitly in terms of the spectral data.
- The classical counterpart is the *classical billiard* of shape Ω, which is an area-preserving diffeomorphism on a surface.
- The dynamical properties of the classical billiard has a major impact on high-frequency quantum dynamics (and on high energy eigenfunctions).

Principle: *The spectrum encodes much geometric information on the domain Ω.*

Example: **Weyl's law** / heat kernel asymptotics / wave invariants:

$$\sum_{n=0}^{\infty} e^{-t\mu_n} = \frac{|\Omega|}{4\pi} t^{-1} + \frac{|\partial\Omega|}{8\sqrt{8}} + t^{-\frac{1}{2}} \int_{\partial\Omega} \frac{\kappa}{12\pi} ds + t^{\frac{1}{2}} \int_{\partial\Omega} \frac{\kappa^2}{256\sqrt{\pi}} ds + \cdots \qquad \text{(W)}$$

Here κ is the curvature. Similarly for Dirichlet / on Riemannian manifolds.

Furthermore, isometric domains are isospectral.

Spectral geometry

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Question (Kac, 1967): ***Can one hear the shape of a drum?***

- The key question *(which we will* **not** *consider here)* of when a domain is determined by its spectrum (modulo a rigid motion) is **wide open**.
- **True** within the class of analytic $\mathbb{Z}_2$-symmetric convex domains (Zelditch, 2009). The only known **counterexamples** are non-convex polygons (Gordon, Webb & Wolpert, 1992).
- **Disks are "special"**: in particular, their spectral determination follows immediately from (W) and the *isoperimetric inequality*.
  A sort of "three-term isoperimetric inequality" shows that regular N-sided polygons are spectrally determined too (E. & Gómez-Serrano, 2022).

Reveling in roundness: eigenfunctions of the disk

Disks are indeed special. For starters, they are the extremizers of the first eigenvalue (Faber–Krahn, Szegö):

$$|\Omega|\,\mu_1(\Omega) \le |\mathbb{D}|\,\mu_1(\mathbb{D}).$$

Furthermore, everything can be written down explicitly. If we label the eigenvalues of the disk $\mathbb{D}$ with two integers, $\{\mu_{km}\}$, then $\sqrt{\mu_{km}}$ is the m^{th} positive zero of the function J_k' (Bessel). The corresponding eigenspace has dim 2 and is spanned by

$$u_{km} := J_k(\sqrt{\mu_{km}}r)\cos k\theta\,, \qquad \widetilde{u}_{km} := J_k(\sqrt{\mu_{km}}r)\sin k\theta$$

Obviously u_{0m} is **radial**, and therefore **constant** on $\partial\mathbb{D}$, for any m.

But what we are really interested in is Schiffer's conjecture.

Reveling in roundness: Schiffer's conjecture

Schiffer's conjecture (1950s):

On a bounded domain $\Omega \subset \mathbb{R}^2$, suppose that there exists a Neumann eigenvalue μ and a corresponding Neumann eigenfunction u that is **constant** *on $\partial\Omega$. Then Ω is a* **disk** *and u is* **radial**.

Schiffer's conjecture (1950s):

On a bounded domain $\Omega \subset \mathbb{R}^2$, suppose that there exists a Neumann eigenvalue μ and a corresponding Neumann eigenfunction u that is **constant** *on $\partial\Omega$. Then Ω is a* **disk** *and u is* **radial**.

The known **rigidity** properties are essentially the following:

- $\partial\Omega$ must be analytic (Kinderlehrer & Nirenberg, 1977).
- If *infinitely many* l.i. eigenfunctions are constant on the boundary, $\Omega =$ disk (Berenstein, 1980).
- If μ is small enough (specifically, $\mu \leq \mu_6(\Omega)$), $\Omega =$ disk (Avilés, 1986).

Furthermore, it is connected with the **Pompeiu problem**: a simply connected domain Ω satisfies the Schiffer overdetermined condition iff there exists a function $f \in C(\mathbb{R}^2)\backslash\{0\}$ (here, e.g. $f(x) := \sin(\mu^{1/2} x_1)$) such that

$$\int_{\mathcal{R}(\Omega)} f\, dx = 0 \qquad \text{for any rigid motion } \mathcal{R}.$$

Alas, we do not have much to say about the Schiffer conjecture.

The Neumann spectrum of an annulus

On an annulus $\Omega_a := \{a < r < 1\}$, with $a \in (0,1)$, we can compute the spectrum pretty much as in the case of disk:

- We label the eigenvalues with two indices: $\{\mu_{km}(a)\}$. Then $\mu_{km}(a)$ is the m^{th} smallest positive number for which the Bessel ODE

$$\mathcal{J}_{km}''(r) + \frac{\mathcal{J}_{km}'(r)}{r} + \left(\mu_{km}(a) - \frac{l^2}{r^2}\right)\mathcal{J}_{km}(r) = 0$$

  has a solution with $\mathcal{J}_{km}'(a) = \mathcal{J}_{km}'(1) = 0$.
- The corresponding basis of eigenfunctions is then

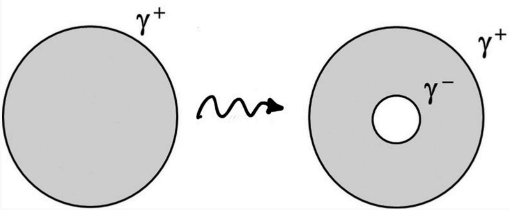
$$u_{km}^a := \mathcal{J}_{km}(r)\cos k\theta\,, \qquad \widetilde{u}_{km}^a := \mathcal{J}_{km}(r)\sin k\theta\,.$$

  Yet *the eigenspaces are not necessarily 2-dimensional:* one can have $\mu_{km}(a) = \mu_{k'm'}(a)$ with $(k,m) \neq (k',m')$.
- Anyhow, u_{0m}^a is **radial**, and therefore **locally constant** on the boundary (i.e., constant of each connected component of $\partial\Omega_a$), for any m and any a.

Relaxing the hypotheses: the “almost Schiffer” problem

A conjecture for disks and annuli:

On a bounded domain $\Omega \subset \mathbb{R}^2$, suppose that there exists a Neumann eigenvalue μ and a corresponding Neumann eigenfunction u that is **locally constant** *on $\partial\Omega$ (i.e., constant on each component of the boundary). Then Ω is a* **disk or annulus** *and u is* **radial**.



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One can show the **rigidity** properties are essentially the same:

- $\partial\Omega$ must be analytic (by Kinderlehrer & Nirenberg, 1977).
- If *infinitely many* l.i. eigenfunctions are locally constant on the boundary, Ω = disk or annulus (essentially as in Berenstein, 1980).
- If μ is small enough (specifically, $\mu \le \mu_3(\Omega)$), Ω = disk or annulus.

Furthermore, it is connected with a nontrivial **Pompeiu-type integral**: if Ω is doubly connected, the function $f(x) := \sin(\mu^{1/2} x_1)$ satisfies

$$\int_{\mathcal{R}(\Omega)} f\,dx - c\int_{\mathcal{R}(\Omega')} f\,dx = 0 \qquad \text{for any rigid motion } \mathcal{R},$$

where Ω' is the “hole” and c is certain constant.

Yet we can show this conjecture is as false as a 3 dollar bill!

Main result

Theorem (E., Fernández, Sicbaldi & Ruiz, 2023)

There are nonradial domains Ω diffeomorphic to an annulus where the equation

$$\Delta u + \mu u = 0 \quad \text{in } \Omega\,, \qquad\qquad \nabla u = 0 \quad \text{on } \partial\Omega\,.$$

admits a nontrivial solution, for some Neumann eigenvalue μ.

(The domains are small $\mathbb{Z}_l$-symmetric perturbations of thin annuli.)

Idea of the proof:

- In each annulus $\Omega_a := \{a < r < 1\}$, take a *radial Neumann* eigenfunction $u^a_{0m}(r)$, with eigenvalue $\mu_{m0}(a)$. This is a **family of trivial (i.e., radial) solutions** smoothly depending on the parameter $a \in (0, 1)$.
- So use **bifurcation** to get nontrivial solutions! (Note the difference with Schiffer's conjecture)

Pitchfork bifurcation on infinite dimensional spaces

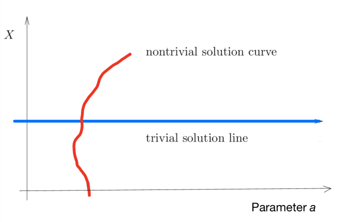
Theorem (Crandall & Rabinowitz, 1971)

"Under suitable technical hypotheses, a family of **nontrivial** *solutions can bifurcate from a family of* **trivial** *solutions."*
Key example: $F(v,a) := v(a-v^2)$, bifurcating from $a=0$.

Specifically: given a C^2 function $F(v,a)$ between Banach spaces, assume that:

- $F(0,a)=0$ for all values of the parameter $a \in (0,1)$.
- $\dim \operatorname{Ker}(D_v F)_{(0,a_*)} = \operatorname{codim} \operatorname{Ran}(D_v F)_{(0,a_*)} = 1$ for some $a_* \in (0,1)$.
- $(D_v D_a F)_{(0,a_*)}[\operatorname{Ker}(D_v F)_{(0,a_*)}] \not\subset \operatorname{Ran}(D_v F)_{(0,a_*)}$.

Then there is a branch ($\equiv C^1$ curve) of nontrivial solutions $\{(v_s,a_s) : |s| < s_0\}$ to the equation $F(v,a)=0$ which bifurcate from the point $(v_0,a_0)=(0,a_*)$.



Attempt to formalize the argument (a.k.a. the devil is in the details)

- To *deform the annuli* $\Omega_a := \{a < r < 1\}$, for each "small" $V \in C^{k,\alpha}(\mathbb{T}, \mathbb{R}^2)$, set
$$\Omega_a^V := \{(a + V_1(\theta) < r < 1 + V_2(\theta)\}.$$
If $u_a(r)$ is the third radial Neumann eigenfunction (with eigenvalue μ_a), the IFT ensures the existence of the corresponding eigenfunction $u_a^V(r, \theta)$ on Ω_a^V with eigenvalue μ_a^V.

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- Since we want to choose $a, V \neq 0$ so that u_a^V is constant on Ω_a^V, we look for zeros of $F(V,a) := u_a^V|_{\partial\Omega_a^V}$. By elliptic regularity, F maps an open subset of $C^{k,\alpha}(\mathbb{T}, \mathbb{R}^2) \times (0,1)$ to $C^{k,\alpha}(\mathbb{T}, \mathbb{R}^2)$ (but no better).
- We have $F(0,a) = 0$ for all a. The derivative is (with some $c_j(a) \in \mathbb{R}$):

$$(D_V F)_{(0,a)} W := (c_1(a)\psi_W(a,\theta), c_2(a)\psi_W(a,\theta))$$

  where $\psi_W(r,\theta)$ is the only solution to the Neumann problem

$$\Delta\psi_W + \mu_a \psi_W = 0 \quad \text{in } \Omega_a, \qquad \partial_r \psi_W(r,\theta) = \begin{cases} W_1(\theta) & \text{on } r = a, \\ W_2(\theta) & \text{on } r = 1. \end{cases}$$

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  Big bad problem — loss of derivatives: $(D_V F)_{(0,a)}$ maps $C^{k,\alpha} \to C^{k+1,\alpha}$, so $\operatorname{codim} \operatorname{Ran}(D_V F)_{(0,a)} = \infty$. All bifurcation theorems break down. (And this is a really bad case; we do not how to fix it using Nash–Moser.)

A great idea of Mindlend, Fall & Weth — Schiffer on spheres and cylinders!

How to avoid the loss of derivatives problem (Fall, Mindlend & Weth, 2023)

The basic unknown is not $V(\theta) \in C^{k,\alpha}(\mathbb{T}, \mathbb{R}^2)$ (used to parametrize the boundary), but a "Dirichlet" function $v(r,\theta) \in C^{k,\alpha}_{\mathrm{D}}(\Omega_{1/2})$ (i.e., $v|_{\partial\Omega_{1/2}} = 0$).

In turn, we use v to parametrize the boundary by a map $v \mapsto V_v \in C^{k,\alpha}(\mathbb{T}, \mathbb{R}^2)$ and to correct the eigenfunction using a "Dirichlet + Neumann" function $v \mapsto w_v \in C^{k,\alpha}(\Omega_{1/2})$ with $w = \partial_r w = 0$ on $\partial\Omega_{1/2}$.

- The function we consider is
$$F(v,a) := \left\{[\Delta + \mu_{02}(a)]\left[(u^a_{02} \circ \Phi^0_a + w_v) \circ (\Phi^{V_v}_a)^{-1}\right]\right\} \circ \Phi^{V_v}_a\,,$$
where $\Phi^V_a : \Omega^V_a \to \Omega_{1/2}$ is a family of diffeomorphisms.
- Then $F(v,a) = 0 \implies \exists$ solution to "almost Schiffer" on Ω^V_a.
- $F(0,a) = 0$ for all a and the differential is
$$(D_v F)_{(0,a)} w = \left\{[\Delta + \mu_{02}(a)]\left[w \circ (\Phi^0_a)^{-1}\right]\right\} \circ \Phi^0_a$$
- **Key observation — no loss of derivatives:** *For any integer l, $F : \mathcal{X} \to \mathcal{Y}$ is Fredholm of index 0, with the* **anisotropic** *spaces of* $\mathbb{Z}_l$**-symmetric** *functions*
$$\mathcal{X} := \{u \in C^{k,\alpha}_{\mathrm{D}} : \partial_r u \in C^{k,\alpha}\}/\mathbb{Z}_l\,, \qquad \mathcal{Y} := \left(C^{k-1,\alpha} + C^{k-2,\alpha}_{\mathrm{D}}\right)/\mathbb{Z}_l\,.$$

Verifying the hypotheses of Crandall–Rabinowitz

$$F(v,a) := \left\{[\Delta + \mu_{0,2}(a)]\left[(u_{02}^a \circ \Phi_a^0 + w_v) \circ (\Phi_a^{V_v})^{-1}\right]\right\} \circ \Phi_a^{V_v}, \qquad v \in C_{\mathrm{D}}^{k,\alpha}(\Omega_{1/2}),$$

$$(D_v F)_{(0,a)} w = \left\{[\Delta + \mu_{0,2}(a)]\left[w \circ (\Phi_a^0)^{-1}\right]\right\} \circ \Phi_a^0, \qquad w \in C_{\mathrm{D}}^{k,\alpha}(\Omega_{1/2}).$$

1. $F(0,a) = 0$ for all $a \in (0,1)$, by definition. ✓
2. $\dim \operatorname{Ker}(D_v F)_{(0,a)} = \operatorname{codim} \operatorname{Ran}(D_v F)_{(0,a)}$ for all $a \in (0,1)$, because $D_v F$ is Fredholm. ✓
3. **One-dimensional kernel — TO DO:** $\dim \operatorname{Ker}(D_v F)_{(0,a_{l,*})} = 1$ for some $a_l \in (0,1)$. Since $\mathcal{X}$ consist of "Dirichlet" functions, to have $\dim \operatorname{Ker}(D_v F)_{(0,a_l)} = 1$ we will show that the Dirichlet eigenvalues $\lambda_{ln}(a_l)$ of an annulus Ω_{a_l} satisfy the crossing condition

   $$\mu_{0,2}(a_l)) = \lambda_{l,0}(a_l) \neq \lambda_{ml,n}(a_l) \qquad \forall\,(m,n) \neq (1,0).$$

   Note we need $l \neq 1$ to ensure (by elementary computations) that the resulting "almost Schiffer" domains will not be radially symmetric.
4. **Transversality — TO DO**: $(D_v D_a F)_{(0,a_l)}[\operatorname{Ker}(D_v F)_{(0,a_l)}] \not\subset \operatorname{Ran}(D_v F)_{(0,a_l)}$. This boils down to demanding $\mu_{0,2}'(a_{l,*}) \neq \lambda_{l,0}'(a_{l,*})$.

So we just have to analyze the spectrum of annuli!

The crossing condition

Lemma

For all $l \geq 4$, there exists some $a_l \in (0,1)$ such that

$$\mu_{0,2}(a_l)) = \lambda_{l,0}(a_l) \neq \lambda_{ml,n}(a_l) \qquad \forall\,(m,n) \neq (1,0).$$

- It is easy to prove that the Dirichlet and Neumann eigenvalues of Ω_a converge to those of the disk as $a \to 0$ (essentially because points have zero capacity). Therefore, $\lambda_{l,0}(a) \geq \lambda_{4,0}(a) > \mu_{0,2}(a)$ for a close to 0.
- Suppose now that a is close to 1, so $h := (1-a)/\pi \ll 1$. Then, with $\varphi \in H_0^1((0,\pi)\backslash\{0\}$ and $x := (1-r)/h \in (0,\pi)$,

$$\lambda_{l,0}(a) = \inf_{\varphi} \frac{h^{-2}\displaystyle\int_0^\pi \psi'(x)^2(1-hx)\,dx + l^2\int_0^\pi \frac{\psi(x)^2}{(1-hx)}\,dx}{\displaystyle\int_0^\pi \psi(x)^2(1-hx)\,dx}$$

$$= h^{-2}\left[\inf_{\varphi} \frac{\int_0^\pi \psi'^2\,dx}{\int_0^\pi \psi^2\,dx} + O(h)\right] + l^2(1+O(h)) = h^{-2} + O(l^2 + h^{-1}).$$

Likewise, $\mu_{0,2}(a) = 4h^{-2} + O(h^{-1})$. Hence $\lambda_{l,0}(a) < \mu_{0,2}(a)$ for any fixed l and a close to 1, and the eigenvalues must cross.

The transversality condition

Lemma

For $l \gg 1$ (or $l = 4$), $\mu'_{0,2}(a_l) \neq \lambda'_{l,0}(a_l)$. Furthermore, $a_l = 1 - \frac{\sqrt{3}\pi}{l} + O(l^{-2})$.

- From the equation $\mu_{0,2}(a_l) = \lambda_{l,0}(a_l) > l^2 \gg 1$, it is easy to see that $h_l := (1 - a_l)/\pi \ll 1$. In fact, one has $\mu_{0,2}(a_l) = 4h_l^{-2} + O(h_l^{-1})$ and $\lambda_{l,0}(a_l) = h_l^{-2} + l^2 + O(h_l^{-1} + l)$, so $h_l = \sqrt{3}\, l^{-2} + O(l^{-1})$.
- The result is proven by means of a "second order asymptotic expansion" in l for $\mu_{0,2}(a_l)$ and $\lambda_{l,0}(a_l)$, which relies on the analytic dependence of the Dirichlet and Neumann eigenvalues of the operators

$$T_{\eta,\delta} := \partial_x^2 - \frac{\eta(1-\frac{\pi}{2}\eta(1+\delta))}{1-\eta(1-\frac{\pi}{2}\eta(1+\delta))x}\,\partial_x - \frac{3(1-\frac{\pi}{2}\eta(1+\delta))^2}{(1-\eta(1-\frac{\pi}{2}\eta(1+\delta))x)^2},$$

$$\widetilde{T}_{\eta,\delta} := \partial_x^2 - \frac{\eta(1-\frac{\pi}{2}\eta(1+\delta))}{1-\eta(1-\frac{\pi}{2}\eta(1+\delta))x}\,\partial_x,$$

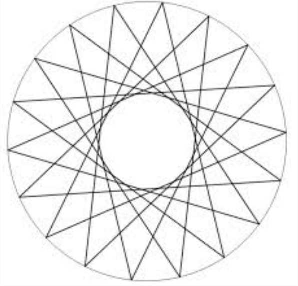
on the parameters (η, δ).

A Schiffer problem for classical billiards?

Question:

Let Ω be a smooth convex billiard. Suppose that there is a sequence of periodic trajectories $\left\{\left(s_n^{(j)}, \theta_n^{(j)}\right)\right\}_{n=1}^{N_j}$, with minimal periods $N_j \to \infty$, for which the angles are constant: $\theta_n^{(j)} = c_j \ \forall n$.

Does this imply $\Omega = \text{disk}$?



Muchas gracias!