# A $Q$-curvature positive energy theorem with applications to rigidity phenomena

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1. R. Avalos, P. Laurain and J.H. Lira, *A positive energy theorem for fourth-order gravity*, Calc. Var. 61, 48 (2022).
2. R. Avalos, P. Laurain and N. Marque, *Rigidity Theorems for Asymptotically Euclidean $Q$-singular Spaces*, arXiv:2204.03607 (2022);
3. R. Avalos, J.H. Lira and N. Marque, *Energy in fourth order gravity*, arXiv:2102.00545 (2021),

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# Layout of the talk

- Introduction - Motivations and preliminaries intuitions from GR
  - Possible interaction between modified gravity/effective field theories and $Q$-curvature analysis;
- The $Q$-curvature/stationary limit of a fourth order energy
  - A positive energy theorem (PET);
  - Positive mass for the Paneitz operator.
- AE $Q$-singular manifolds - some rigidity results.

# Motivations - The General Problem

During this talk we shall be interested in the analysis of a notion of energy canonically associated to certain *fourth-order* gravitational theories. These are theories on space-times of the form $(V \doteq M \times \mathbb{R}, \bar{g})$ described by the *action functionals*

$$S(\bar{g}) = \int_V \left(\alpha R_{\bar{g}}^2 + \beta \langle \mathrm{Ric}_{\bar{g}}, \mathrm{Ric}_{\bar{g}} \rangle_{\bar{g}}\right) dV_{\bar{g}}, \tag{1}$$

where $\alpha$ and $\beta$ are free parameters of the problem. Imposing asymptotic conditions so that the functional $\bar{g} \mapsto S(\bar{g})$ is well-defined, we have its Euler-Lagrange equations:

$$\begin{aligned} A_{\bar{g}} \doteq\, & \beta \Box_{\bar{g}} \mathrm{Ric}_{\bar{g}} + (\frac{1}{2}\beta + 2\alpha) \Box_{\bar{g}} R_{\bar{g}}\, \bar{g} - (2\alpha + \beta) \bar{\nabla}^2 R_{\bar{g}} - 2\beta \mathrm{Ric}_{\bar{g}}.\mathrm{Riem}_{\bar{g}} \\ & + 2\alpha R_{\bar{g}} \mathrm{Ric}_{\bar{g}} - \frac{1}{2}\alpha R_{\bar{g}}^2 \bar{g} - \frac{1}{2}\beta \langle \mathrm{Ric}_{\bar{g}}, \mathrm{Ric}_{\bar{g}} \rangle_{\bar{g}} \bar{g} = 0. \end{aligned} \tag{2}$$

In this setting, among other things, one is interested in the analysis of conserved ("ADM-like") quantities related to these theories.

# Intuitions from GR I

**Definition**

An $(n+1)$-dimensional general relativistic space-time is defined to be an $(n+1)$-dimensional Lorentzian manifold $(V, \bar{g})$ satisfying the Einstein equations:

$$Ric(\bar{g}) - \frac{1}{2}R(\bar{g})\bar{g} = T(\bar{g}, \bar{\psi}) \tag{3}$$

where, $Ric(\bar{g})$ and $R(\bar{g})$ represent the Ricci tensor and Ricci scalar of $\bar{g}$, respectively, and $T$ represents some $(0,2)$-tensor field, called the energy-momentum tensor, depending on $\bar{g}$ and (possibly) on a collection of fields, collectively denoted by $\bar{\psi}$, representing other *physical* fields.

- The idea of conserved quantities in GR is quite complex in general. It is particularly well-understood for isolated systems, which are systems with controlled asymptotic geometry;
- Furthermore, considering the above space-times $(V, \bar{g})$ as globally hyperbolic developments of initial data sets $(M^n, g, K)$ on *a Cauchy-hypersurface* $M$, such conserved quantities are expected to be evaluated on the initial data set;

# Intuitions from GR II

- Recall: An initial data set $(M^n, g, K)$ for the Einstein equations consists of a Riemannian manifold $(M^n, g)$ and a symmetric $(0,2)$-tensor field $K$, satisfying the Einstein constraint equations:

$$\begin{aligned} R(g) - |K|_g^2 + (\mathrm{tr}_g K)^2 &= 2\epsilon \\ \mathrm{div}_g K - d\mathrm{tr}_g K &= J, \end{aligned} \tag{4}$$

where $\epsilon \doteq T(n,n)|_{t=0}$ and $J = -T(n,\cdot)$.

- It is a remarkable fact that, for many sources of interest (scalar fields, perfect fluids, Einstein-Maxwell among others), the constraint equations (4) are a necessary and sufficient condition to guarantee a (short-time) development of the initial data set into a space-time satisfying the associated Einstein equations;
- In such a space-time $(V = M^n \times [0,T), \bar{g})$, $(g, K)$ stand as the induced metric and extrinsic curvature of the initial manifold $M^n$ as an embedded hypersurface in $(V, \bar{g})$.

# Intuitions from GR III

In this context, an isolated system is modelled by initial data sets which are asymptotically Euclidean:

**Definition (AE manifolds)**

We will say that $(M, g)$ is asymptotically Euclidean (AE) of order $\tau > 0$ if there is a compact set $\mathcal{K} \subset M$ and a diffeomorphism $\Phi : M \backslash \mathcal{K} \mapsto \mathbb{R}^n \backslash \bar{B}$ such that, in these coordinates, $g_{ij} - \delta_{ij} = O(|x|^{-\tau})$.

Remark: The above definitions refers to Riemmanian manifolds $(M^n, g)$ and makes no reference to $K$. Along the lines of this definition, an AE initial data set involves also a decay assumption for $K$, typically of the form $K_{ij} = O(|x|^{-\tau-1})$.

- Typically, we will also demand derivatives of the metric (and $K$) to decay at certain rates.
- Since such rates can depend on the specific problem at hand, we will impose these requirement explicitly whenever necessary.

# Intuitions from GR IV

In the case of AE initial data sets, there are well-established notions of energy and momentum:

$$E \doteq \frac{1}{16\pi} \lim_{r\to\infty} \int_{S_r} \left(\partial_i g_{ij} - \partial_j g_{ii}\right) \nu^j d\omega_r \ \ ; \ \ P_i \doteq \frac{1}{8\pi} \lim_{r\to\infty} \int_{S_r} \pi_{ij}\nu^j d\omega_r, \tag{5}$$

where $\pi = K - \mathrm{tr}_g K\, g$.

**Remark (Well-known...)**

*Although $(E, P)$ don't seem to be well-defined geometric quantities, for AE manifolds of order $\tau > \frac{n}{2} - 1$ with $L^1$-integrable sources, they are well-defined, independent of the sequence of compacts and "independent" of the structure of infinity used to compute them.*

The analysis of the above ADM quantities has had a remarkable impact within geometric analysis. For instance, through the positive mass (energy) theorems (PMTs) in GR; the resolution of the Yamabe problem; rigidity phenomena associated to scalar curvature; geometric foliations associated to center of mass; gluing constructions, etc.

# The fourth order problem - Conserved Quantities

**Motivations:**

1. The interplay between conserved quantities in GR and deep problems in geometric analysis motivates a conjectured interplay between the corresponding quantities in higher-order theories and higher-order problems in geometric analysis;
2. Higher-order gravitational theories seem to be well-motivated by modern theoretical physics (effective field theories, improving renormalisation of GR, conformal gravity, etc);
3. Also, fourth-order problems in geometric analysis have proven to be highly interesting. In particular, $Q$-curvature analysis has received plenty of attention due to analytical subtleties and its role in understanding the link between a conformal class and the topology of the underlying manifold.

**Remark**

*The GR approach to conserved quantities can be paralleled in any theory produced via a Lagrangian. Doing this for our quadratic action functional produces a set of energies $\mathcal{E}_{\alpha,\beta}(\bar{g})$. In special limits, these energies are tractable and their analysis relates to known problems.*

# A $Q$-curvature PET - Stationary Solutions

Let us consider stationary solutions $(M^n \times I, \bar{g})$, with $I \subset \mathbb{R}$, such that

$$\bar{g} = -N^2 dt^2 + \tilde{g},$$

where $\tilde{g}$ stands for a time-independent tensor field which restricts to the same Riemannian metric $g$ when applied to tangent vectors to $M$, so that $(M, g, N)$ is a Riemannian manifold, with metric $g$ and $N : M \mapsto \mathbb{R}$ is a positive function. If we also set $\beta = -2\alpha$, we get a purely Riemannian situation, where

$$\mathcal{E}_\alpha(g) \doteq \mathcal{E}_{\alpha,-2\alpha}(\bar{g}) = -\alpha \lim_{r\to\infty} \int_{S_r^{n-1}} \left(\partial_j \partial_i \partial_i g_{aa} - \partial_j \partial_u \partial_i g_{ui}\right) \nu^j \, d\omega_r. \qquad (6)$$

We will fix $\alpha = -1$ and denote by $\mathcal{E}(g) \doteq \mathcal{E}_{-1}(g)$. In this context, the following results deal with the basic properties of $\mathcal{E}(g)$. First, let us recall the definition of the $Q$-curvature of a Riemannian manifold $(M, g)$:

$$Q_g = -\frac{1}{2(n-1)} \Delta_g R_g - \frac{2}{(n-2)^2} |\mathrm{Ric}_g|_g^2 + \frac{n^3 - 4n^2 + 16n - 16}{8(n-1)^2(n-2)^2} R_g^2. \qquad (7)$$

## Proposition

*Let $(M^n, g)$ be an AE manifold of dimension $n \geq 3$ satisfying the following conditions*

1. *There are rectangular end coordinates, given by a structure of infinity $\Phi$, where $g_{ij} = \delta_{ij} + O_4(r^{-\tau})$, where $\tau > \tau_n \doteq \max\{0, \frac{n-4}{2}\}$;*
2. *The Q-curvature of $g$ is in $L^1(M, dV_g)$.*

*Then, given an exhaustion of $M$ by compact sets $\Omega_k$ such that $S_k \doteq \Phi(\partial\Omega_k)$ are smooth connected $(n-1)$-dimensional manifolds without boundary in $\mathbb{R}^n$ satisfying*

$$\begin{aligned} R_k \doteq \inf\{|x| : x \in S_k\} \xrightarrow[k\to\infty]{} \infty, \\ R_k^{-(n-1)}\mathrm{area}(S_k) \text{ is bounded as } k \to \infty, \end{aligned} \tag{8}$$

*the limit*

$$\mathcal{E}^{(\Phi)}(g) = \lim_{k\to\infty} \int_{S_k} (\partial_j\partial_i\partial_i g_{aa} - \partial_j\partial_u\partial_i g_{ui})\, \nu^j \, dS, \tag{9}$$

*exists and is independent of the sequence of $\{S_k\}$ used to compute it.*

## Theorem (Positivity/Rigidity)

*Let $(M^n, g)$ be an n-dimensional AE manifold, with $n \geq 3$, which satisfies the decaying conditions i) and ii) of Proposition 1 and such that $Q_g \geq 0$ and $Y([g]) > 0$, then $\mathcal{E}(g) \geq 0$ with equality holding if and only if $(M, g)$ is isometric to $(\mathbb{R}^n, \cdot)$.*

Remark: In this AE-setting, the Yamabe invariant is defined as the following conformal invariant:

$$Y([g]) \doteq \inf_{u \in C_0^\infty(M)} \frac{\int_M (a_n |\nabla u|_g^2 + R_g u^2) dV_g}{\|u\|^2_{L^{\frac{2n}{n-2}}}}, \tag{10}$$

where $a_n \doteq \frac{4(n-1)}{n-2}$ and $[g]$ denotes the conformal class of $g$.

**Idea of the proof:**

- The Yamabe condition implies the existence of a conformal deformation $\tilde{g} = u^{\frac{4}{n-2}} g$ to zero scalar curvature. That is, $u$ satisfies

$$\Delta_g u - c_n R_g u = 0, \tag{11}$$

where $c_n = \frac{n-2}{4(n-1)}$.

- Consider the case $n \neq 4$. Set $\Phi \doteq u^{-\frac{n-4}{n-2}}$ so that $g = \Phi^{\frac{4}{n-4}} \tilde{g}$.

- Apply the conformal transformation rule for $Q$-curvature:

$$\frac{n-4}{2}\Phi^{\frac{n+4}{n-4}}Q_g = P_{\tilde{g}}\Phi,$$

where $P_g$ is the Paneitz operator defined by

$$P_{\tilde{g}}u = \Delta_{\tilde{g}}^2 u + \mathrm{div}_{\tilde{g}}\left(\left(\frac{4}{n-2}\mathrm{Ric}_{\tilde{g}} - \frac{n^2-4n+8}{2(n-1)(n-2)}R_{\tilde{g}}\,\tilde{g}\right)(\nabla u,\cdot)\right) + \frac{n-4}{2}Q_{\tilde{g}}u, \text{ for any } u \in C^\infty(M).$$

- Since $R_{\tilde{g}} = 0$, we have that $Q_{\tilde{g}} = -\frac{2}{(n-2)^2}|\mathrm{Ric}_{\tilde{g}}|^2_{\tilde{g}}$, and the above implies:

$$\int_{D_r}\frac{n-4}{2}\Phi^{\frac{n+4}{n-4}}Q_g + \frac{(n-4)}{(n-2)^2}|\mathrm{Ric}_{\tilde{g}}|^2_{\tilde{g}}\Phi\,dv_{\tilde{g}} = \int_{S_r}\tilde{g}(\tilde{\nabla}\Delta_{\tilde{g}}\Phi,\tilde{\nu})d\omega_{\tilde{g}} + \underbrace{\frac{4}{n-2}\int_{S_r}\mathrm{Ric}_{\tilde{g}}(\nabla\Phi,\tilde{\nu})d\omega_{\tilde{g}}}_{\xrightarrow[r\to\infty]{}\frac{n-4}{4(n-1)}\mathcal{E}(g)}$$

- Passing to the limit, we get that $\mathcal{E}(g) \geq 0$ with equality iff $Q_g \equiv 0$ and $g$ is conformal to $(\mathbb{R}^n, \cdot)$.
- In the rigidity case, the above conditions together with the decay conditions and the maximum principle imply that $g = \delta$.
- The case $n = 4$ is similar, but using the appropriate relation between $Q$ and Paneitz.

□

Obs: 1) The $Q$-curvature condition cannot be relaxed while keeping the rigidity;

2) The above theorems imply some curvature-topology rigidity corollaries in dimensions 3 and 4.

# The positive mass theorem of Paneitz I

Let us consider a closed manifold $(M^n, g)$ with $n \geq 5$.

- If the Panietz operator satisfies

$$Y_4([g]) = \inf_{u \in H^2(M)\,;\, \|u\|_{2^\#}=1} \int_M P_g(u)u\, dv_g > 0,$$

  where $2^\# = \frac{2}{n-4}$, then it admits a Green function $G_{P_g}$.
- This last infimum is conformally invariant and it plays a similar role to the Yamabe invariant for the $Q$-curvature.
- Through this section, we will assume that $Y_4([g]) > 0$ and therefore $G_{P_g}$ exists for every $g \in [g]$.
- Contrary to the conformal Laplacian, nothing here guarantees that $G_{P_g}$ is positive. This will be one of our assumptions, which is in particular satisfied if $Y([g]) \geq 0$ and $Q_{\tilde{g}}$ is semi-positive for some conformal metric $\tilde{g}$.

# The positive mass theorem of Paneitz II

Let us focus on the expansion of the Green function around a singularity. If we assume that $5 \leq n \leq 7$ or $g$ is locally conformally flat, then, in conformal normal coordinates $\{x^i\}$ for the conformal metric $\tilde{g}$, the Green function $G_P$ of $P_{\tilde{g}}$ admits an expansion of the form

$$G_P(p,x) = \frac{\gamma_n}{r^{n-4}} + \alpha + O_4(r), \tag{12}$$

where $r(x) \doteq d_{\tilde{g}}(p,x)$, $\gamma_n \doteq \frac{1}{2(n-2)(n-4)\omega_{n-1}}$ and $\alpha$ is a constant **called the mass**.

**Obs:** The sign of $\alpha$ is conformally invariant.

- In the above context, the sign of $\alpha$ is crucial for the conformal prescription problem for $Q$-curvature.
- The importance is analogous to that of the corresponding constant for the conformal Laplacian in Schoen's resolution of the Yamabe problem.
- In the Yamabe problem, the sign of the mass is a consequence of the PMT of GR.
- In the $Q$-curvature case, the positivity of $\alpha$ has been established through the work of different authors and the following theorem is known to hold.

# The positive mass theorem of Paneitz III

**Theorem (Hang-Yang)**

*Let $(M, g)$ be a closed n-dimensional Riemannian manifold, with $5 \leq n \leq 7$ or $n \geq 8$ and locally conformally flat around some point $p \in M$. If $Y([g]) \geq 0$ and $(M, g)$ admits a conformal metric with semi-positive Q-curvature, then the mass of $G_P$ at p is non-negative and vanishes if and only if $(M, g)$ is conformal to the standard sphere.*

- The above theorem was initially proven by Humbert-Raulot in the conformally-flat case:
    - E. Humbert and S. Raulot, Positive mass theorem for the Paneitz-Branson operator, Calc. Var. Partial Differential Equations, 36, 4, 525-531, (2009).
- Under the conditions $R_g \geq 0$ and $Q_g$ semi-positive, this was generalised by Gursky-Malchiodi to incorporate $5 \leq n \leq 7$:
    - M. J. Gursky and A. Malchiodi, A strong maximum principle for the Paneitz operator and a non-local flow for the Q-curvature, J. Eur. Math. Soc., 17, 9, 2137-2173 (2015).
- Finally,the scalar curvature condition was relaxed to $Y([g]) > 0$ by Hang-Yang:

# The positive mass theorem of Paneitz IV

- F. Hang and P. C. Yang, Sign of Green's function of Paneitz operators and the Q curvature, Int. Math. Res. Not., 19, 9775-9791 (2015).

- The above Theorem was used by Hang-Yang to *solve* the positive conformal $Q$-curvature prescription in an analogue manner to Schoen's solution to the Yamabe problem:
  - F. Hang and P. C. Yang, Q-curvature on a class of manifolds with dimension at least 5, Comm. Pure Appl. Math., LXIX:1452-1491 (2016).
- The above Theorem is an easy consequence of the fourth order positive energy theorem!

In fact, the following two propositions hold:

**Proposition**

*Let $(M^n, g)$ be a closed manifold satisfying $n \geq 5$ and whose Panietz operator admits a positive Green function $G_P$ with an expansion as (12) around some point $p \in M$. Then, the manifold $(\hat{M} \doteq M\backslash\{p\}, \hat{g} \doteq G_P(p,\cdot)^{\frac{4}{n-4}}g)$ is an AE manifold of order $\tau = 1$ if $n = 5$ and $\tau = 2$ if $n > 5$. Furthermore, either if $5 \leq n \leq 7$ or $g$ is flat around $p$, then $\mathcal{E}(\hat{g}) = 8(n-1)(n-2)\omega_{n-1}\gamma_n\alpha$.*

- If we knew that $\hat{g} = G_P(p,\cdot)^{\frac{4}{n-4}} g$ satisfies the hypotheses of the PET, then we could deduce the positivity and rigidity properties of $\alpha$ (the mass of Paneitz) from those of $\mathcal{E}$;
- The main non-trivial property to be checked is that $Y([\hat{g}]) > 0$, which is established in the proposition below.

**Proposition**

*Consider a closed Riemannian manifold $(M^n, g)$, with $n \geq 5$, which admits a conformal metric with positive Q-curvature such that $Y([g]) \geq 0$. Then, there exists a conformal metric $\tilde{g}$ such that the AE manifold $(\hat{M} = M \setminus \{p\}, \hat{g} = G_{P_{\tilde{g}}}^{\frac{4}{n-4}} \tilde{g})$ satisfies $Y([\hat{g}]) > 0$ and $Q_{\hat{g}} \equiv 0$.*

The above two proposition imply that $(\hat{M} = M \setminus \{p\}, \hat{g} = G_{P_{\tilde{g}}}^{\frac{4}{n-4}} \tilde{g})$ satisfies all the hypotheses of the positive energy theorem and thus $8(n-1)(n-2)\omega_{n-1}\gamma_n \alpha = \mathcal{E}(\hat{g}) \geq 0$ with equality iff $(M, g)$ is conformal to $(\mathbb{S}^n, g_{\mathbb{S}^n})$.

# Rigidity of Q-singular spaces I

Let first introduce a fourth-order tensor field, which serves as a fourth-order analogue of the Ricci-tensor. Considering the $Q$-curvature as a non-linear fourth-order operator on the set $\mathrm{Met}(M)$ of Riemannian metrics on $M$:

$$\begin{aligned} Q : \mathrm{Met}(M) &\to C^\infty(M), \\ g &\mapsto Q_g. \end{aligned}$$

if we denote by $S_2M$ the bundle of symmetric $(0,2)$-tensor fields over $M$, then the linearisation $DQ_g$ of $Q$ at the metric $g$ is given by a map

$$DQ_g : S_2M \to C^\infty(M),$$

and its formal $L^2$-adjoint is then given by a map $DQ_g^* : C^\infty(M) \to S_2M$. In this setting, one has a $(0,2)$-tensor field canonically associated to $Q$-curvature, given by

$$J_g \doteq -\frac{1}{2} DQ_g^*(1). \tag{13}$$

# Rigidity of Q-singular spaces II

This tensor field satisfies local conservation law (Schur-type lemma)

$$\mathrm{div}_g(J_g - \frac{1}{4}Q_g\, g) = 0. \tag{14}$$

**Remark**

*When one applies the above procedure to the scalar curvature map $g \mapsto R_g$ instead of $Q_g$, one obtains $D^* R_g(1) = -\mathrm{Ric}_g$.*

The above relations *make* $J_g$ a fourth order "analogue" of the Ricci tensor, and thus one defines

$$G_{J_g} \doteq J_g - \frac{1}{4}Q_g\, g \tag{15}$$

as the *J-Einstein tensor*. To make things more explicit, let us write down $J_g$ in terms of known tensor:

$$J_g \doteq \frac{1}{n}Q_g g - \frac{1}{n-2}B_g - \frac{n-4}{4(n-1)(n-2)}T_g, \tag{16}$$

# Rigidity of Q-singular spaces III

where above $B_g$ stands for the Bach tensor, while $T_g$ is defined as

$$\begin{aligned} T_g = &(n-2)(\nabla^2 \mathrm{tr}_g S_g - \frac{1}{n} g \Delta_g \mathrm{tr}_g S_g) \\ &+ 4(n-1)(S_g \times S_g - \frac{1}{n}|S_g|_g^2 g) - n^2 (\mathrm{tr}_g S_g) \mathring{S}_g) \end{aligned} \tag{17}$$

where $S_g \doteq \frac{1}{n-2}\left(\mathrm{Ric}_g - \frac{1}{2(n-2)} R_g g\right)$ stands for the Schoutten tensor, $\mathring{S}_g$ stands for its traceless part, and $(S_g \times S_g)_{ij} \doteq S_i^k S_{kj}$.

In this setting, Riemannian manifolds for which $\mathrm{Ker}(DQ_g^*) \neq \{0\}$ are called *Q-singular*. Rigidity of $Q$-singular manifolds has been recently studied, partially inspired by rigidity properties of **static manifolds**, *i.e*, $\mathrm{Ker}(DR_g^*) \neq \{0\}$.

In this context, and pressing on the analogy between $J_g$ and $\mathrm{Ric}_g$, we intend to prove a fourth-order analogue of the following well-known result:

A Ricci-flat AE manifold is isometric to Euclidean space.

# Rigidity of Q-singular spaces IV

Remark: The above statement is strongly related to the rigidity part of the positive energy theorem in GR.

The fist important ingredient to pursue our rigidity claim for $J$-flat AE manifolds is the following theorem:

**Theorem**

*Let $(M^n, g)$ be an AE manifold of order $\tau > \max\{0, \frac{n-4}{2}\}$ satisfying $Q_g \in L^1(M, dV_g)$. Then, the following identity holds*

$$\frac{n-4}{8(n-1)}\mathcal{E}(g) = -\lim_{r\to\infty}\int_{S_r} G_{J_g}(X, \nu_\delta)d\omega_\delta, \tag{18}$$

*where $X = r\partial_r$. In particular, the limit in the right-hand side exists and is finite.*

**Remark**

*In the case of GR, there is an exact analogue of this result, where $G_{J_g}$ is replaced by the usual Einstein tensor.*

**Idea of the proof.**

The proof starts by transferring the problem to $\mathbb{R}^n$. For this, consider the annulus $\Omega_R = B_R(0)\backslash\overline{B_{\frac{R}{4}}(0)} \subset \mathbb{R}^n\backslash\overline{B_1(0)}$ and one choose a cut-off function $\chi_R$ satisfying ($R$ taken to be a large number)

$$\chi_R(x) = \begin{cases} 0, & \text{if } |x| < \frac{R}{2}, \\ 1, & \text{if } |x| > \frac{3R}{4}. \end{cases}$$

Then, denote by $\hat{g} = \chi_R g + (1-\chi_R)\delta$ an associated AE metric on $\mathbb{R}^n$, which by construction:

- is exactly Euclidean in a neighbourhood of the inner boundary of $\Omega_R$;
- It agrees with $g$ in a neighbourhood of the outer boundary of $\Omega_R$.

From to the local conservation law obeyed by $G_{J_{\hat{g}}}$ we find that

# Rigidity of Q-singular spaces VI

$$\int_{\partial\Omega_R} G_{J_{\hat{g}}}(X,\nu)d\omega_{\hat{g}} = \frac{1}{2}\int_{\Omega_R}\langle G_{J_{\hat{g}}}, \pounds_{\hat{g},conf}X\rangle_{\hat{g}}dV_{\hat{g}} + \frac{4-n}{4n}\int_{\Omega_R} Q_{\hat{g}}\mathrm{div}_{\hat{g}}XdV_{\hat{g}}, \tag{19}$$

for any $X \in \Gamma(TM)$. Taking $X = r\partial_r$ and appealing to the AE condition, one finds

$$\Big|\int_{\Omega_R}\langle G_{J_{\hat{g}}}, \pounds_{\hat{g},conf}X\rangle_{\hat{g}}dV_{\hat{g}}\Big| = O(R^{n-2\tau-4}) = o(1). \tag{20}$$

Dealing with the last term in (19) is more subtle and relies on explicit expressions for $\mathrm{Ker}(DQ^*_\delta)$. In the end, one gets

$$\int_{\Omega_R} Q_{\hat{g}}\mathrm{div}_{\hat{g}}XdV_{\hat{g}} \xrightarrow[R\to\infty]{} \frac{n}{2(n-1)}\mathcal{E}(g) \tag{21}$$

□

# Rigidity of Q-singular spaces VII

**Corollary**

*Under the same conditions as in the above theorem, if $J_g = 0$, then $\mathcal{E} = 0$.*

**Proof.**

Since $\mathrm{tr}_g(J_g) = Q_g$, $J_g = 0$ implies $Q_g = 0$ and thus that $G_{J_g} = 0$, which yields $\mathcal{E}(g) = 0$ from (18). □

Remark: If one guarantees $J_g = 0$ and $g$ satisfies the hypotheses of the PET, then $(M^n, g) \cong \mathbb{E}^n$ from the statement of the PET. Therefore, below, we will actually work towards proving that $J_g$ controls the asymptotic decay of $g$. In particular, if $J_g = 0$, then $g$ satisfies the PET. Once more, the Ricci-tensor has this same optimal control.

To address the problem presented above, one needs to appeal to some elliptic theory on weighted spaces, and thus, we shall refine our definition of AE manifolds as follows.

# Rigidity of Q-singular spaces VIII

## Definition (Weighted Sobolev spaces)

Let $E \to \mathbb{R}^n$ be vector bundle over $\mathbb{R}^n$. The weighted Sobolev space $W^{k,p}_{\delta}$, with $k$ a non-negative integer, $1 < p < \infty$ and $\delta \in \mathbb{R}$, of sections $u$ of $E$, is defined as the subset of $W^{k,p}_{loc}$ for which the norm

$$\|u\|_{W^{k,p}_{\delta}(\mathbb{R}^n)} \doteq \sum_{|\alpha| \leq k} \|\sigma^{-\delta - \frac{n}{p} + |\alpha|} \partial^{\alpha} u\|_{L^p(\mathbb{R}^n)} \tag{22}$$

is finite, where $\sigma(x) \doteq (1 + |x|^2)^{\frac{1}{2}}$ and $\alpha$ denotes an arbitrary multi-index.

## Definition ($W^{k,p}_{-\tau}$-AE manifolds)

Let $(M^n, g)$ be a complete, smooth, connected, $n$-dimensional Riemannian manifold and let $\tau > 0$. We say that $(M, g)$ is an Asymptotically Euclidean (AE) manifold of class $W^{k,p}_{-\tau}$ if:

1. There exists a compact set $K \subset M$ and a diffeomorphism $\Phi : M \backslash K \mapsto \mathbb{R}^n \backslash \overline{B_1(0)}$;
2. For each integer $i \leq m$, $k, l \leq n$ $(\Phi^* g)_{kl} - \delta_{kl} \in W^{k,p}_{-\tau}(\mathbb{R}^n \backslash \overline{B_1(0)})$.

# Rigidity of Q-singular spaces IX

Remark: Using a partition of unity one extends the definition of $W^{k,p}_{\delta}$-spaces to an arbitrary AE manifold.

**Theorem (Rigidity for J-flat AE spaces)**

*Let $(M^n, g)$ be a smooth AE manifold of class $W^{3,p}_{-\tau}$ with $p > n$. If $J_g = 0$ and $Y([g]) > 0$, then $(M^n, g)$ is isometric to the Euclidean space $(\mathbb{R}^n, \delta)$.*

**Idea of the proof**:

The proof relies on a decay improvement due to elliptic theory, which has the following form:

Let $(M^n, g)$ is an AE manifold of class $W^{k-1,p}_{-\tau}$, $k \geq 2$ and $p > n$. Then

$$\text{If a function } f \in L^p{}_{-\delta}(M) \text{ and } \Delta_g f \in W^{k-2,p}_{-\delta-2}(M) \Longrightarrow f \in W^{k,p}_{-\delta}(M), \quad (23)$$

combined with

Assume that $(M, g)$ is AE of class $W^{k,p}_{-\tau}$, $k \geq 2$, $0 < \delta < n - 2$ and $p > \frac{n}{k}$.

$$\text{If, } \tau \leq \delta,\ f \in W^{2,p}_{-\tau}(M),\ \Delta_g f \in W^{k-2,p}_{-\delta-2}(M) \text{ then } f \in W^{k,p}_{-\delta}(M). \quad (24)$$

# Rigidity of Q-singular spaces X

The moral of the above two results is that once $f$ has some decay, if $\Delta_g f$ has better decay, this bootstraps (up to a critical value) the decay on $f$!

Then, recall the structure of $J_g$

$$J_g \doteq \frac{1}{n} Q_g g \underbrace{- \frac{1}{n-2} B_g - \frac{n-4}{4(n-1)(n-2)} T_g}_{\text{traceless}}, \qquad (25)$$

$$Q_g = \mathrm{tr}_g J_g$$

- The proof then relies on a successive improvement of the decay first of $Q_g$, then $R_g$, then $T_g$, then $B_g$, then $\mathrm{Ric}_g$ and finally $g_{ij} - \delta_{ij}$;
- In this sequence, the bootstrap (23) improves the number of derivatives that decay, while (24) improves the order of decay;
- The first key point is that an improvement on the decay on $Q_g$ translates into an improvement on $\Delta_g R_g$, which then bootstraps to $R_g$.
- Using the structure of $T_g$, such an improvement on $R_g$ translates into $T_g$.

# Rigidity of Q-singular spaces XI

- Having better decay for $J, Q$ and $T$, implies better decay for $B_g$, which reads:

$$B_{uv} = \frac{1}{n-2}\Delta_g \mathrm{Ric}_{uv} - \frac{1}{2(n-1)(n-2)}\Delta_g R_g g_{uv} - \frac{1}{2(n-1)}\nabla_{uv} R_g$$
$$- 2\mathrm{Riem}_{auvb} S^{ab} - (n-4) S^a_u S_{av} - |S_g|^2 g_{uv} - 2\mathrm{Tr}(S_g) S_{uv},$$

- All the above translates into an improvement on $\Delta_g \mathrm{Ric}_{uv}$, which bootstraps into $\mathrm{Ric}_g$.
- Finally, using harmonic coordinates where the Ricci reads

$$\mathrm{Ric}_{ij} = g^{ab}\partial_{ab} g_{ij} + f_{ij}(g, \partial g)$$

with $f_{ij}(g, \partial g)$ quadratic on $\partial g$. One has a priori control of $f_{ij}$ due to Sobolev multiplication properties, and then on $g^{ab}\partial_{ab}(g_{ij} - \delta_{ij})$, which bootstraps into an improved control on $g_{ij} - \delta_{ij}$;

- Once we achieve $g_{ij} - \delta_{ij} \in W^{5,p}_{-\sigma}$, with $\max\{0, \frac{n-4}{2}\} < \sigma < n-2$, we apply the PET to conclude.

□

Thank you for your attention!