Local Hölder Stability in the Inverse Steklov and Calderón Problems for Radial Schrödinger operators and Quantified Resonances

Niky Kamran

McGill University

Regensburg, December 13, 2023

Joint work with T. Daudé (Besançon) and F. Nicoleau (Nantes)
arXiv:2212.03148, to appear in AHP

The Dirichlet-to-Neumann map and the Steklov spectrum

Let

$$M = [0,1] \times S^{d-1}\,, \tag{1}$$

and consider the Dirichlet problem

$$\begin{cases} -\triangle u + q\,u = 0, & \text{on } M\,, \\ u = \psi \in H^{\frac{1}{2}}(\partial M), & \text{on } \partial M\,, \end{cases} \tag{2}$$

When $q \in L^{\infty}(M)$ and zero is not a Dirichlet eigenvalue, (2) has a unique solution $u \in H^1(M)$.

The **Dirichlet-to-Neumann (DN) map** $\Lambda_q : H^{1/2}(\partial M)$ to $H^{-1/2}(\partial M)$ is defined by

$$\Lambda_q \psi = (\partial_\nu u)_{|\partial M}\ , \tag{3}$$

The DN map is a self-adjoint operator on $L^2(\partial M = S^{d-1}, dS_g)$.

Its spectrum, the **Steklov spectrum**, is discrete

$$0 = \sigma_0 < \sigma_1 \leq \sigma_2 \leq \cdots \leq \sigma_k \to \infty \, .$$

Assume q is **radial**,

$$q = q(r) \, ,$$

and replace $r \in (0, 1]$ by

$$x = -\log r \in [0, \infty) \, ,$$

so that ∂M corresponds to $x = 0$. The Euclidean metric

$$g = dr^2 + r^2 d\Omega^2 \, ,$$

then takes the form

$$g = f(x)^4 (dx^2 + d\Omega^2) \, , \quad f(x) = \exp(-x/2) \, .$$

Setting $v = f^{d-2}u$, the Dirichlet problem (2) becomes

$$\begin{cases} [-\partial_x^2 - \triangle_S + Q(x)]v = -\frac{(d-2)^2}{4}v, & \text{on } M\,, \\ v = f^{d-2}\psi, & \text{on } \partial M\,, \end{cases} \tag{4}$$

where $\triangle_S$ denotes the Laplacian on the boundary sphere S^{d-1}, where

$$Q(x) := e^{-2x}q(e^{-x})\,.$$

We use separation of variables and the Fourier decomposition of $L^2(S^{d-1})$ to reduce (4) to an infinite sequence of radial ODEs. Let $\{Y_k,\ k \geq 0\}$ an orthonormal basis of $L^2(S^{d-1})$ of eigenfunctions of $\triangle_S$,

$$-\triangle_S Y_k = \alpha_k Y_k\,, \quad \alpha_k = k(k+d-2)\,,$$

and let

$$v = \sum_{k=0}^{\infty} v_k(x)\ Y_k\,.$$

We obtain

$$-v_k'' + Qv_k = -(\alpha_k + \frac{(d-2)^2}{4})v_k = -\kappa_k^2 v_k\,,$$

where

$$\kappa_k := k + \frac{d-2}{2}\,, \quad k \geq 0\,.$$

The Weyl-Titchmarsh function M

We now assume that

$$Q \in L^2(0,\infty)\,. \tag{5}$$

Under this assumption, it is well-known that

$$L = -\frac{d^2}{dx^2} + Q\,,$$

is of **limit point-type at infinity**, *i.e.* for all $z \in \mathbb{C} \setminus [-\beta,\infty)$ with $\beta >> 1$, there exists, up to a zon-zero multiplicative constant, a unique solution $u(x,z)$ of

$$-u'' + Qu = zu\,, \quad z \in \mathbb{C}\,,$$

which is L^2 at ∞. The **Weyl-Titchmarsh (WT)** function $M(z)$ is then defined by

$$M(z) := \frac{u'(0,z)}{u(0,z)} \text{ for all } z \in \mathbb{C} \setminus [-\beta,\infty)\,. \tag{6}$$

Note that the L^2 hypothesis (5) on Q does not guarantee that the initial potential $q \in L^\infty(M)$. Thus, the definition (3) of the DN map is not directly applicable in this L^2 setting. We overcome this problem by using the separation of variables:

$$\psi = \sum_{k=0}^{\infty} \psi_k Y_k \,,$$

and define the DN map Λ_q as a sum of operators Λ_q^k by

$$\Lambda_q \psi = \sum_{k=0}^{\infty} (\Lambda_q^k \psi_k) Y_k \,, \tag{7}$$

where the Λ_q^k are **multiplication operators**

$$\Lambda_q^k \psi_k = -\frac{(d-2)}{2} v_k(0) - v_k'(0)\,.$$

The Λ_q^k, $k \geq 0$ can be expressed in terms of the WT function M evaluated at the points $-\kappa_k^2$,

$$\Lambda_q^k \psi_k = \left(-\frac{(d-2)}{2} - M(-\kappa_k^2) \right) \psi_k\,,$$

thus providing the expression of the Steklov spectrum $\{\sigma_k,\, k \geq 0\}$ in terms of M

$$\sigma_k = -\frac{(d-2)}{4} - M(-\kappa_k^2)\,.$$

The Simon amplitude function A

There is an important representation formula due to B. Simon for M in terms of the Laplace transform of a unique **amplitude function** A, under the hypothesis that $Q \in L^1(0, \infty)$:

$$M(-\kappa^2) = -\kappa - \int_0^\infty A(\alpha) e^{-2\kappa\alpha} d\alpha \,, \quad \forall \kappa > \frac{1}{2} ||Q||_1 \,,$$

see B. Simon, *A new approach to inverse spectral theory, I. Fundamental formalism*, Annals of Mathematics **150**, (1999), 1029-1057.

We shall use a slightly refined version of this formula which applies in our L^2-setting.

The stability problem - Main result

We denote $\{\tilde{\sigma}_k, k \geq 0\}$ the Steklov spectrum associated to a potential $\tilde{Q}$. Since $M(-\kappa_k^2) = -\kappa_k + o(1)$, as $k \to \infty$, we have $\{\sigma_k - \tilde{\sigma}_k, k \geq 0\} \in \ell^\infty(\mathbb{N})$. We assume that

$$||\tilde{\sigma}_k - \sigma_k||_{\ell^\infty(\mathbb{N})} =: \epsilon. \tag{8}$$

Our main goal is to estimate the difference $\tilde{Q} - Q$ of the potentials.

We shall obtain stability estimates in the space $L^2(0, T)$ for any $T > 0$, meaning that

$$||\tilde{Q} - Q||_{L^2(0,T)} \leq C_T \; g(\epsilon)\,, \tag{9}$$

where $g(\epsilon) \to 0$ when $\epsilon \to 0$, and C_T is a constant depending only on T. More precisely, we have:

Theorem

Let $Q \in L^2(0,\infty)$ with Simon amplitude A. Let $\delta \geq 3-d$ and $\mu_k := \lambda_k + \delta$ where $\lambda_k = 2k+d-3$. Let $\{c_k \leq 0,\ k \geq 0\}$ be such the series $\sum_{k\geq 0} c_k t^{\lambda_k}$ has radius of convergence $R > 1$. Then the function

$$\tilde{A}(\alpha) = A(\alpha) + \sum_{k\geq 0} c_k e^{-\mu_k \alpha}\,, \quad \alpha > 0\,, \tag{10}$$

is the Simon amplitude of a potential $\tilde{Q} \in L^2(0,\infty)$. Moreover, for any fixed $T > 0$, there exists $C_T > 0$ such that

$$||\tilde{Q} - Q||_{L^2(0,T)} \leq C_T \left(||\tilde{\sigma}_k - \sigma_k||_{\ell^\infty(\mathbb{N})} \right)^{\theta}\,, \tag{11}$$

where the Hölder exponent $\theta \in (0, \frac{1}{2}]$ is independent of T and is given by

$$\theta = \frac{1}{2}\min\{1, \log R / \log(\frac{9M_0}{2})\}, \quad M_0 = \max\{2, 4(d-3+\delta)+1\}\,. \tag{12}$$

Remarks:

- When $R \geq \frac{9M_0}{2}$, we can take as Hölder exponent $\theta = \frac{1}{2}$.
- When the initial potential is the trivial potential $Q = 0$, the perturbed potentials $\tilde{Q}$ can be seen as a generalization of the so-called Bargmann potentials.
- With respect to the original Schrödinger operator, the type of perturbation being considered for the amplitude function A amounts to the introduction of a **finite number of negative eigenvalues** $-\frac{\mu_k^2}{4}$ for $k = 1, ..., N$, (corresponding to the case where μ_k is negative), and of a **countable set of real resonances** $-\frac{|\mu_k|}{2}$ which are equally spaced on the negative real axis (for k greater than some k_0). These resonances are quantified explicitly in terms of the parameter δ and the eigenvalues of the Laplace Beltrami operator Δ_S on the boundary sphere.

- Generically, the best one can expect are **logarithmic stability results** for the inverse Steklov problem, see T. Daudé, N. Kamran N and F. Nicoleau, *Stability in the inverse Steklov problem on warped product Riemannian manifolds* J. Geom. Anal. **31** (2021), no. 2, 1821-1854.

Hölder stability for the Calderón problem

As a byproduct, we also obtain local Hölder stability estimates for the Calderón problem for radial Schrödinger operators on the unit ball.

Corollary

Let $q \in L^2((0,1), r^3 dr)$ be a fixed radial potential and let $\tilde{q}$ be the potential associated with $\tilde{Q}$ given in Theorem 1. Then, $\Lambda_q - \Lambda_{\tilde{q}}$ is a bounded operator on $L^2(S^{d-1})$, and for any fixed $T > 0$, there exists a positive constant C_T such that

$$||\tilde{q} - q||_{L^2((e^{-T},1),r^3)dr} \leq C_T \, ||\Lambda_{\tilde{q}} - \Lambda_q||^{\theta}_{B(L^2(S^{d-1}))},$$

where $\theta \in (0, \frac{1}{2}]$ is the Hölder exponent given above.

Sketch of proof

Making the change of variables $\alpha = -\log t$, our hypothesis on the difference of the Steklov spectra takes the form

$$| \int_0^1 t^{-\delta} \big(A(-\log t) - \tilde{A}(-\log t) \big) \, t^{2k+d-3+\delta} dt | \leq \epsilon \, .$$

We see that this is a **Hausdorff moment problem**, so we do not expect better than **logarithmic stability estimates**.

Nevertheless, one can approach the stability problem differently by working directly with perturbations of the amplitude A by exponential series obtained from power series of **Müntz type**. We shall see that his leads to our Hölder stability results.

We set for $k \geq 0$

$$\lambda_k := 2k + d - 3 + \delta\,,$$

where $\delta \geq 3 - d$ is an arbitrary fixed real parameter (so that $\lambda_k \geq 0$), and

$$h(t) = t^{-\delta}\left(\tilde{A}(-\log t) - A(-\log t)\right)$$

We define formally a new amplitude $\tilde{A}$ by adding to A a power series

$$\tilde{A}(\alpha) = A(\alpha) + \sum_{k\geq 0} c_k e^{-(\lambda_k+\delta)\alpha}\,, \tag{13}$$

or equivalently

$$h(t) = \sum_{k \geq 0} c_k t^{\lambda_k} \, .$$

We assume that the series defining $h(t)$ has a radius of convergence $R > 1$, so that $h \in C^0([0,1])$. Furthermore we assume that h is such that our starting hypothesis holds, that is

$$|\int_0^1 h(t)\, t^{\lambda_k} dt| \leq \epsilon \, , \quad \forall k \geq 0 \, .$$

Our goal is to obtain a good approximation of Hölder type for $||h||_2^2$. We use results from G. Still, *On the approximation of Müntz series by Müntz polynomials*, J. Approx. Theory **45**, (1985), 26-54, and the polynomial approximation techniques of our 2021 JGA paper.

Theorem

Given $\epsilon > 0$ *and* $R > 1$ *as above and letting* $M_0 = \max\{2, 4(d-3+\delta)+1\}$, *we have, for some universal constant* $B > 0$, *the estimate*

$$||h||_2^2 \leq B^2 \epsilon + R^{1-d} \epsilon^{\frac{\log R}{\log(\frac{9M_0}{2})}}. \tag{14}$$

We note that the estimate (14) is generally a Hölder type estimate for $||h||_2^2$, but that if $R > \frac{9M_0}{2}$, this estimate is Lipschitz.

Given a sequence $\Lambda_\infty := (\lambda_n)_{n\geq 0}$ of integers such that $0 \leq \lambda_0 < \lambda_1 < \cdots$ and $\lambda_k \to \infty$ as $k \to \infty$, we define for fixed $n \geq 1$ the finite sequence

$$\Lambda_n := 0 \leq \lambda_0 < \lambda_1 < \ldots < \lambda_n \,,$$

giving rise to the vector space $\mathcal{M}(\Lambda)$ of "Müntz polynomials of degree λ_n":

$$\mathcal{M}(\Lambda_n) = \{P \mid P(t) = \sum_{k=0}^{n} a_k \; t^{\lambda_k}\} \,.$$

Recall that according to the Müntz-Szász's Theorem, if Λ_∞ is a sequence of positive real numbers as above, then span $\{t^{\lambda_0}, t^{\lambda_1}, \ldots\}$ is dense in $L^2([0,1])$ if and only if

$$\sum_{k=1}^{\infty} \frac{1}{\lambda_k} = \infty \,.$$

Given now a function f in $C^0([0,1])$ or in $L^2([0,1])$, the error of approximation of f with respect to $\mathcal{M}(\Lambda_n)$ is defined by

$$E_k(f,\Lambda_n) := \inf_{P \in \mathcal{M}(\Lambda_n)} ||f - P||_k \,,$$

where $k = 2$ or $k = \infty$ depending on whether $f \in C^0([0,1])$ or $f \in L^2([0,1])$. For our application, we have $\lambda_k := 2k + d - 3 + \delta$, giving $\lambda_{k+1} - \lambda_k = 2 > 0$, so by Theorem 2 of Still, we know that

$$E_\infty(h,\Lambda_n) \leq CR^{-\lambda_{n+1}} \,, \tag{15}$$

for some positive constant C.

We have, denoting by π_n the orthogonal projection onto the subspace $\mathcal{M}(\Lambda_n)$,

$$||h||_2^2 = ||\pi_n(h)||_2^2 + ||h - \pi_n(h)||_2^2 \,.$$

Our next step is to combine a certain estimate from our JGA paper with the estimate (15) to obtain an estimate for the norm of $\pi_n(h)$. In order to do so, we use the Gram-Schmidt process to obtain polynomials $(L_m(t))$ with $L_0(t) = 1$, and for $m \geq 1$,

$$L_m(t) = \sum_{j=0}^{m} C_{mj} t^{\lambda_j} ,$$

where we have set

$$C_{mj} = \sqrt{2\lambda_m + 1} \, \frac{\prod_{r=0}^{m-1}(\lambda_j + \lambda_r + 1)}{\prod_{r=0, r\neq j}^{m}(\lambda_j - \lambda_r)} .$$

The family $(L_m(t))$ defines an orthonormal Hilbert basis of $L^2([0,1])$.

We may now recall the following estimate from JGA,

$$||\pi_n(h)||_2^2 \leq \epsilon^2 \sum_{k=0}^{n} \left(\sum_{p=0}^{k} |C_{kp}| \right)^2 ,$$

which gives immediately

$$||h||_2^2 \leq \epsilon^2 \sum_{k=0}^{n} \left(\sum_{p=0}^{k} |C_{kp}| \right)^2 + CR^{-\lambda_{n+1}} , \tag{16}$$

using (15) and the inequality

$$||h - \pi_n(h)||_2^2 = E_2(h, \Lambda_n) \leq E_\infty(h, \Lambda_n) .$$

Now, according to another estimate from JGA, we have

$$||\pi_n h||_2^2 \leq B^2 \epsilon^2 \; g(n)^2 \,,$$

where B is a positive constant and $g : [0, +\infty)$ is a monotone increasing function defined for $t \in [0, +\infty)$ by

$$g(t) = \frac{3}{2} \frac{1}{\sqrt{\left(\frac{9M_0}{2}\right)^2 - 1}} \sqrt{2t+1} \; \left(\frac{9M_0}{2}\right)^{t+1} \,, \tag{17}$$

where

$$M_0 = \max\{2, 4(d - 3 + \delta) + 1\}$$

Now, we choose n as a function of ϵ so as to control the norm of the projection $||\pi_n h||_2^2$ of h and thus set

$$n(\epsilon) := [\ (g^{-1}(\frac{1}{\sqrt{\epsilon}}))]$$

where square brackets denote the integral part function. Since g is a monotone increasing function, we have

$$g(n(\epsilon)) \leq \frac{1}{\sqrt{\epsilon}}\,, \tag{18}$$

so we obtain immediately:

$$||\pi_{n(\epsilon)} h||_2^2 \leq B^2 \epsilon. \tag{19}$$

Our next task is now to estimate the size of $n(\epsilon)$ relative to ϵ so as to obtain the Hölder estimate we seek for $||h||_2^2$. From (17), we obtain that

$$g(t) \sim (t+1)\log(\frac{9M_0}{2}),$$

as $t \to \infty$, which combined with (18) leads to

$$n(\epsilon) = \frac{\log(\frac{1}{\sqrt{\epsilon}})}{\log(\frac{9M_0}{2})}.$$

Plugging this into (16) gives

$$||h||_2^2 \leq B^2\epsilon + CR^{-\lambda_{n(\epsilon)+1}}.$$

Now, using the expression $\lambda_k = 2k + d - 3 + \delta$, we have

$$R^{-\lambda_{n(\epsilon)+1}} = R^{1-d-\delta} R^{-2n(\epsilon)} \sim R^{1-d-\delta} R^{-\frac{2\log\frac{1}{\sqrt{\epsilon}}}{\log(\frac{9M_0}{2})}} \sim R^{1-d-\delta} \epsilon^{\frac{\log R}{\log(\frac{9M_0}{2})}},$$

and obtain In terms of the amplitude function A in the variable $\alpha \in (0, \infty)$, using the relation

$$||h||_2^2 = \int_0^1 t^{-2\delta} (A(-\log t) - \tilde{A}(-\log t))^2 \, dt\,,$$

we obtain

$$\int_0^\infty e^{(2\delta-1)\alpha} (A(\alpha) - \tilde{A}(\alpha))^2 \; d\alpha \leq B^2 \epsilon + R^{1-d-\delta} \epsilon^{\frac{\log R}{\log(\frac{9M_0}{2})}},$$

as claimed.

From the perturbed amplitude $\tilde{A}$ to $\tilde{Q} \in L^2(0, \infty)$

We want to prove the existence of square-integrable potentials $\tilde{Q}$ associated to perturbed amplitudes $\tilde{A}$ as defined in (13). This will require additional hypotheses on the perturbation.

We set for $k \geq 0$,

$$\mu_k := \lambda_k + \delta = 2k + d - 3 + 2\delta$$

so that

$$\tilde{A}(\alpha) = A(\alpha) + \sum_{k \geq 0} c_k e^{-\mu_k \alpha}, \quad \alpha > 0 . \tag{20}$$

If $d \geq 3$, the μ_k may be negative; so we split the series in (20) as

$$\tilde{A}(\alpha) = A(\alpha) + \sum_{k=0}^{N-1} c_k e^{-\mu_k \alpha} + \sum_{k \geq N} c_k e^{-\mu_k \alpha}, \quad \alpha > 0, \tag{21}$$

such that for $k = 0, ..., N-1,\ \mu_k < 0$ and for $k \geq N+1,\ \mu_k \geq 0$, with the convention that the first sum in (21) does not appear if all the μ_k's are positive (i.e if $N = 0$). Rewrite (21) as:

$$\tilde{A}(\alpha) - A(\alpha) = \sum_{k=0}^{N-1} 2c_k \ \sinh(|\mu_k|\alpha) + \sum_{k \geq 0} c_k e^{-|\mu_k| \ \alpha}. \tag{22}$$

We shall see that the first sum in the (RHS) of (22) corresponds to the introduction of **negative eigenvalues** $-\frac{\mu_k^2}{4}$ for $k = 0, ..., N-1$, while the second sum corresponds to the introduction of **real resonances** $-\frac{|\mu_k|}{2}$ for $k \geq 0$.

We have:

Theorem

Let $Q \in L^2(0, \infty)$ be a square-integrable potential with amplitude function A, let $\{c_k,\ k \geq 0\}$ be a sequence of real numbers such that

- *i) For all $k \geq 0,\ c_k \leq 0$.*
- *ii) the power series $\sum_{k\geq 0} c_k t^{\lambda_k}$ has a radius of convergence $R > 1$.*

Then the function $\tilde{A}$ defined by

$$\tilde{A}(\alpha) = A(\alpha) + \sum_{k\geq 0} c_k e^{-\mu_k \alpha}\,, \quad \alpha > 0\,,$$

is the amplitude function of a potential $\tilde{Q} \in L^2(0, \infty)$.

To prove the theorem, we first compute the difference of the spectral measures of Q and $\tilde{Q}$ in terms of the data contained in $\tilde{A}$.

The WT function M is a function of **Herglotz type**, that is $\operatorname{Im} z > 0 \implies \operatorname{Im} M(z) > 0$. This implies

$$M(z) = c + dz + \int_{\mathbb{R}} \left(\frac{1}{\lambda - z} - \frac{1}{1+\lambda^2}\right) d\rho(\lambda) \,,$$

where $c = Re(M(i)$, $d = \lim_{y \to +\infty} \frac{M(iy)}{y} = 0$, and $d\rho(\lambda)$ is the (positive) spectral measure associated to L,

$$d\rho(E) = \lim_{\epsilon \downarrow 0} \frac{1}{\pi} \operatorname{Im}\left(M(E + i\epsilon)\right) dE \,. \tag{23}$$

We denote by $\tilde{M}(-\kappa^2)$ the putative WT function associated with the amplitude $\tilde{A}(\alpha)$, so we have formally and for suitable κ,

$$\tilde{M}(-\kappa^2) - M(-\kappa^2) = -\int_0^\infty \left(\tilde{A}(\alpha) - A(\alpha)\right) \, e^{-2\kappa\alpha} \, d\alpha. \tag{24}$$

Now, using (22), we obtain

$$\tilde{M}(-\kappa^2) - M(-\kappa^2) = -2 \sum_{k=0}^{N-1} c_k \; \frac{|\mu_k|}{4\kappa^2 - \mu_k^2} \; - \; \sum_{k\geq 0} \frac{c_k}{2\kappa + |\mu_k|} \, . \qquad (25)$$

We note that the above series is indeed convergent since by hypothesis the power series $\sum_{k\geq 0} c_k t^{\lambda_k}$ has radius of convergence $R > 1$ so that in particular $\sum_{k\geq 0} |c_k| < \infty$.

Now, using (23), we are able to define the difference of the spectral measures $d\tilde{\rho}(E) - d\rho(E)$

For $E \geq 0$,

$$d\tilde{\rho}(E) = d\rho(E) - \frac{2}{\pi} \sum_{k\geq 0} c_k \; \frac{\sqrt{E}}{4E + \mu_k^2} \; dE,$$

and for $E < 0$,

$$d\tilde{\rho}(E) = d\rho(E) - \frac{1}{2} \sum_{k=0}^{N-1} c_k \; |\mu_k| \; \delta(\cdot + \frac{\mu_k^2}{4}) \; dE,$$

It is proved in F. Gesztesy and B. Simon, *A new approach of inverse spectral theory, II. General potentials and the connection to the spectral measure*, Annals of mathematics **152**, (2000), 593-643 that this corresponds to the introduction of a finite number of **negative eigenvalues** $-\frac{\mu_k^2}{4}$ for $k = 0, ..., N-1$, and of **real resonances** $-\frac{|\mu_k|}{2}$ for $k \geq 0$.

Now we use the results of R. Killip and B. Simon, *Sum rules and spectral measures of Schrödinger operators with L^2 potentials*, Ann. of Math. **170** (2009), no. 2, 739-782, to show that there exists a potential $\tilde{Q} \in L^2(0, \infty)$ associated to the spectral measure $d\tilde{\rho}(E)$, allowing us to define the WT function $\tilde{M}(z)$ for $z \in \mathbb{C} \backslash [-\tilde{\beta}, +\infty[$ for $\tilde{\beta} >> 1$. The amplitude function associated to $\tilde{M}(z)$ is automatically given by $\tilde{A}(\alpha)$ thanks to the uniqueness of the inverse Laplace transform and analytic continuation.

First Example

We define for $\alpha \geq 0$,

$$\tilde{A}(\alpha) = 2(\gamma^2 - \beta^2)\ e^{-2\gamma\alpha},$$

where $\beta > 0$ and $\gamma \in [0, \beta)$. It corresponds to a Müntz series with a single term, $c_0 = 2(\gamma^2 - \beta^2) < 0$ and $\mu_0 = 2\gamma \geq 0$.

Thus, we take $\delta = \gamma + \frac{3-d}{2} \geq 3 - d$ since $d \geq 3$ by hypothesis and obtain

$$\tilde{Q}(x) = -8\beta^2 \left(\frac{\beta - \gamma}{\beta + \gamma}\right) \frac{e^{-2\beta x}}{(1 + \frac{\beta-\gamma}{\beta+\gamma}\ e^{-2\beta x})^2}$$

The associated Jost function is given in the variable $\kappa = -ik$ by

$$\psi(0, \kappa) = \frac{\kappa + \gamma}{\kappa + \beta},$$

and is holomorphic in Re $\kappa > -\beta$. The unique root of the Jost function is given by $\kappa = -\gamma$ which is a **real resonance**.

Second Example

For $\alpha > 0$, we define the amplitude

$$\tilde{A}(\alpha) = -\frac{2c_1}{\kappa_1} \, \sinh(2\kappa_1 \alpha),$$

where $c_1 > 0$ and $\kappa_1 > 0$. It corresponds to a Müntz series with two terms and with two μ_k of different sign. The associated potential is given by

$$\tilde{Q}(x) = -2\frac{d^2}{dx^2} \, \log\left(1 + \frac{c_1}{\kappa_1^2} \, \int_0^x \sinh^2(\kappa_1 y) \, dy\right),$$

The Jost function has the form in the κ variable

$$\psi(0,\kappa) = \frac{\kappa - \kappa_1}{\kappa + \kappa_1}.$$

We note that the Jost function is vanishing at $\kappa = \kappa_1$ which corresponds to the single **negative eigenvalue** $-\kappa_1^2$.

Local Hölder stability for the potentials

In this section, we deduce from the estimates for the difference of the amplitudes $A - \tilde{A}$ a set of new Hölder **local** stability estimates for the difference of the associated potentials $Q - \tilde{Q}$. By local stability, we mean that we are able to control the norm $||Q - \tilde{Q}||_{L^2(0,T)}$ with respect to ϵ, if the Steklov spectra of the underlying Schrödinger operators are close up to ϵ as in (8).

We choose a fixed $Q \in L^2(0,\infty)$ is *fixed* and assume that $\tilde{Q} \in L^2(0,\infty)$ belongs to the infinite dimensional class for which

$$\tilde{A}(\alpha) = A(\alpha) + \sum_{k \geq 0} c_k e^{-\mu_k \alpha}, \quad \alpha > 0,$$

Moreover, we assume that $c_k \leq 0$ for all $k \geq 0$ and the power series $\sum_{k\geq 0} c_k t^{\lambda_k}$ has a radius of convergence $R > 1$.

To obtain thelocal stability estimates, we use of the local version of the classical **Gel'fand-Levitan equations**. For $0 \leq x \leq t \leq T$, we consider the integral equation

$$V(x,t) + \int_x^T K(t,s)V(x,s)\, ds = -K(x,t), \tag{26}$$

where the integral kernel $K(t,s)$ is given by

$$K(t,s) = p(2T - t - s) - p(|t - s|), \tag{27}$$

and

$$p(t) = -\frac{1}{2}\int_0^{\frac{t}{2}} A(\alpha)\, d\alpha.$$

These integral equations are uniquely solvable for all $x \in (0, T)$ and we can recover the underlying potential using the relation:

$$Q(T - x) = -2\frac{d}{dx}\left(V(x,x)\right). \tag{28}$$

We have:

Lemma

Under the hypotheses of Theorem 1, there exists a constant C_T depending only on T such that

$$||p - \tilde{p}||_{(C^0(0,2T),||\cdot||_\infty)} \leq C_T \; f(\epsilon), \tag{29}$$

where

$$f(\epsilon) = \left(B^2 \epsilon + R^{1-d} \; \epsilon^{\frac{\log R}{\log(\frac{9M_0}{2})}} \right)^{\frac{1}{2}}$$

Now, let us introduce some notation to simplify the presentation below. In what follows, the parameters x and T are assumed to be fixed and t is a variable lying in the interval $[x, T]$. We denote by $\mathcal{K}$ the integral operator on $L^2(x, T)$ with kernel $K(t, s)$,

$$(\mathcal{K}f)(t) = \int_x^T K(t, s)\, f(s)\, ds,$$

and set

$$d(t) := p(t - x) - p(2T - x - t).$$

Thus, the solution $V(x, .)$ of the integral equation (26) can be written as

$$V := V(x, .) = (I + \mathcal{K})^{-1} d. \tag{30}$$

Using (30) and the usual resolvent identity, one obtains

$$\tilde{V} - V = (I + \tilde{\mathcal{K}})^{-1} \left(\tilde{d} - d + (\mathcal{K} - \tilde{\mathcal{K}})(I + \mathcal{K})^{-1} d \right) \tag{31}$$

By Lemma 5, one has the uniform estimate for $t, s \in [0, T]$,

$$|\tilde{d}(t) - d(t)| \leq C_T \ f(\epsilon) \quad , \quad |\tilde{K}(t,s) - K(t,s)| \leq C_T \ f(\epsilon),$$

thus using Schur's lemma, one gets

$$||\tilde{\mathcal{K}} - \mathcal{K}|| \leq C_T \ f(\epsilon), \tag{32}$$

in the sense of the operator norm on $L^2(x, T)$. As a consequence for $\epsilon > 0$ sufficiently small, the operator $I + (I + \mathcal{K})^{-1}(\tilde{\mathcal{K}} - \mathcal{K})$ is invertible, and using again the resolvent identity, one obtains easily

$$(I + \tilde{\mathcal{K}})^{-1} = \left(I + (I + \mathcal{K})^{-1}(\tilde{\mathcal{K}} - \mathcal{K})\right)^{-1} (I + \mathcal{K})^{-1}.$$

It follows that, for $\epsilon << 1$, the operator norm of $(I + \tilde{K})^{-1}$ is uniformly bounded:

$$||(I + \tilde{\mathcal{K}})^{-1}|| \leq 2 \ ||(I + \mathcal{K})^{-1}||. \tag{33}$$

Thus, thanks to (31) , (32) and (33), one has:

$$||\tilde{V} - V||_{L^2(x,T)} \leq C_T \ f(\epsilon)$$

In the same way, differentiating the integral equation (26) with respect to x, one obtains

$$||\frac{\partial \tilde{V}}{\partial x} - \frac{\partial V}{\partial x}||_{L^2(x,T)} \leq C_T \ f(\epsilon).$$

Finally, one has for all $0 \leq x \leq T$,

$$||\frac{d}{dx}\left(\tilde{V}(x,x)\right) - \frac{d}{dx}\left(V(x,x)\right)||_{L^2(x,T)} \leq C_T \ f(\epsilon).$$

Then taking $x = 0$ and using (28), we see that

$$||\tilde{Q} - Q||_{L^2(x,T)} \leq C_T \ f(\epsilon),$$

and the proof of Theorem 1 is complete.

Proof of Corollary

First, it is easy to see that $\Lambda_q - \Lambda_{\tilde{q}} \in B(L^2(S^{d-1}))$. Indeed, the orthogonal projection of the DN map onto the space of homogeneous harmonic polynomials od degree k in $\mathbb{R}^d$ restricted to S^{d-1} the spherical harmonics space of homogeneous harmonic polynomials of degree k satisfies

$$\Lambda_q^k \psi_k = \sigma_k \psi_k.$$

Thus,

$$\begin{aligned} ||(\Lambda_{\tilde{q}} - \Lambda_q)\psi||_{L^2}^2 &= ||\sum_k (\tilde{\sigma}_k - \sigma_k)\psi_k Y_k||_{L^2}^2 \\ &= \sum_k |\tilde{\sigma}_k - \sigma_k||\psi_k|^2 \\ &\leq ||\tilde{\sigma}_k - \sigma_k||_{l^\infty(\mathbb{N})} \; ||\psi||_{L^2}. \end{aligned}$$

So, we deduce that $||\Lambda_{\tilde{q}} - \Lambda_q||_{B(L^2(S^{d-1}))} = ||\tilde{\sigma}_k - \sigma_k||_{l^\infty(\mathbb{N})}$. It follows that the local Hölder stabililty estimates obtained in Theorem 1 imply that for any $T > 0$, there exists a positive constant C_T such that

$$||\tilde{Q} - Q||_{L^2(0,T)} \leq C_T \Big(\, ||\Lambda_{\tilde{q}} - \Lambda_||_{B(L^2(S^{d-1}))} \Big)^\theta,$$

or equivalently,

$$||\tilde{q} - q||_{L^2((e^{-T},1),r^3dr)} \leq C_T \Big(\, ||\Lambda_{\tilde{q}} - \Lambda_q||_{B(L^2(S^{d-1}))} \Big)^\theta.$$

Hölder stability for regular metrics on closed deformed balls

Consider on $M = (0,1] \times S^{d-1}$ the warped product metric

$$g = c^4(r)[dr^2 + r^2 g_S]\,. \tag{34}$$

The metric g is regular if and only if $c^{(2k+1)}(0) = 0$ and $g_S = d\Omega^2$.

Change coordinates to $x = -\log r \in [0, +\infty)$. In these new coordinates, the metric g has the form

$$g = f^4(x)[dx^2 + g_S]\,, \tag{35}$$

where $f(x) = c(e^{-x})e^{-\frac{x}{2}}$.

Laplace's equation $-\triangle_g u = 0$ reads

$$[-\partial_x^2 - \triangle_{g_S} + q_f(x)]v = -\frac{(d-2)^2}{4} v\,, \tag{36}$$

where $v = f^{d-2}u$ and q_f is given by

$$q_f(x) = \frac{(f^{d-2})''(x)}{f^{d-2}(x)} - \frac{(d-2)^2}{4}\,.$$

Question: Can we find $Q \in L^2(0,\infty)$ such that $Q = q_f$ with c defined by $f(x) = c(e^{-x})e^{-\frac{x}{2}}$ satisfying the regularity conditions $c^{(2k+1)}(0) = 0$?

The answer is **yes**! In fact one has **explicit parametrized families**. These give Riemannian metrics corresponding to a deformation of the closed Euclidean unit ball for which the Steklov spectrum is **Hölder stable**.

Thank you for your attention!