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The Categorical Connection Between Geometric and Algebraic Formulations of GR

Studies in History and Philosophy of Science Part B: Studies in History and Philosophy of Modern Physics

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On Einstein algebras and relativistic spacetimes

☆

Sarita Rosenstock [a], Thomas William Barrett [b], James Owen Weatherall [a]

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Abstract

In this paper, we examine the relationship between general relativity and the theory of Einstein algebras. We show that according to a formal criterion for theoretical equivalence recently proposed by Halvorson, 2012, Halvorson, 2015 and Weatherall (2015a), the two are equivalent theories.

Commun. math. Phys. 26, 271—275 (1972)

Einstein Algebras

ROBERT GEROCH
The Enrico Fermi Institute, Chicago, Illinois, USA

Received January 31, 1972

Abstract. An approach to quantization of general relatively using a reformulation of the classical theory in which the events of space-time play essentially no role is discussed.

Manifold $M \mapsto \mathbb{R}$-algebra ~~$C^\infty(M) = \{f : M \to \mathbb{R}\}$~~ A

derivation on M = $\mathbb{R}$-linear map $\hat{X}$: ~~$C^\infty(M)$~~ A $\to$ ~~$C^\infty(M)$~~ A

s.t. $\forall f, g \in$ ~~$C^\infty(M)$~~ A, $\hat{X}(f \circ g) = f \circ \hat{X}(g) = g \circ \hat{X}(f)$

$\Rightarrow$ module of derivations $\mathscr{D}$ and dual module $\mathscr{D}^*$

$\Rightarrow$ metric $g : \mathscr{D} \to \mathscr{D}$ symmetric module isomorphism

$+$ Ricci tensor, Riemann tensor, etc.

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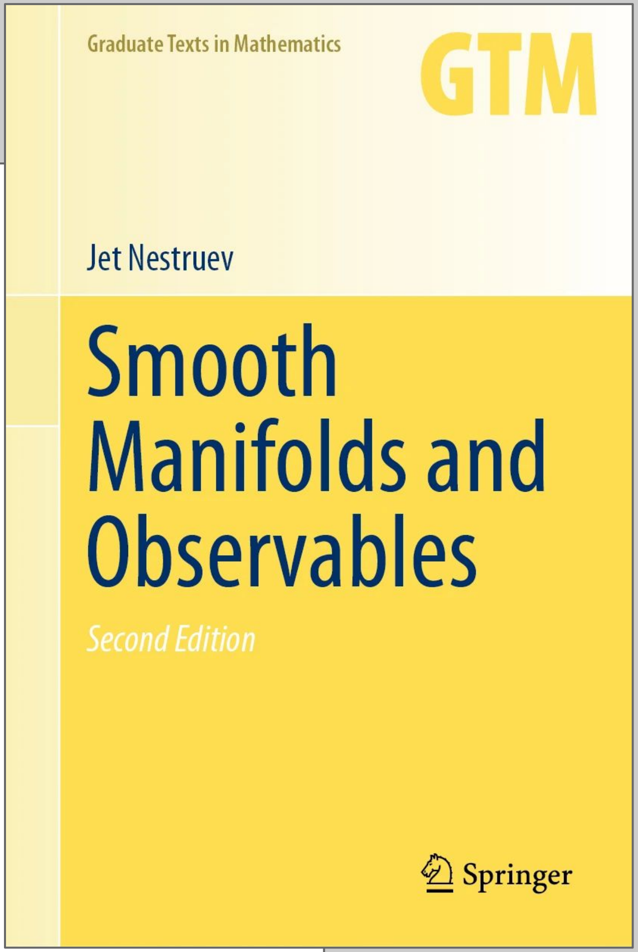
Abstract

In this paper, we examine the relationship between general relativity and the theory of Einstein algebras. We show that according to a formal criterion for theoretical equivalence recently proposed by Halvorson, 2012, Halvorson, 2015 and Weatherall (2015a), the two are equivalent theories.

$|A| =$ "points" of $A =$

"smooth" homomorphisms $\hat{x} : A \to \mathbb{R}$

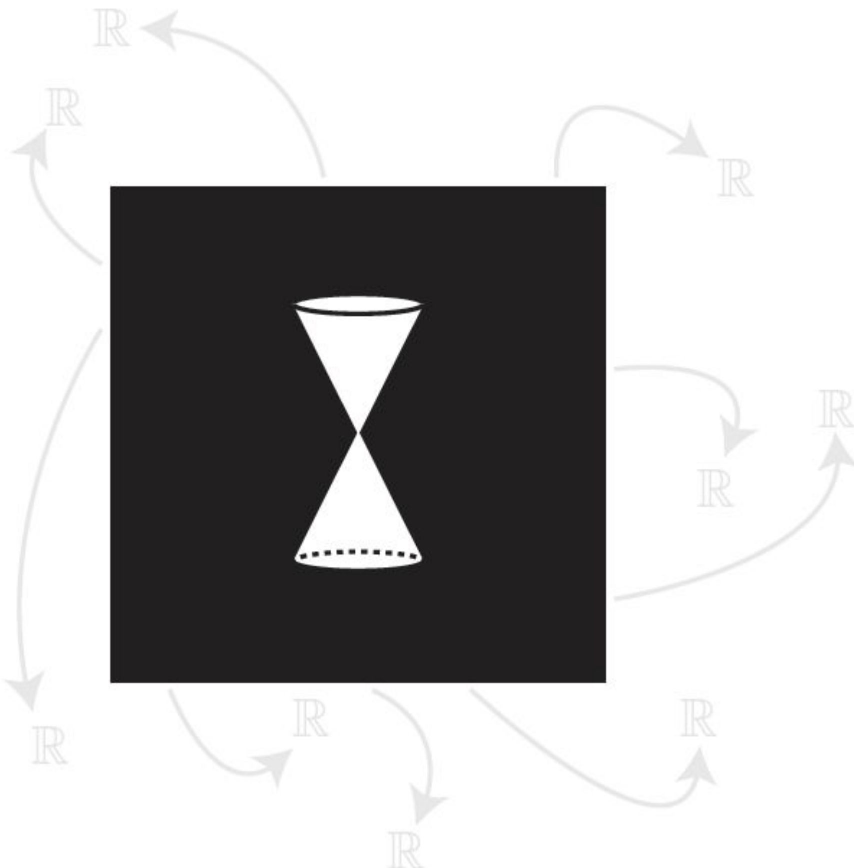
$$\hat{x}(f) \leftrightarrow f(x)$$



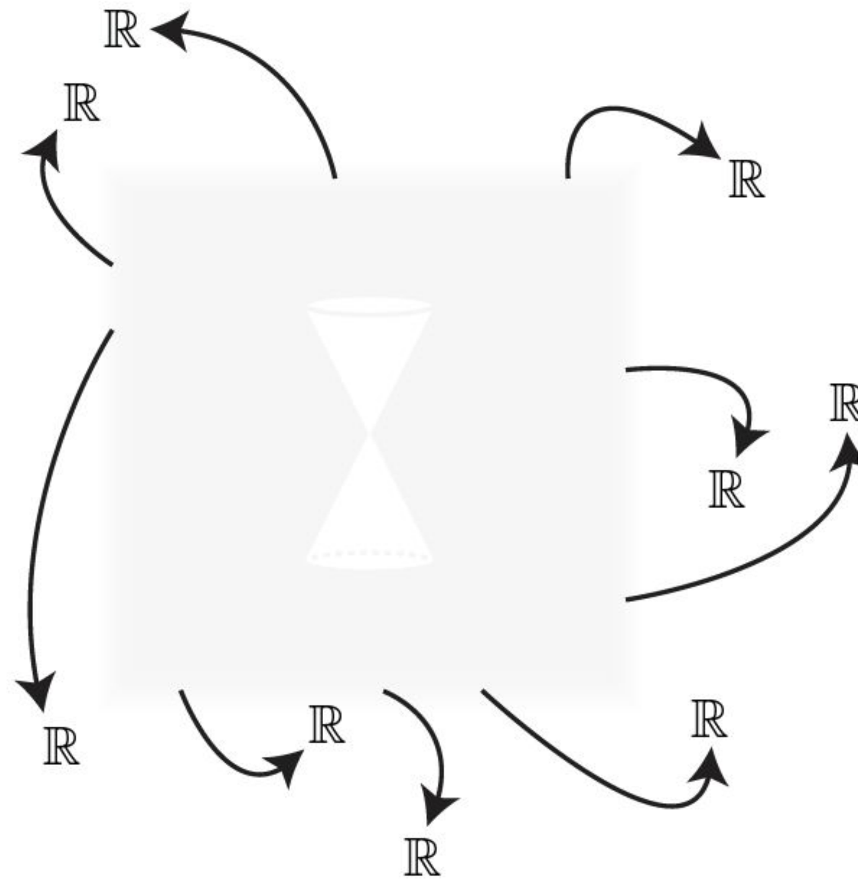
$\to$ Topology on A

$\to$ Conditions on topological algebra to construct manifold

$\to$ + Algebraic definitions of derivations & tensor fields



Manifold + Metric



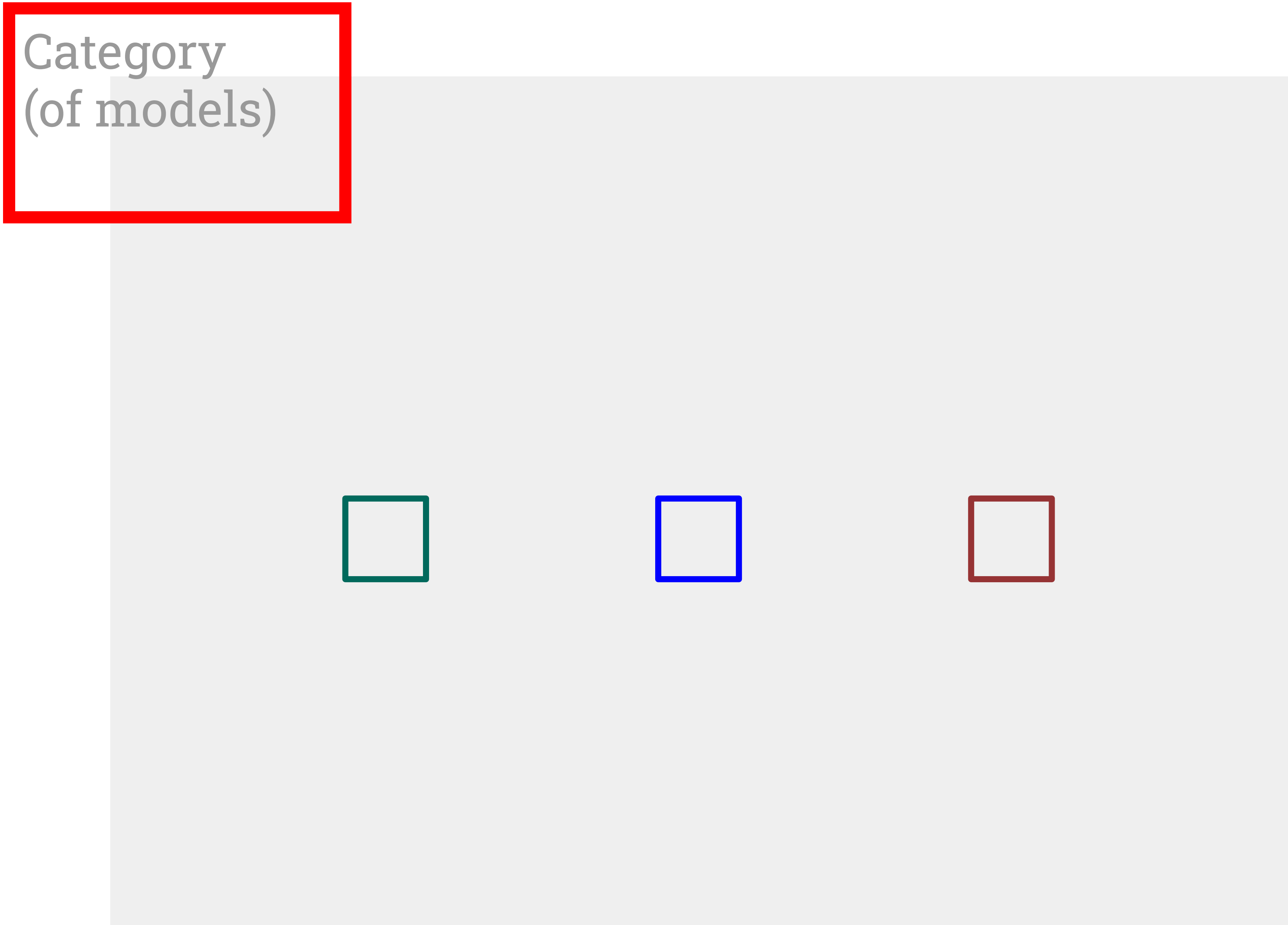
Einstein Algebra

Categorical relationship between theories

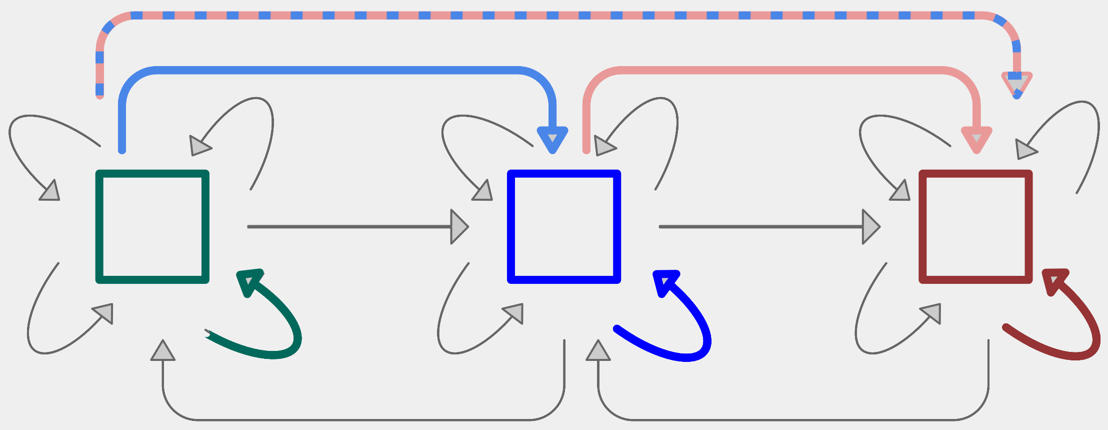
Q: Do T_1 and T_2 have "the same" content, or does one involve stronger ontological commitments than the other?

[some category theory]

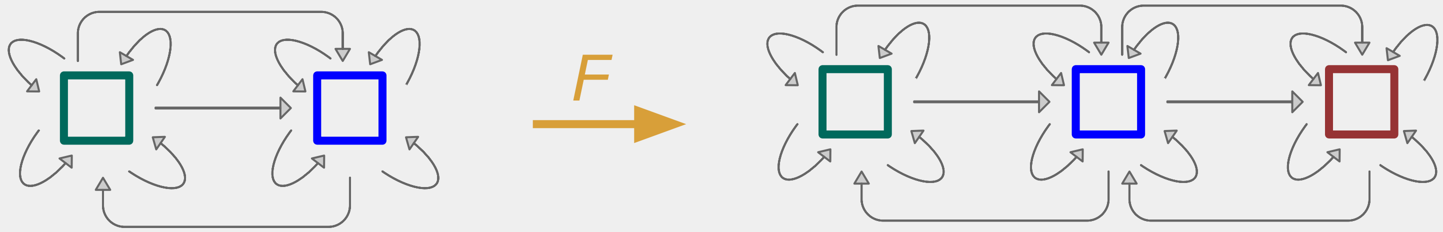
A: The physically relevant relationship between T_1 and T_2 is that T_1 has [more/less/the same] [structure/properties/stuff] compared to T_2.


Category
(of models)

Category
(of models)



Functor



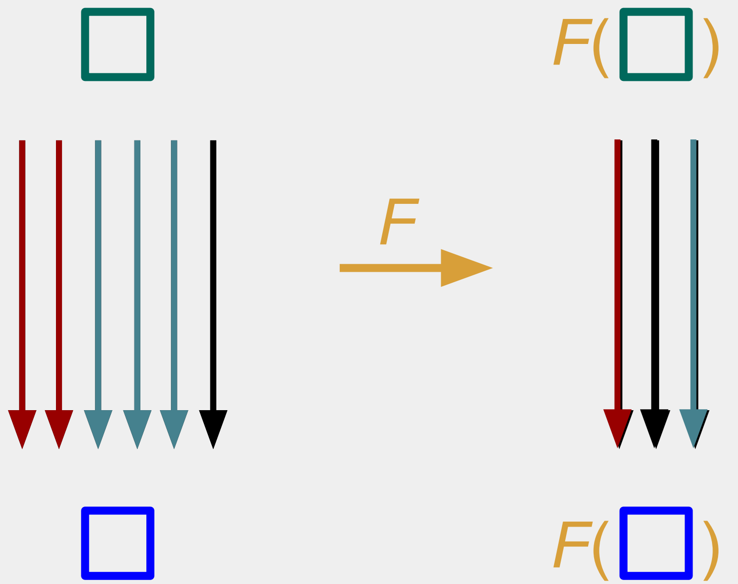
Q: Do T_1 and T_2 have "the same" content, or does one involve stronger metaphysical commitments than the other?

1. T_1 can be understood as category $\mathbf{C}_1$.
2. T_2 can be understood as category $\mathbf{C}_2$.
3. The relationship between T_1 and T_2 can be understood as a functor $F: \mathbf{C}_1 \to \mathbf{C}_2$.
4. F has xyz properties.

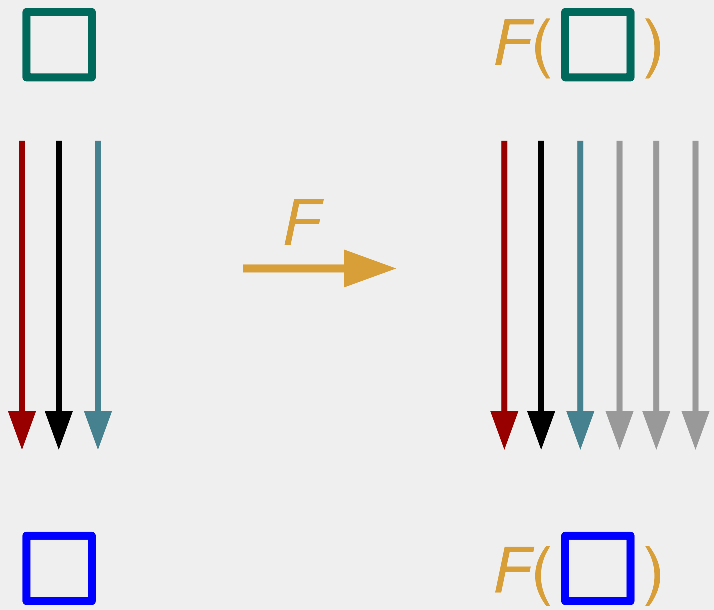
from which we infer

A: The physically relevant relationship between T_1 and T_2 is that T_1 has [more/less/the same] [structure/properties/stuff] compared to T_2.

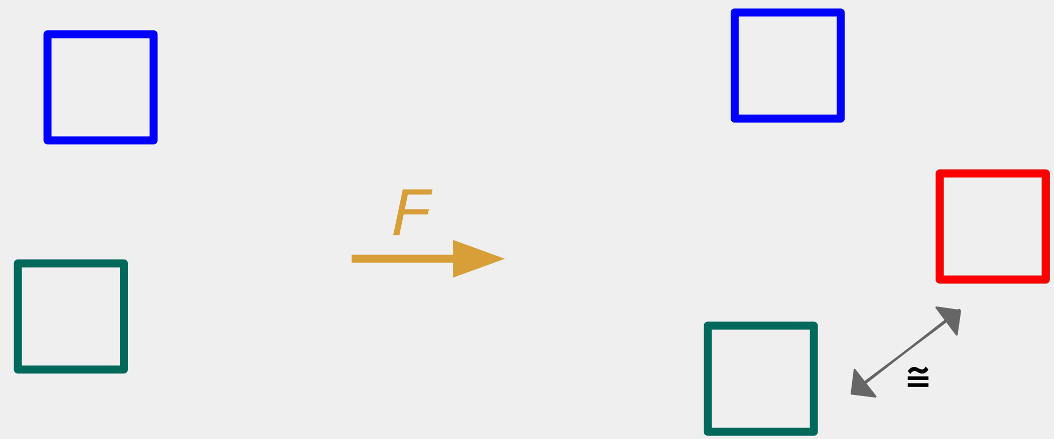
Full



Faithful



Essentially Surjective



The PSS heuristic

If a functor is full, faithful, and essentially surjective, it realizes an equivalence of categories.

Heuristic:

- → not essentially surjective = forgets "properties"
- → not full = forgets "structure"
- → not faithful = forgets "stuff"

Origin of PSS Heuristic

Toby Bartels 11/13/98

Just Categories now (Was: Symplectic forms and Categories)

John Baez <ba...@galaxy.ucr.edu> wrote parenthetically:

>I will
>leave it to James Dolan to explain the technical distinction between
>"extra properties", "extra structure", and "extra stuff" - there is
>a nice category-theoretic way of making this precise.

Ooh, let me guess!

Given a functor U: C -> D, interpret U as a forgetful functor.
Then C is D with extra *structure* if U is surjective on the objects
and, given a pair of objects, injective on the morphisms between them;
and C is D with extra *properties* if U is injective on the morphisms
(meaning injective on the objects and on the morphisms between a given pair);
Otherwise, I guess C is just D with extra *stuff*
if, given a pair of objects, U is injective on the morphisms between them.

For example, the forgetful functor Groups -> Sets
shows that groups are sets with extra structure,
while the forgetful functor Abelian Groups -> Groups
shows that Abelian groups are groups with extra properties.
Or you can turn around and use the free functor Sets -> Groups
and say that sets are groups with extra properties
(to wit, the property of being free).
OTOH, the Abelianization functor Groups -> Abelian groups
is surjective on the objects (and on the morphisms for that matter),
but groups are not Abelian groups with extra structure,
because the functor isn't injective on the morphisms between a given pair.

-- Toby
to...@ugcs.caltech.edu

Inspiration for PSS

"Mathematical gadgets" can be defined by specifying

Some stuff: set(s), space(s),

Equipped with structure: subset(s), elements, relations

Satisfying properties: equations, inequalities, inclusions

Example

A function is

a pair of sets X and Y (stuff)

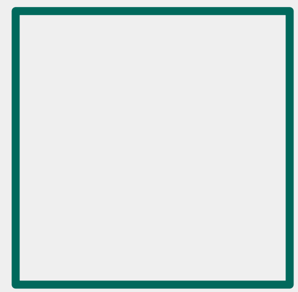
Equipped with $f \subseteq X \times Y$ (structure)

Satisfying $\forall x \in X \;\; \exists ! y \in Y$ s.t. $(x, y) \in f$. (property)

Ex: forgetting structure

Category **Sq**

Object:

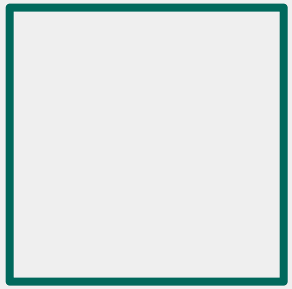


Arrows: 90^0, 180^0, 270^0 rotations; reflections over hor, vert, and diag axes

Ex: forgetting structure

Category **Sq**

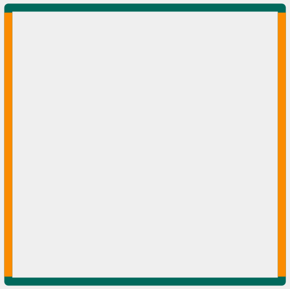
Object:



Arrows: 90^0, 180^0, 270^0 rotations; reflections over hor, vert, and diag axes

Category **Sq***

Object:

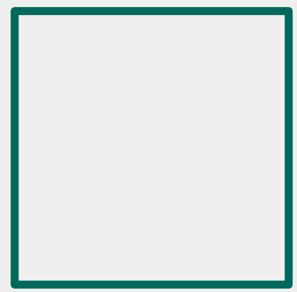


Arrows: 180^0 rotation; hor & vert reflection

Ex: forgetting structure

Category **Sq**

Object:

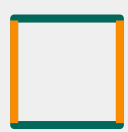
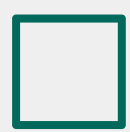


Arrows: 90^0, 180^0, 270^0 rotations; reflections over hor, vert, and diag axes

Category **Sq***

Object:

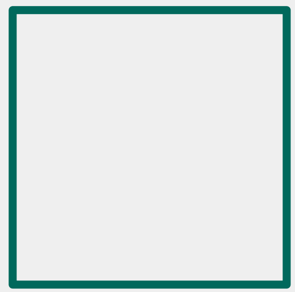
Arrows: 180^0 rotation; hor & vert reflection

Functor F: **Sq*** $\to$ **Sq** takes  to , arrows to themselves

Ex: forgetting stuff

Category **Sq**

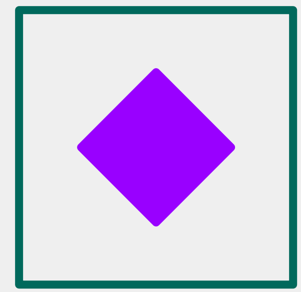
Object:



Arrows: 90^0, 180^0, 270^0 rotations; reflections over hor, vert, and diag axes

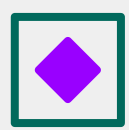
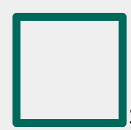
Category **Sq****

Object:



Arrows:

outside square arrows + inside square arrows

Functor F: **Sq**** $\rightarrow$ **Sq** takes  to , arrows to “themselves”

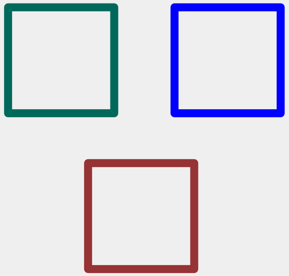
Ex: forgetting properties

Category **Sq**

Object: 

Arrows: 90^0, 180^0, 270^0 rotations; Reflections over hor, vert, and diag axes

Category **ColSq**

Objects: 

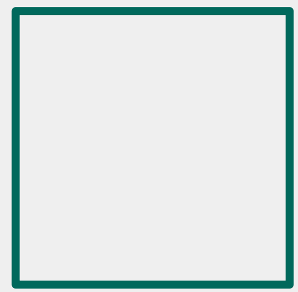
Arrows: ← same x 3

Functor F: **Sq** → **ColSq** takes  to , arrows to themselves

Ex: equivalence

Category **Sq**

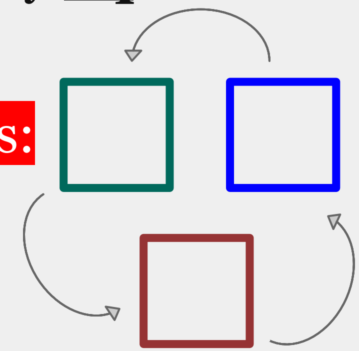
Object:



Arrows: 90^0, 180^0, 270^0 rotations; Reflections over hor, vert, and diag axes

Category **Sq**

Objects:



Arrows: ← + color swaps

Functor F: **Sq** → **Sq** takes  to , arrows to themselves

Intuitive significance of PSS

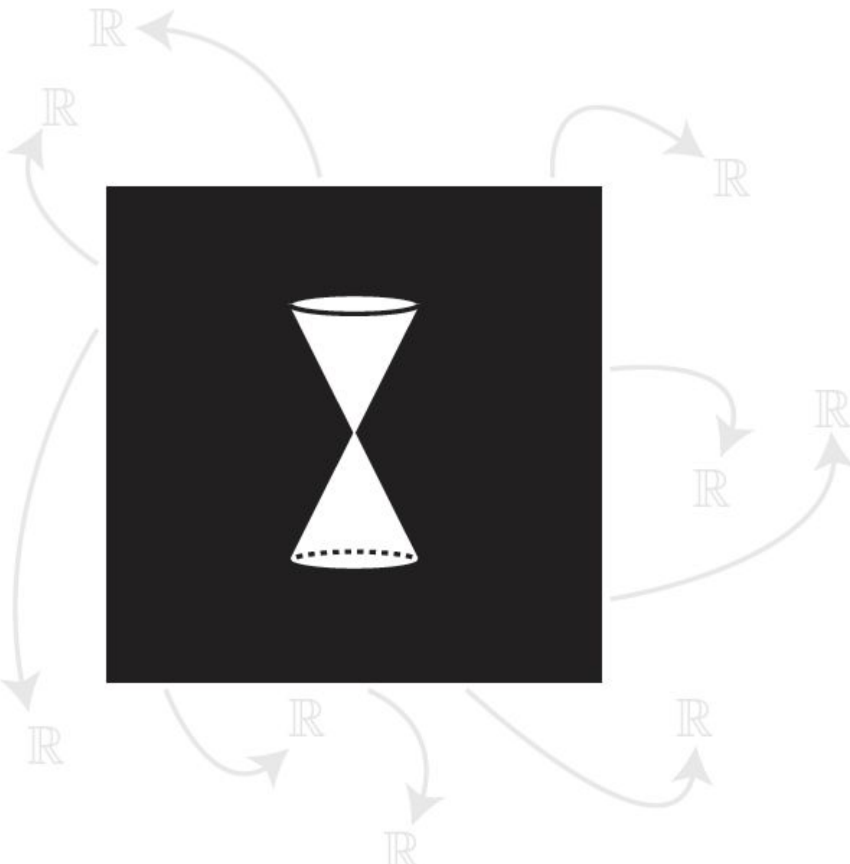
forgetting properties → expanding scope

forgetting stuff → reducing dimension

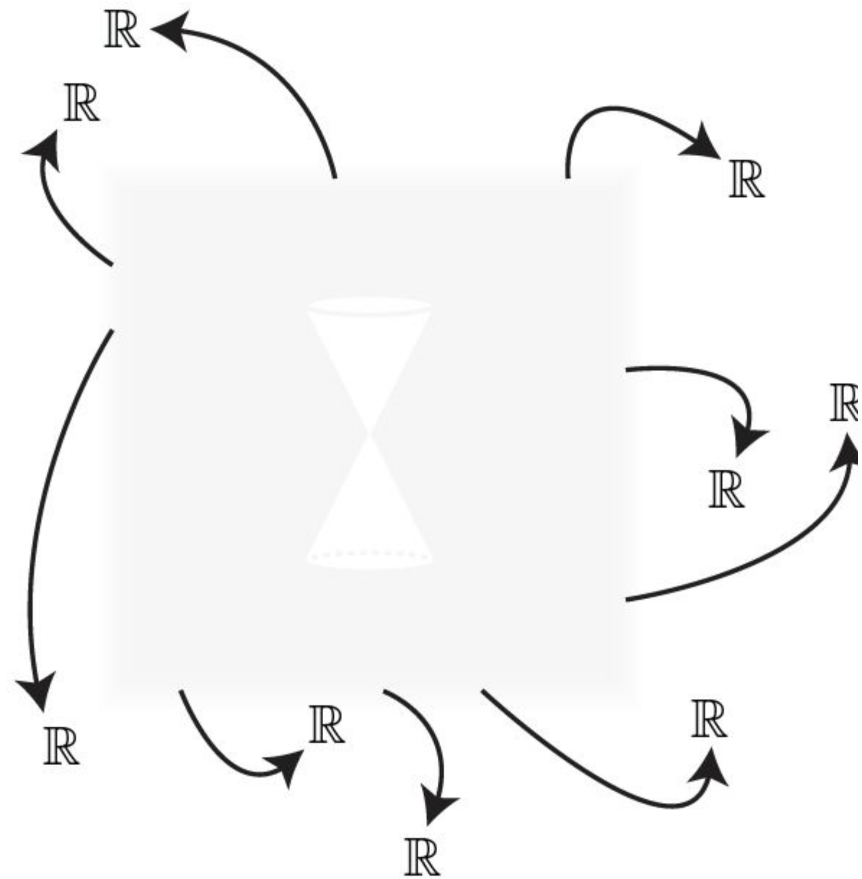
forgetting structure → adding noise or eliminating artifacts

Category **Man**	Category **EA**
Objects:	Objects:
Manifolds with metric	“Smooth” algebras + “metric”
Arrows:	Arrows:
Metric preserving diffeomorphisms of the manifold	“Smooth” isomorphisms that preserve “metric”

Functor F: **Man** → **EA** is an equivalence of categories



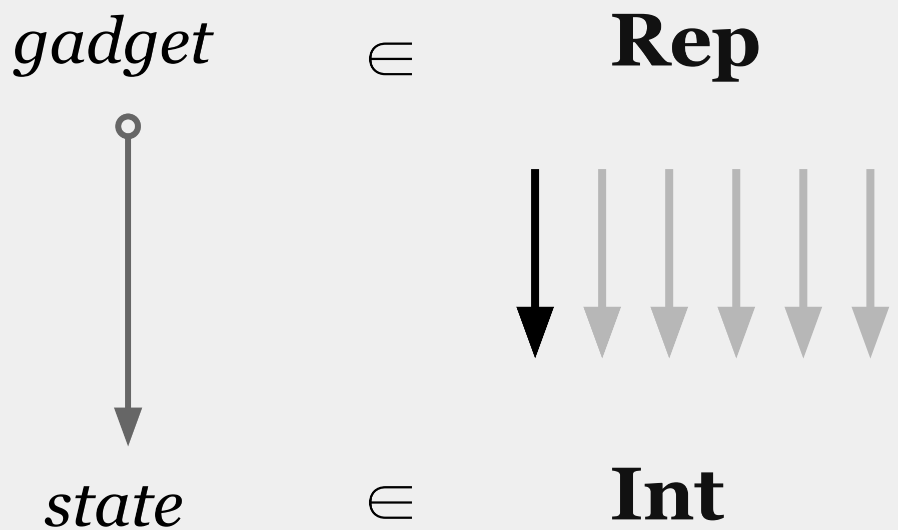
Manifold + Metric



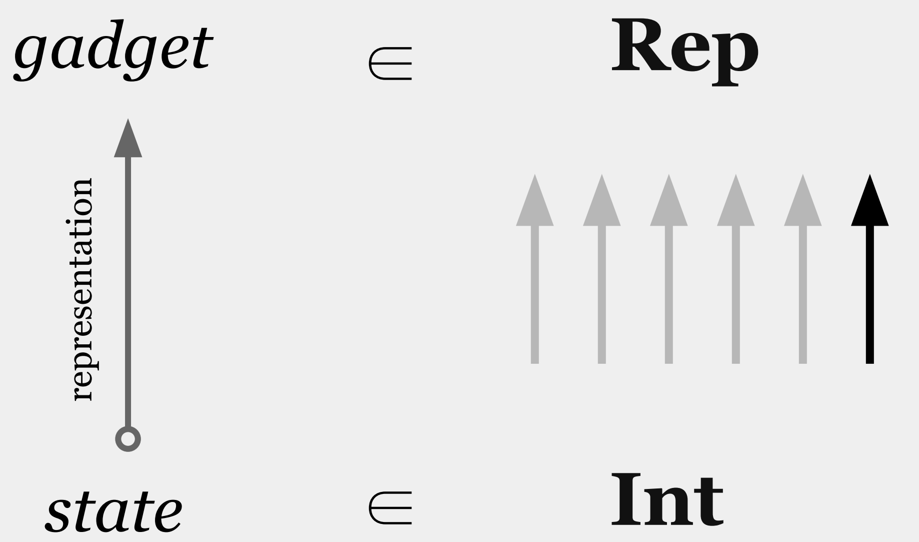
Einstein Algebra

Problem:

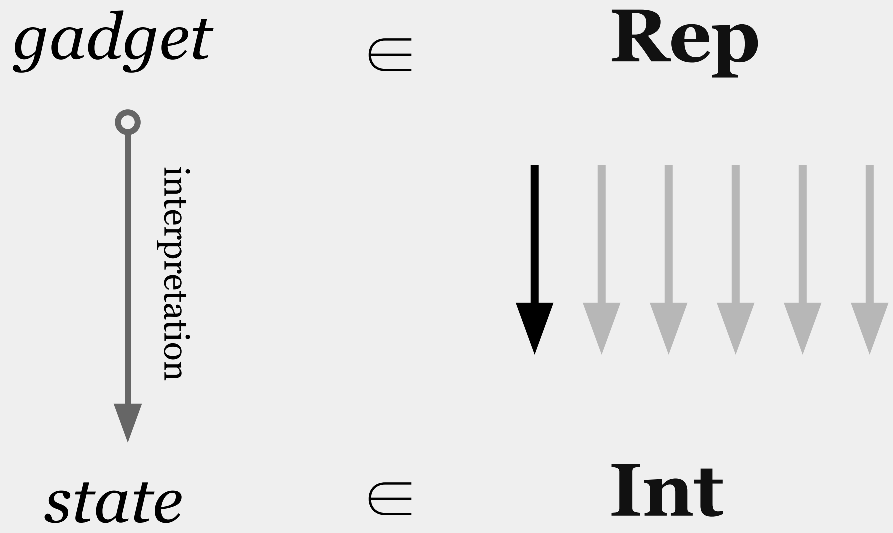
Which categories & functors


gadget ∈ Rep
state ∈ Int

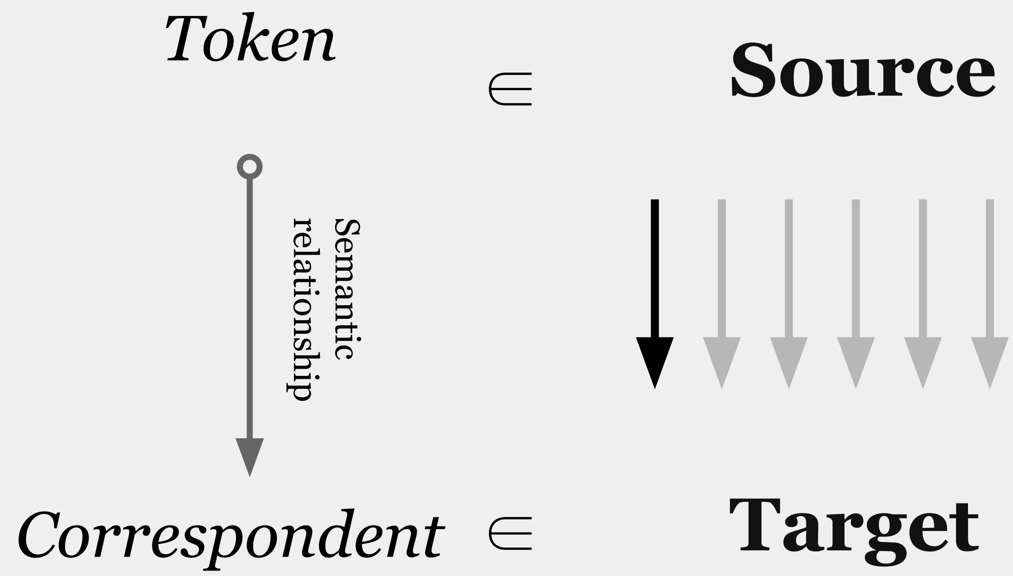
Representation
Diagram



Interpretation
Diagram

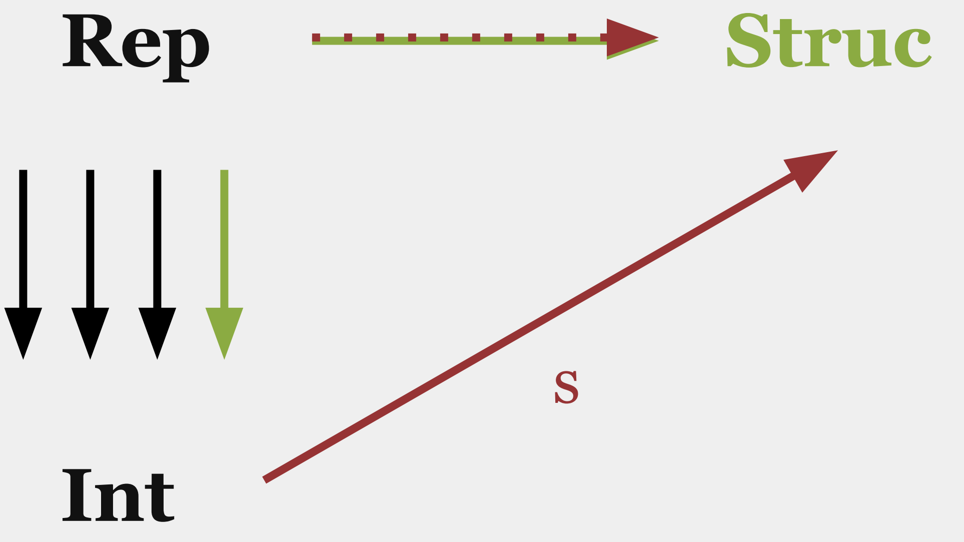


Semantic
Diagram



Structure

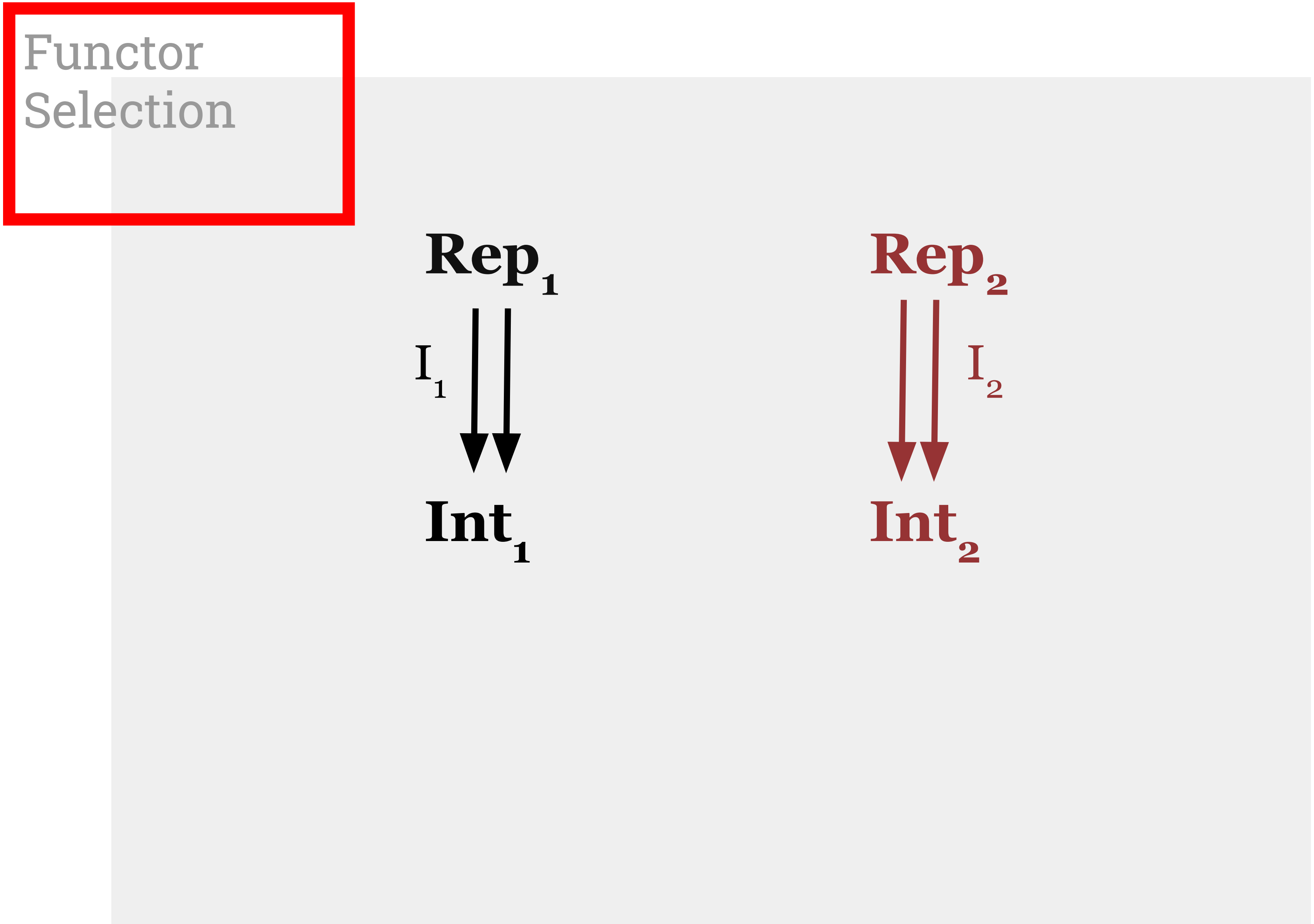
Category



“

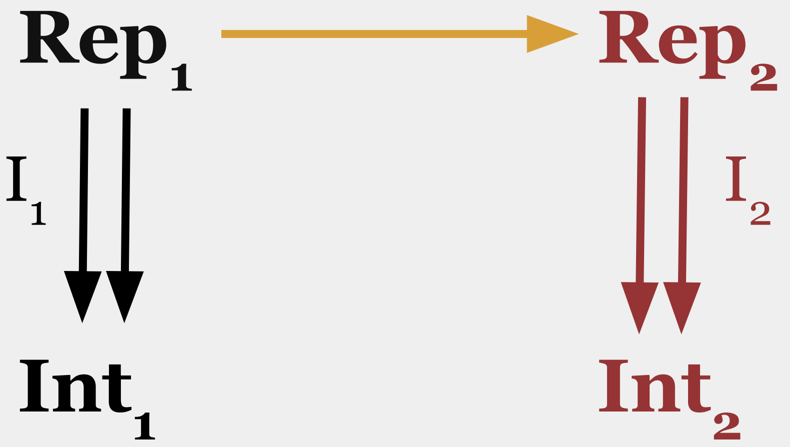
The intuitive general idea of a colimit is that it defines an object obtained by sewing together the objects of the diagram, according to the instructions given by the morphisms of the diagram.

nLab

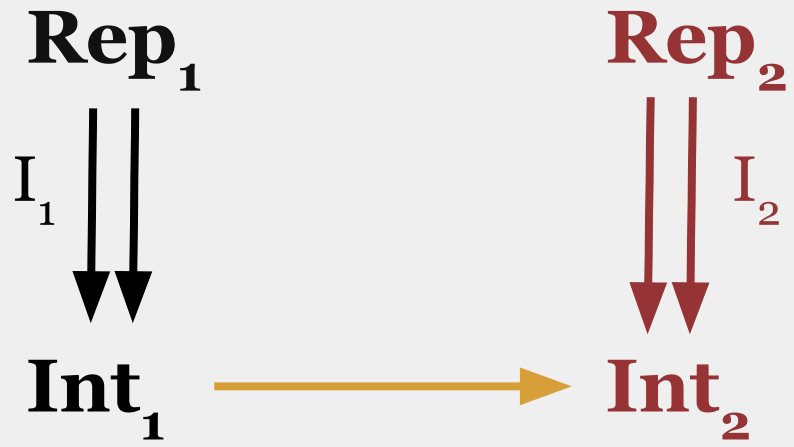


Functor
Selection
Rep1
I1
Int1
Rep2
I2
Int2

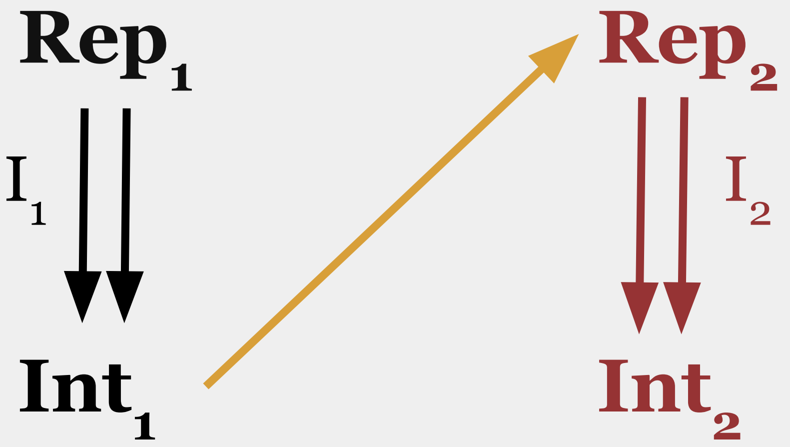
Functor Selection

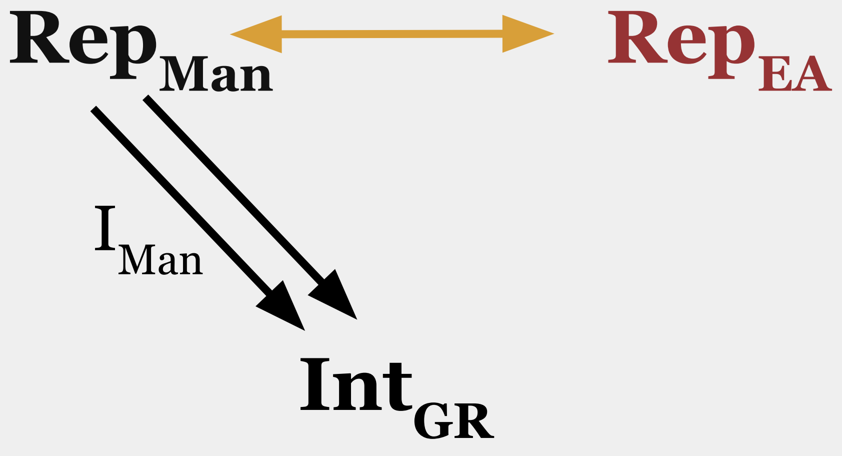


Functor Selection

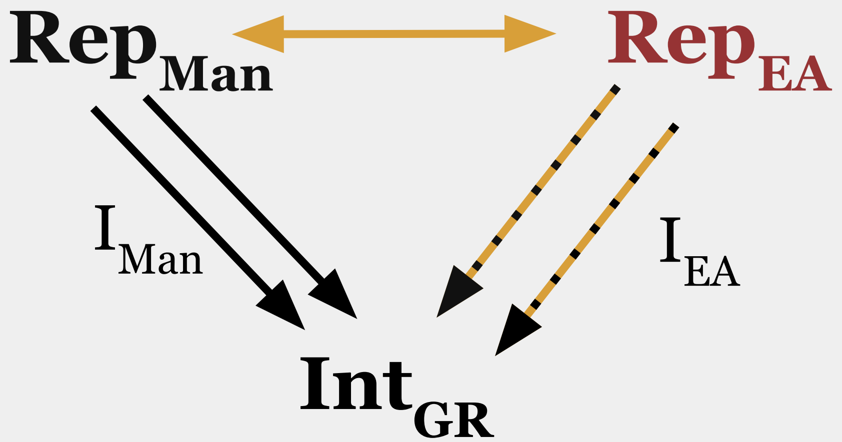


Functor
Selection

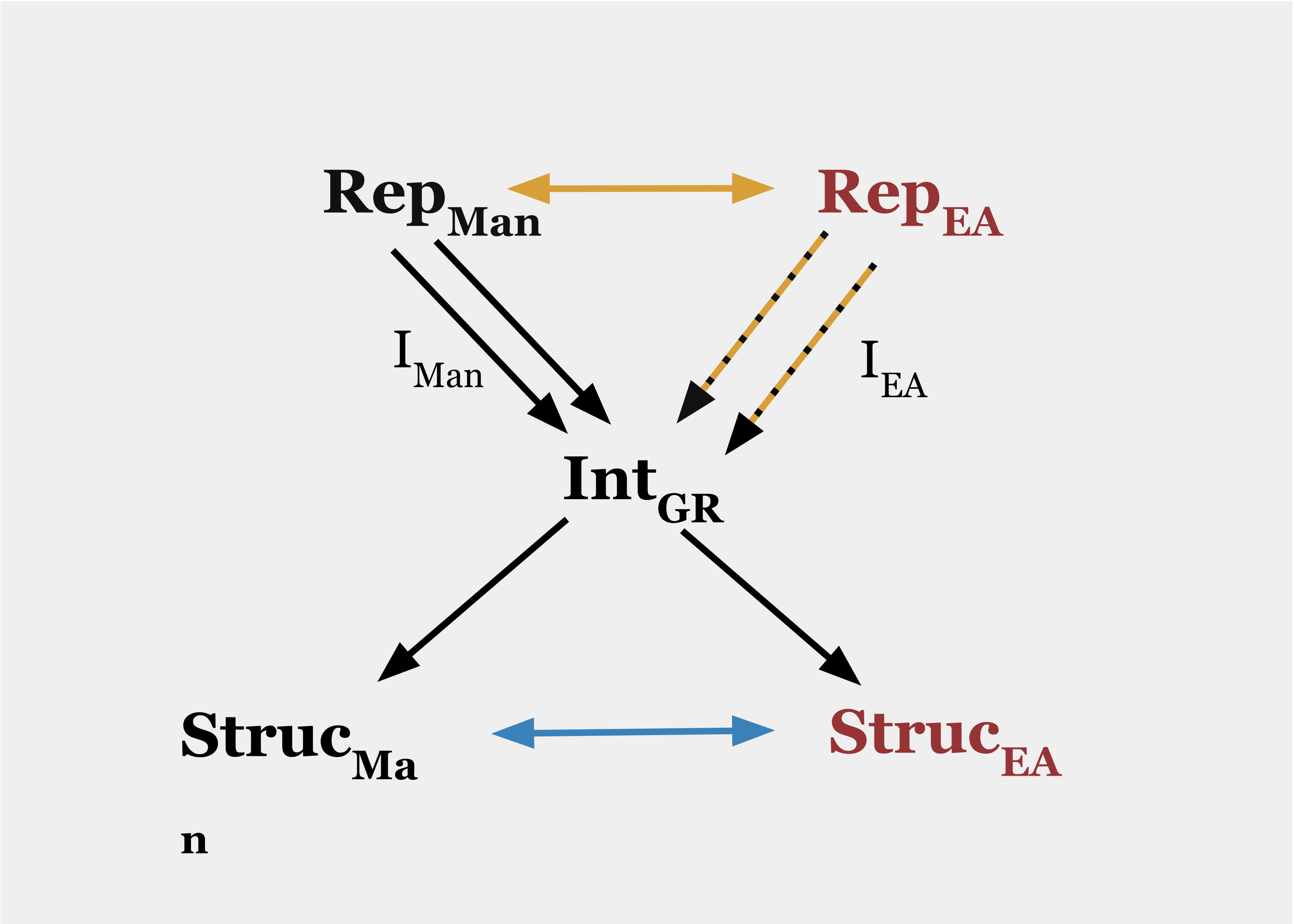


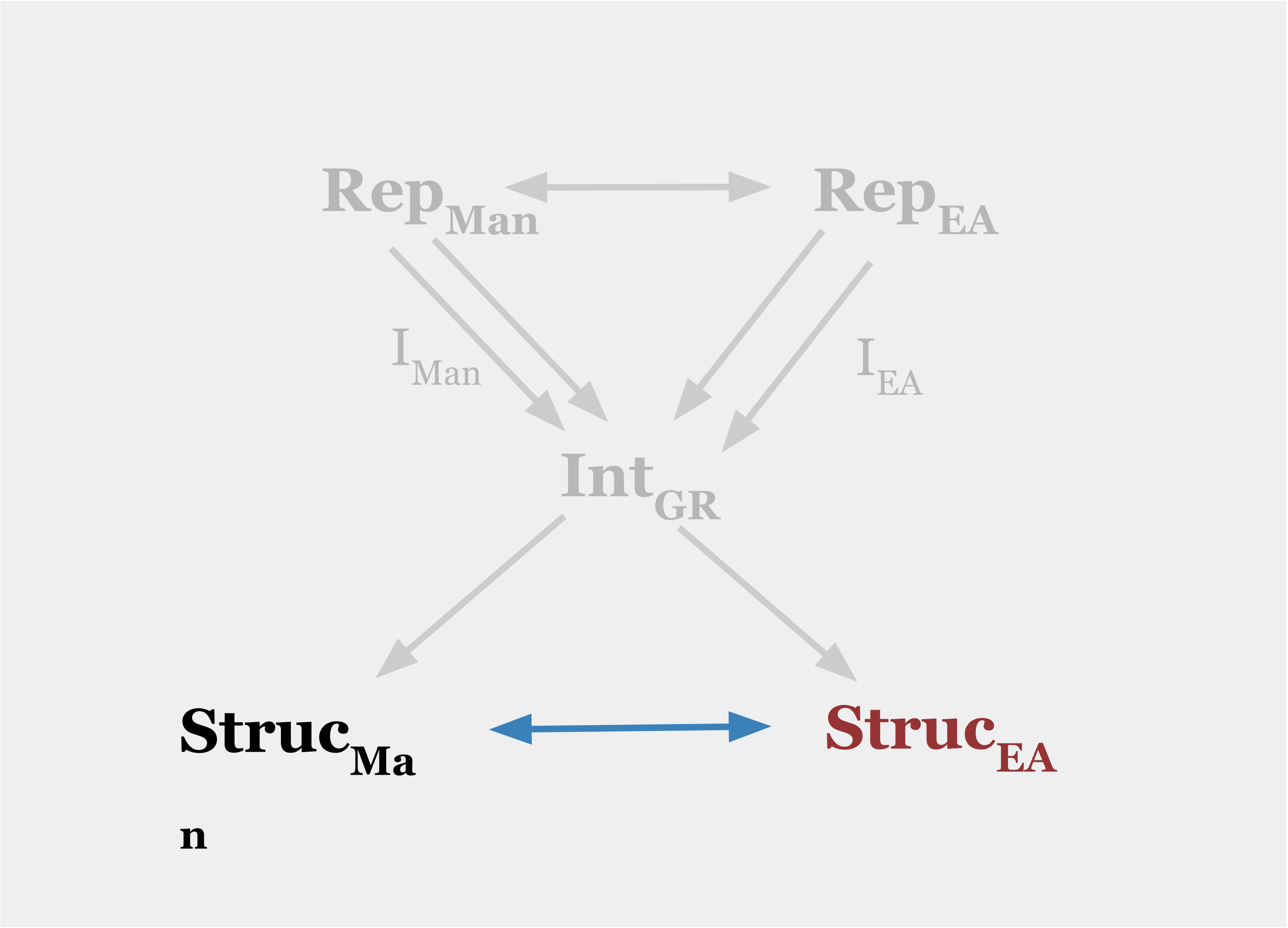


Rep_{Man}
Rep_{EA}
I_{Man}
Int_{GR}

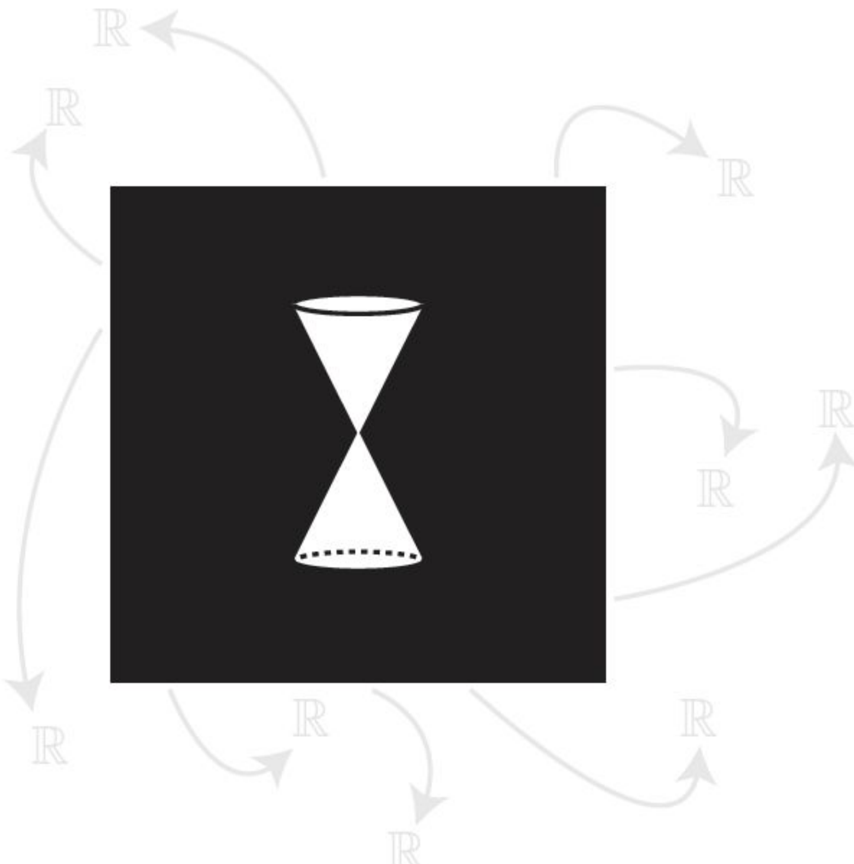


Rep_Man
Rep_EA
I_Man
I_EA
Int_GR

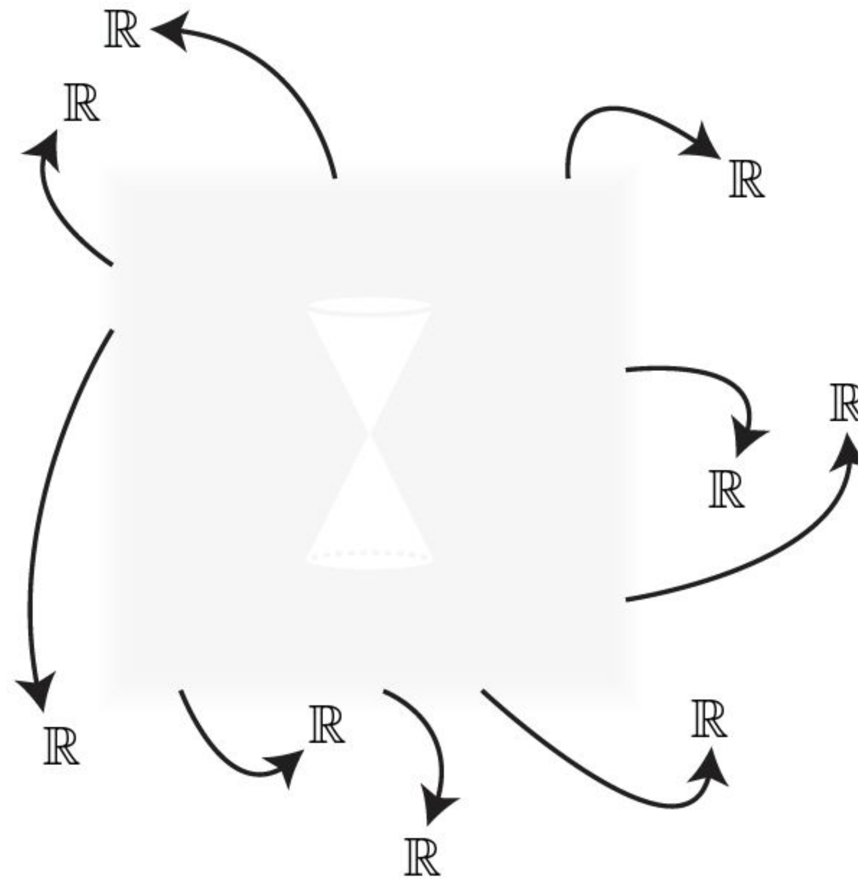

Rep_Man
Rep_EA
I_Man
I_EA
Int_GR
Struc_Man
Struc_EA



Rep_{Man}
Rep_{EA}
I_{Man}
I_{EA}
Int_{GR}
$Struc_{Man}$
$Struc_{EA}$



Manifold + Metric



Einstein Algebra

Summary

Category theory provides a scaffold for telling a story about the relationship between theoretic formalisms that is constrained to be consistent with how we use them.

- Categories offer a precise way of (provisionally) defining a formalism
- Functors express relationships between formalisms
- Semantic diagrams express how we are interpreting formalisms
- PSS heuristic summarises key features of the proposed relationships between formalisms *as interpreted*

Thanks!

Any questions?

?