THE QUANTIZATION OF MAXWELL THEORY IN THE CAUCHY RADIATION GAUGE

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MOTIVATION

- Consider a *globally hyperbolic* 4-*manifold* $\mathsf{M} = (\mathbb{R} \times \Sigma, g = -\beta^2 dt^2 + h_t)$
- A *gauge theory* is a quadruple $(\mathsf{V}_0, \mathsf{V}_1, \mathsf{P}, \mathsf{K})$ consisting of:

(I) two Hermitian bundles $\mathsf{V}_0, \mathsf{V}_1$ over M;

(II) a formally self-adjoint differential operator $\mathsf{P} \colon \Gamma(\mathsf{V}_1) \to \Gamma(\mathsf{V}_1)$;

(III) a linear differential operator $\mathsf{K} \colon \Gamma(\mathsf{V}_0) \to \Gamma(\mathsf{V}_1)$ with $\mathsf{K} \neq 0$ such that
- (i) $\mathsf{P} \circ \mathsf{K} = 0$ *(gauge transformation)*
- (ii) $\mathsf{D}_1 := \mathsf{P} + \mathsf{K}\mathsf{K}^* \colon \Gamma(\mathsf{V}_1) \to \Gamma(\mathsf{V}_1)$ is Green hyperbolic;
- (iii) $\mathsf{D}_0 := \mathsf{K}^*\mathsf{K} \colon \Gamma(\mathsf{V}_0) \to \Gamma(\mathsf{V}_0)$ is Green hyperbolic,

Equivalently, we can work at the level of *initial data*

(I') V_{ρ_i} are the bundle of initial data for D_i

(III') $\mathsf{K}_\Sigma := \rho_1 K U_0 \quad \mathsf{K}_\Sigma^\dagger := \rho_0 K^* U_1 \quad \mathsf{G}_i = (\rho_i \mathsf{G}_i)^* \mathsf{G}_{i,\Sigma} (\rho_i \mathsf{G}_i)$

where ρ_i and U_i are the Cauchy data and the Cauchy evolution operator and $(\rho_i \mathsf{G}_i)^*$ is the adjoint of $(\rho_i \mathsf{G}_i)$

HOW CAN WE QUANTIZE IT?

Step 1: Construct the classical phase space

$$
\begin{array}{ccccc}
\mathrm{i}(\cdot, G_1 \cdot)_{V_1} =: q_1 \,,\ \mathcal{V}_{\mathrm{P}} := \dfrac{\ker(K^*)}{\mathrm{ran}(P)} & \xrightarrow{[G_1]} & \dfrac{\ker(P)}{\mathrm{ran}(K)} \\
\text{unitary} \Big\downarrow [\rho_1 G_1] & \searrow^{[G_1]} & \Big\uparrow [j] \\
\mathrm{i}(\cdot, G_{1_\Sigma} \cdot)_{V_{\rho_1}} =: q_{1_\Sigma} \,,\ \mathcal{V}_{\Sigma} := \dfrac{\ker(K_\Sigma^\dagger)}{\mathrm{ran}(K_\Sigma)} & \xrightarrow{[\mathcal{U}_1]} & \dfrac{\ker(D_1) \cap \ker(K^*)}{K(\ker(D_0))}
\end{array}
$$

and assign $\forall v \in \mathcal{V}_{\mathrm{P}}$ an element of the abstract unital $*$-algebra $\mathrm{CCR}(\mathcal{V}_{\mathrm{P}}, \mathrm{q}_1)$

generators: $\Phi(v) \qquad \Phi^*(v) \qquad \mathbb{1}$

CCR relations:
$$[\Phi(v), \Phi(w)] = [\Phi^*(v), \Phi^*(w)] = 0$$
$$[\Phi(v), \Phi^*(w)] = \mathrm{q}_1(v, w)\mathbb{1}$$

Step 2: Construct an **Hadamard states** $\omega : \mathrm{CCR}(\mathcal{V}_{\mathrm{P}}, \mathrm{q}_1) \to \mathbb{C}$ defined by

covariances: $\Lambda^+(v, w) := \omega(\Phi(v)\Phi^*(w)) \qquad \Lambda^-(v, w) := \omega(\Phi^*(w)\Phi(v))$

Hadamard conditions: $\mathrm{WF}'(\Lambda^\pm) \subset \mathcal{N}^\pm \times \mathcal{N}^\pm$ where: $\mathcal{N} = \mathcal{N}^+ \cup \mathcal{N}^-$

Intermezzo I: microlocal methods in AQFT

DEFINITION: Let $u \in D'(M)$ be a distribution. We call

- **singular support** of u

$$singsupp(u) = \{p \in \mathrm{M} \mid \nexists O \ni p \text{ such that } u|_O \in C^\infty(O)\}.$$

- **wavefront set** of u

$$WF(u) = \{(p,k) \in T^*M \setminus \{0\} \mid p \in singsupp(u) \text{ and } k \in \Sigma_p(u)\},$$

where $\Sigma_p(u) = \cap_\rho \Sigma(\rho u)$ with $\rho(p) \neq 0$ and

$$\Sigma(\rho u) = \{k \in \mathrm{R}^n \setminus \{0\} \mid \nexists \text{ a conic } V \ni k \text{ such that } |\widehat{\rho u}|(k') \leq C_N(1+|k'|)^{-N}, \forall N \in \mathrm{N} \text{ and } \forall k' \in V\}.$$

EXAMPLE: Dirac delta distribution $\delta(x)$:

$$\begin{cases} singsupp(\delta) = \{0\} \\ \widehat{(\rho\delta)}(k) = \rho(0) \end{cases} \implies WF(\delta) = \{(0,k)\}$$

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EXAMPLE: Covariance of an Hadamard state $\Lambda^\pm$

$$WF(\Lambda^\pm) = \{(x,k_x,y,k_y) \in T^*M \times T^*M \setminus \{0\} \mid (x,k_x) \sim (y,-k_y), \pm k_x \triangleright 0\}$$

$$WF'(\Lambda^\pm) := \{(x,k_x,y,-k_y) \in T^*M \times T^*M \setminus \{0\} \mid (x,k_x,y,k_y) \in WF(\Lambda^\pm)\}$$

PROPOSITION [Gérard-Wrochna]: let $c^{\pm} : \Gamma(V_{\rho_1}) \to \Gamma(V_{\rho_1})$ be

(i) $c^{\pm}(\mathrm{ran}(K_\Sigma)) \subset \mathrm{ran}(K_\Sigma)$ and $(c^{\pm})^{\dagger} = c^{\pm}$ (w.r.t. $q_{1,\Sigma}$);

(ii) $(c^+ + c^-)\mathfrak{f} = \mathfrak{f} \mod \mathrm{ran}K_\Sigma \ \forall \mathfrak{f} \in \ker(K_\Sigma^{\dagger})$;

(iii) $q_{1,\Sigma}(\mathfrak{f}, c^{\pm}\mathfrak{f}) = i(\mathfrak{f}, G_{1,\Sigma}c^{\pm}\mathfrak{f})_{V_{\rho_1}} \geq 0 \quad \forall \mathfrak{f} \in \ker(K_\Sigma^{\dagger})$.

(iv) $WF'(U_1 c^{\pm}) \subset (\mathcal{N}^{\pm} \cup F) \times T^*\Sigma$ for $F \subset T^*M$

Then $\Lambda^{\pm}([s],[t]) := (s, \lambda^{\pm} t)_{V_1}$ where $\lambda^{\pm} := (\rho_1 G_1)^* i G_{1,\Sigma} c^{\pm}(\rho_1 G_1)$

are pseudo-covariances for a quasifree Hadamard state $\omega : \mathrm{CCR}(\mathcal{V}_P, q_1) \to \mathbb{C}$.

Difficulties:

- the fiber metric on V_{ρ_1} may in general be not positive definite $\Longrightarrow$ the positivity (iii) is difficult to achieve
- pseudodifferential calculus works nice with the Hadamard condition (iv), but interact badly with gauge invariance (i) and positivity (iii)

HOW CAN WE CONSTRUCT HADAMARD STATES?

$\hookrightarrow$ gauge fixing the degrees of freedom

OUTLINE

(I) The Cauchy radiation gauge

(II) Hodge-decomposition in Sobolev spaces

(III) The complete gauge fixing and the pPhase space

(IV) Hadamard states in the Cauchy radiation gauge

`Joint project with Gabriel Schmid, Ph.D. student in Genoa`

MAXWELL THEORY AS A GAUGE THEORY

(I) $\mathsf{V}_0 = \left(\mathsf{M} \times \mathbb{C}, (\cdot\,,\cdot)_{\mathsf{V}_0}\right)$ and $\mathsf{V}_1 = \left(\mathsf{T}^*\mathsf{M} \otimes_{\mathbb{R}} \mathbb{C}, (\cdot\,,\cdot)_{\mathsf{V}_1}\right)$ where

$$(\cdot,\cdot)_{V_1} := \int_{\mathsf{M}} g^{-1}(\bar{\cdot},\cdot)\, \mathrm{vol}_g$$

(II) set $\mathsf{P} =: \delta d$, $\mathsf{K} = d$ and $\mathsf{K}^* = \delta \implies D_1 := \delta d + d\delta$ and $D_0 = \delta d$

Because $\ker \mathsf{P}$ is invariant under conformal rescaling we can set

$$\mathsf{M} = \mathbb{R} \times \Sigma \qquad g = -dt^2 + h_t\,.$$

DEFINITION: $A = A_0 dt + A_\Sigma$ satisfies **Cauchy radiation gauge** on a Σ if

$$\delta A = 0 \textit{ (Lorenz gauge)} \quad \text{and} \quad A_0|_\Sigma = \partial_t A_0|_\Sigma = 0$$

REMARK: On ultrastatic spacetimes, the following gauge are equivalent:

(i) A satisfies the *Cauchy radiation gauge*;

(ii) A satisfies the *temporal gauge* $A_0 = 0$ and the *Coulomb gauge* $\delta_\Sigma A_\Sigma = 0$;

(iii) The fiber metric g^{-1} reduces to h^{-1} in the Cauchy radiation gauge

$$g^{-1}(A,A) = -(A_0,A_0) + h^{-1}(A_\Sigma,A_\Sigma) = h^{-1}(A_\Sigma,A_\Sigma) \geq 0$$

HOW TO ACHIEVE THE CAUCHY RADIATION GAUGE?

- Decompose $A = A_0 \mathrm{d}t + A_\Sigma$

- $A' = A + df$ satisfies the Cauchy radiation gauge if we can solve the system

$$\begin{cases} \mathrm{D}_0 f & = -\delta A \\ \nabla_t f|_\Sigma & = -A_0|_\Sigma \\ \nabla_t^2 f|_\Sigma & = -\nabla_t A_0|_\Sigma \end{cases} \iff \begin{cases} \mathrm{D}_0 f & = -\delta A \\ \nabla_t f|_\Sigma & = -A_0|_\Sigma \\ \Delta_0 f|_\Sigma & = -\delta_\Sigma A_\Sigma|_\Sigma \end{cases}$$

- if Σ is compact, by the Hodge decomposition $A_\Sigma = \delta\alpha + dg + h$

$$\begin{cases} \mathrm{D}_0 f & = -\delta A \\ f|_\Sigma & = -g|_\Sigma \\ \nabla_t f|_\Sigma & = -A_0 \end{cases}$$

- Since D_0 is normally hyperbolic $\Longrightarrow \exists ! \;\; f$ satisfying the Cauchy problem

IF THE MANIFOLD IS NOT COMPACT?

$\hookrightarrow$ Hodge-decomposition in Sobolev space

- (Σ, h) be complete d-dimensional Riemannian manifold and set

$$L^2_k(\Sigma) := \overline{\Omega^k_c(\Sigma;\mathbb{C})}^{\|\cdot\|_2} \qquad \langle\cdot,\cdot\rangle_2 := \int_M h^{-1}_{(k)}(\bar{\cdot},\cdot)\,\mathrm{vol}_\Sigma$$

- $\Delta_k = \delta\mathrm{d} + \mathrm{d}\delta$ is symmetric, positive and essentially self-adjoint and we define

$$\mathsf{E}_k := (\mathbb{1} + \overline{\Delta}_k) \colon \mathcal{D}(\overline{\Delta}_k) \to L^2_k(\Sigma)$$

- We can define the *Sobolev space of degree $s \in \mathbb{R}$* to be the Hilbert space

$$\mathsf{H}^s_k(\Sigma) := \mathcal{D}(\mathsf{E}_k^{s/2}), \qquad \langle\cdot,\cdot\rangle_{\mathsf{H}^s} := \langle \mathsf{E}_k^{s/2}\cdot, \mathsf{E}_k^{s/2}\cdot\rangle_{L^2}\,.$$

- Finally set

$$\Omega^k_s(\Sigma) := \Omega^k(\Sigma;\mathbb{C}) \cap \mathsf{H}^s_k(\Sigma) \qquad \text{and} \qquad \mathsf{H}^\infty_k(\Sigma) := \bigcap_{s\in\mathbb{R}} \mathsf{H}^s_k(\Sigma)$$

THEOREM [M.-Schmid]: For complete Riemannian manifolds (Σ, h) it holds

$$\mathsf{H}^s_k(\Sigma) \cong \mathrm{Har}^s_k(\Sigma) \oplus \overline{\mathrm{d}\Omega^{k-1}_\infty(\Sigma)} \oplus \overline{\delta\Omega^{k+1}_\infty(\Sigma)}\,,$$

where $\mathrm{Har}^s_k(\mathsf{M}) := \ker(\overline{\mathrm{d}_k}) \cap \ker(\overline{\delta_k})$ and the closures are taken w.r.t. $\|\cdot\|_{\mathsf{H}^s}$.

Sketch of the proof

(1) $\mathrm{d} : \Omega^k_\infty \to \Omega^{k+1}_\infty$ and $\delta : \Omega^{k+1}_\infty \to \Omega^k_\infty$ are closable and $H^s_\bullet$-adjoint, i.e.

$$(\overline{\delta}^* A, A')_{\mathrm{H}^s_k} = (A, \overline{d} A')_{\mathrm{H}^s_{k+1}}$$

(2) The sequence

$$0 \to \mathcal{D}_s(\overline{\mathrm{d}_0}) \xrightarrow{\overline{\mathrm{d}_0}} \mathcal{D}_s(\overline{\mathrm{d}_1}) \xrightarrow{\overline{\mathrm{d}_1}} \dots \xrightarrow{\overline{\mathrm{d}_{d-1}}} \mathcal{D}_s(\overline{\mathrm{d}_d}) \xrightarrow{\overline{\mathrm{d}_d}} 0$$

is a well-defined co-chain complex and using that $\overline{d}$ is closed in H^s,

$$\begin{aligned}
\mathrm{H}^s_k(\Sigma) &\cong \ker(\overline{\mathrm{d}}_k)^\perp \oplus \ker(\overline{\mathrm{d}}_k) \\
&= \ker(\overline{\mathrm{d}}_k)^\perp \oplus \overline{\mathrm{ran}(\overline{\mathrm{d}}_{k-1})} \oplus (\ker(\overline{\mathrm{d}}_k) \cap \mathrm{ran}(\overline{\mathrm{d}}_{k-1})^\perp) \\
&= \overline{\mathrm{ran}(\delta_{k+1})} \oplus \overline{\mathrm{ran}(\mathrm{d}_{k-1})} \oplus (\ker(\overline{\mathrm{d}}_k) \cap \ker(\overline{\delta}_k)) .
\end{aligned}$$

where we used $\ker(P^*)^\perp = \overline{\mathrm{ran}(P)}$ and $\mathrm{ran}(P)^\perp = \ker(P^*)$ □

REMARK:

(i) $\Omega^k_s(\Sigma) := \Omega^k \cap \mathrm{H}^s_k(\Sigma) \cong \mathrm{Har}^s_k(\Sigma) \oplus \left(\Omega^k \cap \overline{\mathrm{d}\Omega^{k-1}_\infty(\Sigma)}\right) \oplus \left(\Omega^k \cap \overline{\delta\Omega^{k+1}_\infty(\Sigma)}\right)$

(ii) If (Σ, h) is of bounded geometry, H^s coincides with $\mathrm{W}^{s,2}$

(iii) Any form $\alpha \in \Omega^k(\Sigma) \cap \overline{\mathrm{d}\Omega^{k-1}_\infty(\Sigma)}$ is exact

COROLLARY: For any $\omega \in \Omega^1_s(\Sigma)$, the Poisson equation

$$\Delta_0 f = \delta\omega$$

has a unique solution on the space $\{f \in C^\infty(\Sigma; \mathbb{C}) \mid \mathrm{d}f \in \overline{\mathrm{d}\Omega^0_\infty(\Sigma)}^{\mathrm{H}^s}\}$

For $A = A_0 dt + A_\Sigma$ we introduce the spaces

$$\Gamma_s(\mathrm{V}_k) := \Omega^k_s(\mathrm{M}; \mathbb{C}) := C^\infty(\mathbb{R}, \Omega^{k-1}_s(\Sigma)) \oplus C^\infty(\mathbb{R}, \Omega^k_s(\Sigma))$$

COROLLARY: For any $A \in \Gamma_s(\mathrm{V}_1)$ there exists $f \in \Gamma(V_0)$ w. $df|_{\Sigma_t} \in \overline{\mathrm{d}\Omega^0_\infty(\Sigma_t)}^{\mathrm{H}^s}$ such that $A' = A + df$ satisfies the Cauchy radiation gauge.

THEOREM [M.-Schmid]: - (M, g) is a globally hyperbolic manifold
- (Σ, h_t) are complete Riemannian manifolds
$\Longrightarrow$ The Cauchy problem for D_1 is well-posed: for any

$$\forall (h_0, h_1, f) \in \left(\Omega_s^{k-1}(\Sigma) \oplus \Omega_s^k(\Sigma)\right) \oplus \left(\Omega_s^{k-1}(\Sigma) \oplus \Omega_s^k(\Sigma)\right) \oplus \Gamma_{\mathrm{tc},s-1}(V_k)$$

there exists a unique solution

$$A \in \Gamma_s(V_k) = C^\infty\left(\mathbb{R}, \Omega_s^{k-1}(\Sigma)\right) \oplus C^\infty\left(\mathbb{R}, \Omega_s^k(\Sigma)\right)$$

to the initial value problem

$$\begin{cases} D_k A = f \\ A|_{\Sigma_{t_0}} = h_0 \\ (\mathrm{i}^{-1}\partial_t A)|_{\Sigma_{t_0}} = h_1 \end{cases}$$

Ideas behind the proof

The energy $\mathcal{E}_k(\omega, t) := \|\omega|_{\Sigma_t}\|_{\mathcal{H}^s(\Sigma_t)} + \|\partial_t \omega|_{\Sigma_t}\|_{\mathcal{H}^{s-1}(\Sigma_t)}$ is bounded:

$$\mathcal{E}_k(\omega, t_1) \leq \mathcal{E}_k(\omega, t_0) \cdot e^{C(t_1 - t_0)} + \int_{t_0}^{t_1} e^{C(t_1-\tau)} \|\Box_k \omega|_{\Sigma_\tau}\|_{\mathcal{H}^{s-1}(\Sigma_\tau)} \, \mathrm{d}\tau$$

REMARK: f is unique (up to a constant), so the gauge is fixed completely, *i.e.*

$$\frac{\ker(\mathrm{P})}{\mathrm{ran}(\mathrm{K})} \simeq \ker(\mathrm{D}_1) \cap \ker(\mathrm{K}^*) \cap \ker(\mathrm{R})$$

where $\mathrm{R} = U_1 \mathrm{R}_\Sigma \rho_1$ and $\mathrm{R}_\Sigma(a_0, \pi_0, a_\Sigma, \pi_\Sigma) := (a_0, \pi_0, 0, 0)$

THE GAUGE-FIXED PHASE SPACE

PROPOSITION (phase space): The following diagram is commutative

$$\begin{array}{ccc}
\mathcal{V}_{\mathrm{P}} := \dfrac{\ker(\mathrm{K}^*|_{\Gamma_{tc,s}})}{\mathrm{ran}(\mathrm{P}|_{\Gamma_{tc,s}})} & \xrightarrow{\quad [\mathrm{G}_1] \quad} & \dfrac{\ker(\mathrm{P}|_{\Gamma_s})}{\mathrm{ran}(\mathrm{K}) \cap \Gamma_{s,d}} \\
\Big\downarrow [\rho_1 \mathrm{G}_1] & \searrow^{[\mathrm{G}_1]} & \Big\uparrow \\
\mathcal{V}_\Sigma := \dfrac{\ker(\mathrm{K}_\Sigma^\dagger|_{\mathcal{H}_s})}{\mathrm{ran}(\mathrm{K}_\Sigma|_{\mathcal{H}_s})} & \xrightarrow{\quad [U_1] \quad} & \dfrac{\ker(\mathrm{D}_1|_{\Gamma_s}) \cap \ker(\mathrm{K}^*|_{\Gamma_s})}{\mathrm{K}(\ker(\mathrm{D}_0)) \cap \Gamma_{s,d}} \\
\Big\downarrow \mathrm{T}_\Sigma & & \Big\uparrow \\
\mathcal{V}_{\mathrm{R}} := \ker(\mathrm{K}_\Sigma^\dagger|_{\mathcal{H}_s}) \cap \ker(\mathrm{R}_\Sigma|_{\mathcal{H}_s}) & \xrightarrow{\quad U_1 \quad} & \ker(\mathrm{D}_1|_{\Gamma_s}) \cap \ker(\mathrm{K}^*|_{\Gamma_s}) \cap \ker(\mathrm{R}|_{\Gamma_s})
\end{array}$$

We conclude the classical theory, by endowing $\mathcal{V}_{\mathrm{R}}$ with an Hermitian form $q_{\Sigma,\mathsf{R}}$

- Decomposing $A = A_0 dt + A_\Sigma$, we set

$$\rho_0 \colon f \mapsto \begin{pmatrix} f|_\Sigma \\ \frac{1}{\mathrm{i}}\partial_t f|_\Sigma \end{pmatrix} \qquad \text{and} \qquad \rho_1 \colon A \mapsto \begin{pmatrix} A_0|_\Sigma \\ \frac{1}{\mathrm{i}}\partial_t A_0|_\Sigma \\ A_\Sigma|_\Sigma \\ \frac{1}{\mathrm{i}}\partial_t A_\Sigma|_\Sigma \end{pmatrix}$$

- By construction $[\rho_1 \mathsf{G}_1] \colon (\mathcal{V}_{\mathrm{P}}, q_1) \to (\mathcal{V}_\Sigma, q_{1,\Sigma})$ is an unitary isomorphism

$$q_{1,\Sigma}([\cdot],[\cdot]) = \mathrm{i}([\cdot], \mathsf{G}_{1,\Sigma}[\cdot])_{\mathsf{V}_{\rho_1}} \qquad \mathsf{G}_{1,\Sigma} = \frac{1}{\mathrm{i}} \begin{pmatrix} 0 & -\mathbb{1} & 0 & 0 \\ -\mathbb{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbb{1} \\ 0 & 0 & \mathbb{1} & 0 \end{pmatrix}$$

- We define $\mathrm{q}_{\Sigma,\mathsf{R}}$ such that $\mathsf{T}_\Sigma : (\mathcal{V}_\Sigma, \mathrm{q}_{1,\Sigma}) \to (\mathcal{V}_{\mathrm{R}}, \mathrm{q}_{\Sigma,\mathsf{R}})$ is unitary

$$\mathrm{q}_{\Sigma,\mathsf{R}}(\cdot,\cdot) = \mathrm{i}(\cdot \mathsf{G}_{\Sigma,\mathsf{R}} \cdot)_{\mathsf{V}_{\rho_1}} \qquad \mathsf{G}_{\Sigma,\mathsf{R}} = \frac{1}{\mathrm{i}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbb{1} \\ 0 & 0 & \mathbb{1} & 0 \end{pmatrix}$$

Summing up: unitary isomorphisms $(\mathcal{V}_{\mathrm{P}}, q_1) \simeq (\mathcal{V}_\Sigma, q_{1,\Sigma}) \simeq (\mathcal{V}_{\mathrm{R}}, q_{\Sigma,\mathsf{R}})$

HOW TO CONTROL THE MICROLOCAL BEHAVIOUR OF T_Σ?

To compute T_Σ we follows this ansatz

$$T_\Sigma = \mathbb{1} - \mathsf{K}_\Sigma(\mathsf{R}_\Sigma \mathsf{K}_\Sigma)^{-1}\mathsf{R}_\Sigma$$

PROPOSITION: Let (Σ, h) be a Riemannian manifold and $\pi_\delta := \mathbb{1} - \mathrm{d}_\Sigma \Delta_0^{-1}\delta_\Sigma$. There exists a map $\mathsf{T}_\Sigma \colon \mathcal{V}_\Sigma \to \mathcal{V}_\Sigma$ defined by

$$T_\Sigma = \begin{pmatrix} 0_{2\times 2} & 0_{2\times 2} \\ 0_{2\times 2} & \begin{pmatrix} \pi_\delta & 0 \\ 0 & \pi_\delta \end{pmatrix} \end{pmatrix}$$

satisfies the following properties

(i) $T_\Sigma = \mathbb{1} - \mathsf{K}_\Sigma(\mathsf{R}_\Sigma \mathsf{K}_\Sigma)^{-1}\mathsf{R}_\Sigma$ on $\ker(\mathsf{K}_\Sigma^\dagger)$

(ii) $\mathsf{T}_\Sigma^2 = \mathsf{T}_\Sigma$ and $\mathsf{T}_\Sigma|_{\mathcal{V}_\mathrm{R}} = \mathbb{1}$;

(iii) $\ker(\mathsf{T}_\Sigma) = \mathrm{ran}(\mathsf{K}_\Sigma)$;

(iv) $\mathrm{ran}(\mathsf{T}_\Sigma) = \ker(\mathsf{K}_\Sigma^\dagger) \cap \ker(\mathsf{R}_\Sigma)$.

WE CAN NOW CONSTRUCT HADAMARD STATES

0)By the standard deformation argument, we assume

$$(M, g) \quad \text{to be ultrastatic and of bounded geometry}$$

1) Replace the phase space $(\mathcal{V}_{\mathrm{P}}, \mathrm{q})$ with the space of initial data $(\mathcal{V}_{\Sigma}, \mathrm{q}_{\Sigma})$

$$\rho G : (\mathcal{V}_{\mathrm{P}}, \mathrm{q}) \xrightarrow[\text{unitary}]{\simeq} (\mathcal{V}_{\Sigma}, \mathrm{q}_{\Sigma}) \qquad \mathrm{q}_{\Sigma}(\cdot, \cdot) := (\cdot\,, \mathrm{i}G_{\Sigma} \cdot) \quad G = (\rho G)^* G_{\Sigma}(\rho G)$$

2) Construct an 'approximate' *square root of the Hodge-Laplacian*:

$$\varepsilon^* = \varepsilon \qquad \varepsilon^{-1}\varepsilon = \mathbb{1} \qquad \varepsilon^2 = \Delta + r_{-\infty} \quad (\Psi\text{DO–calculus})$$

$$\Updownarrow \qquad \qquad \searrow$$

microlocal factorization of $\quad \Box = (\partial_t + \mathrm{i}\varepsilon)(\partial_t - \mathrm{i}\varepsilon) - r_{-\infty} \quad$ (smoothing op.)

$$\Updownarrow \qquad \pi^{\pm} := \frac{1}{2}\begin{pmatrix} \mathbb{1} & \pm\varepsilon^{-1} \\ \pm\varepsilon & \mathbb{1} \end{pmatrix}$$

microlocal factorization of $\quad U_{\Box} = U_{(\partial_t + \mathrm{i}\varepsilon)}\pi^+ + U_{(\partial_t - \mathrm{i}\varepsilon)}\pi^-$

Intermezzo II: pseudodifferential calculus I/II

The *differential* operator $d/dx : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ can be written as

$$\frac{d}{dx} f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ikx} k \hat{f}(k)\, dk$$

hence a m-order differential operator A with constant coefficient reads as

$$Pf(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ikx} p(x,k) \hat{f}(k) dk \qquad p(x,k) = \sum_{\alpha \leq m} a_\alpha(x) k^\alpha$$

The Kohn-Nirenberg quantization is the natural generalization

$$S^m_{1,0} \ni p(x,k) \mapsto P\Big(x, \frac{d}{dx}\Big) := Op(p) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{ik(x-y)} p(x,k) f(y) dy\, dk \in \Psi^m(\mathbb{R})$$

where the symbol $p(x,k)$ is promoted to a smooth function in the class

$$S^m_{1,0} := \left\{ p \in C^\infty(\mathbb{R} \times \mathbb{R}) \,\Big|\, \left| \frac{d^\alpha}{dx^\alpha} \frac{d^\beta}{dk^\beta} (p(x,k)) \right| \leq C_{\alpha\beta} \langle k \rangle^{m-|\beta|} \ \ \forall \alpha, \beta \in \mathbb{N} \right\}$$

Intermezzo II: pseudodifferential calculus II/II

NICE PROPERTIES:

- The ΨDO–calculus transforms covariantly under local diffeomorphisms:
  - $\psi : \mathbb{R}^n \to \mathbb{R}^n$ differomorphism
  - $U_i \subset \mathbb{R}^n$ precompact open sets and $\chi_i \in C_c^\infty(\mathbb{R}^n)$ s.t. $\chi_i|_{U_i} = 1$

  $\Rightarrow$ For $A \in \Psi^m(U_1)$ we have $\chi_1 A \psi^*(\chi_2 u) = Bu \in \Psi^m(U_2)$

  $\Rightarrow$ the definition of ΨDO extends on smooth manifolds

- Let $S^{-\infty} := \cap_m S^m_{1,0}$ and $\Psi^{-\infty}(M)$ accordingly:

  $\Rightarrow$ $A : D'(M) \to C^\infty(M)$ is *smoothing* if and only if $A \in \Psi^{-\infty}(M)$

  $\Rightarrow$ $WF(Au) = \emptyset$ for any $u \in \mathrm{D}'(M)$

- If M **compact** and $A \in \Psi^m(M)$ and $B \in \Psi^n(M)$

  $\Rightarrow$ $A \circ B \in \Psi^{m+n}$

  $\Rightarrow$ For polyhomogeneous symbols i.e. $\sigma_P \sim \sum_j \alpha_j k^j \Rightarrow \sigma_{AB} = \sigma_A \circ \sigma_B \in S^{m+n}_{ph}$

The ΨDO–calculus can be extended on *manifolds of bounded geometry*

CONSTRUCTION OF AN 'APPROXIMATE' SQUARE ROOT OF THE LAPLACIAN

(sketch of the proof)

- Let $M = \mathbb{R} \times \Sigma$ with Σ of bounded geometry
- The closure of the Laplacian $\overline{\Delta}$ with domain $H^2(\Sigma)$ is self-adjoint on $L^2(\Sigma)$
- We fix $\chi \in C_c^\infty(\mathbb{R})$ with $\chi(0) = 1$ and set $\chi_R(\lambda) = \chi(R^{-1}\lambda)$ for $R \geq 1$
- We get $\chi_R(\overline{\Delta}) \in \Psi^{-\infty}(\Sigma)$ and we set $r_{-\infty} = R\chi_R(\overline{\Delta})$
- By the spectral calculus we find $R > 1$ s. t. $\overline{\Delta} + r_{-\infty}$ is m-accreative
- By standard results of Kato, $\overline{\Delta} + r_{-\infty}$ has a unique m-accreative square root

$$\varepsilon = \varepsilon^* \qquad \exists! \varepsilon^{-1} \in \Psi^{-1} \qquad \varepsilon^2 = \Delta + r_{-\infty}$$

□

3) The square root ϵ_i of the Hodge-Laplacian Δ_i has to satisfy

$$\epsilon_i \pi_\delta = \pi_\delta \varepsilon_i \quad \text{modulo } \Psi^{-\infty}$$

where again $\pi_\delta = \mathbb{1} - \mathrm{d}_\Sigma \Delta_0^{-1} \delta_\Sigma$

4) Finally consider the pseudodifferential projectors $\pi^\pm$ defined by

$$\pi^\pm := \frac{1}{2} \begin{pmatrix} \mathbb{1} & \pm\varepsilon_0^{-1} & 0 & 0 \\ \pm\varepsilon_0 & \mathbb{1} & 0 & 0 \\ 0 & 0 & \mathbb{1} & \pm\varepsilon_1^{-1} \\ 0 & 0 & \pm\varepsilon_1 & \mathbb{1} \end{pmatrix}$$

THEOREM [S.M., Schmid] Consider the operators $c^\pm := \mathrm{T}_\Sigma \pi^\pm \mathrm{T}_\Sigma$. Then

$$\lambda^\pm := (\rho_1 \mathrm{G}_1)^* \lambda_\Sigma^\pm (\rho_1 \mathrm{G}_1) \qquad \text{where} \qquad \lambda_\Sigma^\pm := \pm i \mathrm{G}_{1,\Sigma} c^\pm$$

are the pseudo-covariances of a quasi-free Hadamard state on $\mathrm{CCR}(\mathcal{V}_\mathrm{P}, \mathrm{q}_1)$.

Sketch of the proof

(i) Since $\varepsilon_i = \varepsilon_i^*$ are formally self-adjoint w.r.t the Hodge-inner product on Σ

$$(\pi^{\pm})^{\dagger} = \mathsf{G}_{1,\Sigma}^{-1}(\pi^{\pm})^{*}\mathsf{G}_{1,\Sigma} = \pi^{\pm}\,,$$

Then $\pi^{\pm}$, T_{Σ} and also $c^{\pm}$ are formally self-adjoint w.r.t. $\sigma_{1,\Sigma}$.

(ii) $\pi^{+} + \pi^{-} = \mathbb{1}$ and hence

$$(c^{+} + c^{-})\mathfrak{f} = T_{\Sigma}^{2}\mathfrak{f} = T_{\Sigma}\mathfrak{f} = \mathfrak{f} \quad \text{mod} \quad \text{ran}(\mathsf{K}_{\Sigma}|_{\Gamma^{\infty}_{\mathbf{H}}})$$

for all $\mathfrak{f} \in \ker(\mathsf{K}_{\Sigma}^{\dagger})$, where in the last step we used that T_{Σ} is a bijection between $\mathcal{V}_{\mathrm{P}}$ and $\mathcal{V}_{\Sigma}$ together with $T_{\Sigma} = \mathbb{1}$ on $\ker\mathsf{R}_{\Sigma}$.

(iii) we compute

$$\pm\mathrm{q}_{1,\Sigma}(\mathfrak{f}, c^{\pm}f) = \pm\mathrm{q}_{1,\Sigma}(\mathfrak{f}, \mathsf{T}_{\Sigma}\pi^{\pm}\mathsf{T}_{\Sigma}f) = \pm\mathrm{q}_{\Sigma,\mathsf{R}}(\mathsf{T}_{\Sigma}\mathfrak{f}, \pi^{\pm}\mathsf{T}_{\Sigma}f) \geq 0$$

(iv) follows because $\pi^{\pm}$ commutes with T_{Σ} modulo a smooth kernel and $\pi^{\pm}$ satisfies the Hadamard condition □

Outlook

WHAT WE HAVE SEEN AND WHAT COMES NEXT?

MAXWELL'S THEORY:

- Gauge fixing is useful for getting positivity and gauge invariance, but "the price to pay" is working with smooth, Sobolev initial data

- For generic manifold , we can construct Hadamard projectors $\pi^{\pm}$, but it is not clear that they commute with T_{Σ} (even modulo smoothing)

FUTURE WORK: LINEARIZED GRAVITY

- Gauge fixing completely the linearized gravity on the level of initial data: *Synchronous, de Donder, traceless-gauge, . . .*

- Constructing T_{Σ} is very challenging from a technical point of view (two-tensors can make life miserable very fast)

- We cannot use the deformation argument, so we need to modify $\pi^{\pm}$ such that the operators $c^{\pm} = T_{\Sigma}\pi^{\pm}\mathrm{T}_{\Sigma}$ satisfies the Hadamard conditions

THANKS for your attention!