The fermionic entanglement entropy and area laws

Felix Finster

Fakultät für Mathematik
Universität Regensburg

Johannes-Kepler-Forschungszentrum
für Mathematik, Regensburg

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The quantum state

- observable algebra $\mathscr{A}$
  (unital $*$-algebra generated by field operators)
- quantum state ω

$$\omega : \mathscr{A} \to \mathbb{C} \qquad \text{with} \qquad \omega(A^* A) \geq 0 \quad \text{for all } A \in \mathscr{A} \,.$$

- assume representation of $\mathscr{A}$ on Fock space $(\mathcal{F}, \langle .|. \rangle_{\mathcal{F}})$
- represent state by density operator W,

$$\omega(A) = \text{tr}_{\mathcal{F}}\big(WA\big) \qquad \text{for all } A \in \mathscr{A}$$

W is positive and has trace one.

  - pure state: $W = |\Psi\rangle\langle\Psi|$, projection operator of rank one

Notions of entropy

- von Neumann entropy

$$S := -\mathrm{tr}\big(W \log(W)\big)$$

vanishes for pure state; quantifies "mixture" of state

- Rényi entropy

$$S_\kappa := \frac{1}{1-\kappa}\,\mathrm{tr}\big(\log(W^\kappa)\big) \qquad (\kappa \neq 1)$$

  - limiting case $\kappa \to 1$ gives back von Neumann entropy
  - most of our results also apply to Renyi entropy
  - in the talk: focus on von Neumann entropy

Relative entropy and entanglement entropy

- Relative entropy: Two density operators W and W_0,

$$S^{\text{rel}} := -\text{tr}_{\mathcal{F}}\big(W\,(\log W - \log W_0)\big)$$

- entanglement entropy: Bi-partite system: $\mathcal{F} = \mathcal{F}_A \otimes \mathcal{F}_B$,

$$W_A := \text{tr}_B(W) : \mathcal{F}_A \to \mathcal{F}_A$$

$$S^{\text{ent}} := \text{tr}\big(W_A\,\log(W_A)\big) - \text{tr}\big(W\,\log(W)\big)$$

If W is a pure state, this simplifies to

$$S^{\text{ent}} := \text{tr}\big(W_A\,\log(W_A)\big)$$

The quasi-free fermionic case

We now specialize the setting.

- Consider fermion field in globally hyperbolic spacetime: $\mathscr{A}$ generated by fermionic field operators satisfying CAR

$$\big\{\Psi(\overline{f}), \Psi^\dagger(g)\big\} = k\big(\overline{f}, g\big) = <f|kg>$$
$$\big\{\Psi(\overline{\phi}), \Psi(\overline{\phi'})\big\} = 0 = \big\{\Psi^\dagger(\phi), \Psi^\dagger(\phi')\big\}$$

  $k := G_+ - G_-$ is causal propagator and $f, g \in C_0^\infty(\mathscr{M}, S\mathscr{M})$.
- Assume quasi-free state: All n-point distributions can be computed in terms of the 2-point distribution using Wick rules.

$$\omega_{2n+1}\big(h_1, \ldots, h_{2n+1}\big) = 0$$
$$\omega_{2n}\big(h_1, \ldots, h_{2n}\big) = \sum_{\sigma \in S'_{2n}} (-1)^{\mathrm{sign}(\sigma)} \prod_{i=1}^n \omega_2\big(h_{\sigma(2i-1)}, h_{\sigma(2i)}\big)$$

  here S'_{2n} denotes the ordered permutations

The quasi-free fermionic case

- Introduce one-particle Hilbert space,

$$\psi := kf \qquad \text{solutions of Dirac equation}$$
$$\langle \psi | \psi' \rangle_{\mathcal{H}} := k(\overline{f}, f') \qquad \text{corresponding scalar product}$$

scalar product can be evaluated on any Cauchy surface

Reduction to the one-particle density operator

- reduced one-particle density operator D is defined by

$$\omega_2\big(\overline{f}, f'\big) = \langle \psi \,|\, D\, \psi' \rangle_{\mathcal{H}} \,,$$

is linear operator on $\mathcal{H}$ with $0 \leq D \leq 1$.

- von Neumann entropy can be expressed in terms of D,

$$S = \mathrm{tr}_{\mathcal{H}}\big(\eta(D)\big)$$
$$\eta(t) := -t \log t - (1-t) \log(1-t)$$

- likewise for relative entropy and entanglement entropy

$$S^{\mathrm{rel}} = -\mathrm{tr}_{\mathcal{H}}\Big(D \log D_0 + (\mathbb{1} - D)\, \log\big(\mathbb{1} - D_0\big)\Big)$$
$$S^{\mathrm{ent}} = \mathrm{tr}_{\mathcal{H}}\big(\eta\big(\chi_\Lambda\, D\, \chi_\Lambda\big) - \chi_\Lambda\, \eta(D)\, \chi_\Lambda\big)$$

where χ_Λ is typically projection operator on $\mathcal{H}$.

Area laws

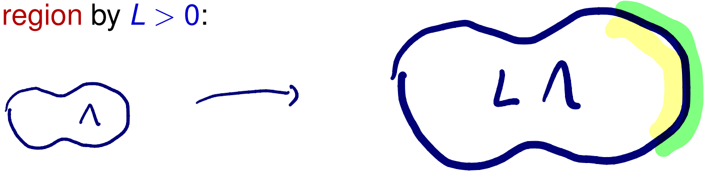
This setting has been studied for the non-relativistic Fermi gas, and area laws have been proved.

- Widom, and later Hajo Leschke, Robert Helling, Alexander Sobolev, Wolfgang Spitzer, ...

The idea of area law:

$\Lambda \subset \mathbb{R}^d$	typically bounded spatial region
$\chi_\Lambda : \Psi(x) \mapsto \chi_\Lambda(x)\Psi(x)$	multiplication by characteristic function

scale region by $L > 0$:



$$S^{\text{ent}}(L\Lambda) = \text{area}(\Lambda)\, L^{d-1} + o(L^{d-1}) \qquad \text{area law}$$

$$S^{\text{ent}}(L\Lambda) = \text{area}(\Lambda)\, L^{d} \log(L) + o(L^{d}) \qquad \text{enhanced area law}$$

Example: Two-Dimensional Rindler Spacetime

- joint work with Magdalena Lottner (Regensburg), Albert Much (Leipzig) and Simone Murro (Genova)

Consider two-dimensional Minkowski space $\mathscr{M} = \mathbb{R}^{1,1}$

$$\text{Dirac equation} \quad (\mathcal{D} - m)\psi = 0$$

$$(\psi|\phi)_{\mathscr{M}} = \int_{-\infty}^{\infty} \prec\psi|\gamma^0\phi\succ|_{(0,x)}\, dx$$

- Choose one-particle density for regularized vacuum,

$$D = \Pi_\varepsilon^-$$

integral operator with integral kernel ($\omega(k) := \sqrt{k^2 + m^2}$)

$$\Pi^\varepsilon(x,y) = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dk}{2\pi}\left(1 - \frac{1}{\omega(k)}\begin{pmatrix} -k & m \\ m & k \end{pmatrix}\right) e^{-\varepsilon\omega(k)}\, e^{ik(x-y)}$$

Example: Two-Dimensional Rindler Spacetime

Note:

- If $\varepsilon = 0$, this is projection operator to all negative-energy solutions, corresponds to a
  quasi-free Hadamard state (usual vacuum state)
- If $\varepsilon > 0$, of regularized Hadamard form

$$\text{Rindler space} \quad \mathscr{R} := \big\{ (t, x) \in \mathscr{M} \quad \text{with} \quad |t| < x \big\}$$

Thus on Cauchy surface at fixed time $t = 0$,

$$\Lambda = \mathbb{R}^+$$

THEOREM

For small ε,

$$S^{ent}(\Pi^\varepsilon, \Lambda) = \frac{1}{6} \log \varepsilon + \mathcal{O}(\varepsilon^0)$$

This is an enhanced area law for a zero-dimensional area.

Example: Spatial subsets in Minkowski space

- joint work with Magdalena Lottner (Regensburg) and Alexander Sobolev (University College London)

Consider four-dimensional Minkowski space $\mathscr{M} = \mathbb{R}^{1,3}$

$$\text{Dirac equation} \quad (\mathcal{D} - m)\psi = 0$$

$$(\psi|\phi)_{\mathscr{M}} = \int_{\mathbb{R}^3} \prec \psi | \gamma^0 \phi \succ |_{(0,\vec{x})} \, d^3x$$

- Again regularized Dirac vacuum,

$$\Pi^\varepsilon(\vec{k}) = \frac{1}{2} \left(\mathbb{1} + \frac{\sum_{\beta=1}^3 k_\beta \gamma^\beta \gamma^0 - m\gamma^0}{\sqrt{|\vec{k}|^2 + m^2}} \right) e^{-\varepsilon \sqrt{|\vec{k}|^2 + m^2}}$$

- Choose Λ as a bounded domain with C^1-boundary

Example: Spatial subsets in Minkowski space

Two parameters L (scaling) and ε (regularization length)

THEOREM

Consider asymptotics $L\varepsilon^{-1} \to \infty$ and $\varepsilon \searrow 0$. Then

$$\lim \frac{\varepsilon^2}{L^2} S^{ent}(\Pi^\varepsilon, L\Lambda) = \mathfrak{M} > 0$$

If $\varepsilon > 0$ stays bounded away from zero, then

$$\lim \frac{\varepsilon^2}{L^2} S^{ent}(\Pi^\varepsilon, L\Lambda) = \mathfrak{M}_\varepsilon > 0$$

and

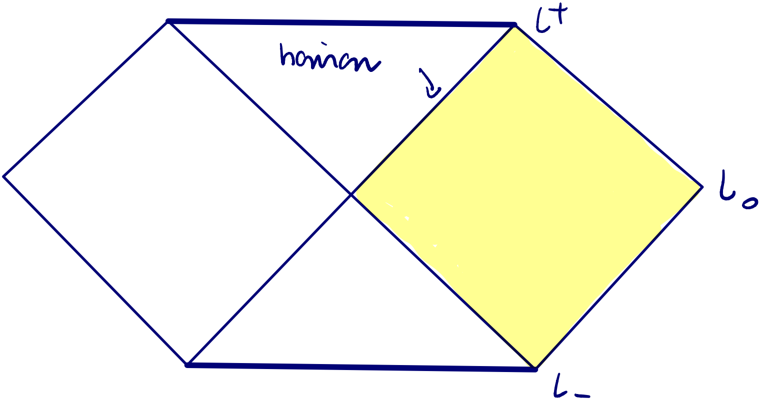
$$\lim_{\varepsilon \searrow 0} \mathfrak{M}_\varepsilon = \mathfrak{M}$$

The constants $\mathfrak{M}$ and $\mathfrak{M}_\varepsilon$ can be given explicitly.

Example: The Schwarzschild event horizon

- F.F., M. Lottner, "The fermionic entanglement entropy of the vacuum state of a Schwarzschild black hole horizon," arXiv:2302.07212 [math-ph]

Consider the exterior Schwarzschild geometry



Example: The Schwarzschild event horizon

- ω vacuum state of observer at infinity
- ultraviolet regularization in Killing direction, and a finite number of angular modes (more details later)
- based on the integral representation of the Dirac propagator derived in
  - F.F., N. Kamran, J. Smoller and S.-T. Yau, "The long-time dynamics of Dirac particles in the Kerr-Newman black hole geometry," arXiv:gr-qc/0005088, *Adv. Theor. Math. Phys.* **7** (2003) 25-52

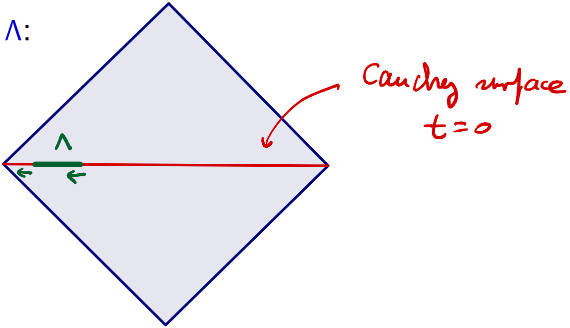
Example: The Schwarzschild event horizon

$$(\Pi^\varepsilon)(x,y) := \frac{1}{\pi} \sum_{k,n} \int_{-\infty}^{0} d\omega \; e^{\varepsilon\omega} \sum_{a,b=1}^{2} t_{ab}^{kn\omega} X_a^{kn\omega}(x) \langle X_b^{kn\omega}(y)|$$

- k, n are angular modes (spin-weighted spherical harmonics)
- X is composed of solutions of radial and angular ODEs
- $t_{ab}^{kn\omega}$ formed of reflection and transmission coefficients.

Example: The Schwarzschild event horizon

- How to choose Λ:



in Regge-Wheeler coordinates:

$$\Lambda = (u_0 - \rho, u_0) \times S^2 \qquad \text{with } u_0 \to -\infty,\ \rho \to \infty$$

Example: The Schwarzschild event horizon

THEOREM

$$\lim_{\varepsilon \searrow 0} \frac{1}{\log M/\varepsilon} \lim_{\rho \to \infty} \lim_{u_0 \to -\infty} S^{ent}(\Pi^\varepsilon, \Lambda) = \sum_{k,\, n \text{ occupied}} \frac{1}{6} \,.$$

Interpretation of this result:

- Entanglement entropy gives the area of even horizon.
- Similar to string theory and loop quantum gravity, area is "quantized"
- Total area is obtained by counting the number of occupied states.

Remark on proofs

- Adapt methods by Widom, Sobolev and others
- Non-smooth pseudo-differential operators, functional analytic methods, estimates of Schatten norms

Connection to modular theory

- The relative ~~entanglement~~ entropy can be computed using modular theory
  Araki, Longo, Bisognano, Wichmann, Hollands, Sanders, Lechner, Cadamuro, Galanda, Much, Verch, …

Here I cannot explain the connection in detail.
General connection to formulation in Fock spaces:

- E. Witten, "Notes on some entanglement properties of quantum field theory," arXiv:1803.04993 [hep-h]

Using this connection for quasi-free states, one use alternatively the above formula

$$S^{\text{rel}} := -\text{tr}_{\mathcal{F}}\big(W\,(\log W - \log W_0)\big)$$

- This should work even in cases when no modular group action present.

Connection to causal fermion systems

Here I cannot give a self-contained introduction. Just a few remarks

- approach to fundamental physics
- novel mathematical model of spacetime
- physical equations are formulated in generalized spacetimes
- Different limiting cases:
  - Continuum limit: Quantized fermionic fields interacting via classical bosonic fields
  - QFT limit: fermionic and bosonic quantum fields (ongoing, more towards the end of the talk)
- For overview, more details (papers, books, videos, online course), applications to cosmology and black holes, …

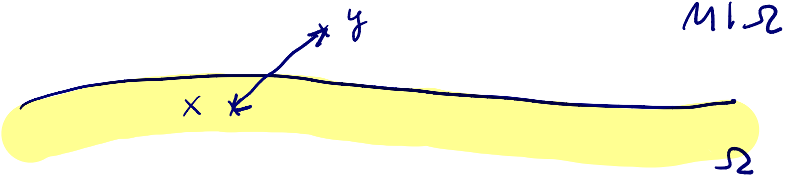
www.causal-fermion-system.com

Connection to causal fermion systems

- spacetime and all structures therein are described by a measure ρ on a set of linear operators on a Hilbert space,

$$\text{spacetime} \quad M := \operatorname{supp} \rho \subset \mathrm{L}(\mathcal{H})$$

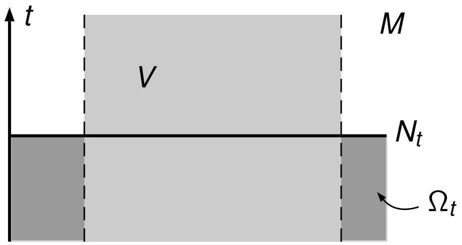
- Hilbert space scalar product formulated by a surface layer integral,



$$(\psi|\phi)^t_\rho = -2i \left(\int_{\Omega^t} d\rho(x) \int_{M\setminus\Omega^t} d\rho(y) - \int_{M\setminus\Omega^t} d\rho(x) \int_{\Omega^t} d\rho(y) \right) \times \prec\psi(x) \,|\, Q(x,y)\,\phi(y)\succ_x$$

Connection to causal fermion systems

Choose $V \subset M$ and "localize" the scalar product,



$$(u|u)^t_{V,\rho} := \left(\int_{\Omega^t \cap V} d\rho(x) \int_{M \setminus \Omega^t} d\rho(y) + \int_{\Omega^t} d\rho(x) \int_{V \setminus \Omega^t} d\rho(y) \right) \times \prec \psi(x) \,|\, Q(x,y)\, \phi(y) \succ_x ,$$

Connection to causal fermion systems

- Represent the "localized" scalar product with respect to the full scalar product,

$$(u|u)^t_{V,\rho} = (u \,|\, \sigma_V \, u)^t_\rho \qquad \text{for all } u \in \mathcal{H}^\text{f} \,;$$

- This gives reduced one-particle density operator
- Now define fermionic entanglement entropy as before,

$$S^\text{ent} = -\text{tr}_{\mathcal{H}^\text{f}} \big(\sigma_V \log(\sigma_V) \big) - \text{tr}_{\mathcal{H}^\text{f}} \Big((\mathbb{1} - \sigma_V) \log \big((\mathbb{1} - \sigma_V) \big) \Big)$$

Connection to causal fermion systems

- Remark: There is another notion of entropy for causal fermion systems,
  - includes bosonic and fermionic parts
  - quantifies the "disorder" of the system at time t.
- Goal: Understand better how all these entropies are related to each other. Which properties do they have, ...

- F.F., "A notion of entropy for causal fermion systems," arXiv:2103.14980 [math-ph], *Lett. Math. Phys.* **111** (2021) 129

www.causal-fermion-system.com

Thank you for your attention!