Applications of Finsler Geometry: from Zermelo navigation to wildfire spread modeling

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UNIVERSIDAD DE MURCIA

Joint work with **M. Á. Javaloyes** (U. Murcia), **M. Sánchez** (U. Granada) and **S. Markvorsen** (DTU):

M. Á. Javaloyes, E. Pendás-Recondo and M. Sánchez.
Applications of cone structures to the anisotropic rheonomic Huygens' principle.
Nonlinear Analysis 209, 112337 (2021).

M. Á. Javaloyes and E. Pendás-Recondo.
Lightlike Hypersurfaces and Time-Minimizing Geodesics in Cone Structures.
In: *Developments in Lorentzian Geometry*, Springer Proceedings in Mathematics & Statistics, vol. 389, Springer, 2022.

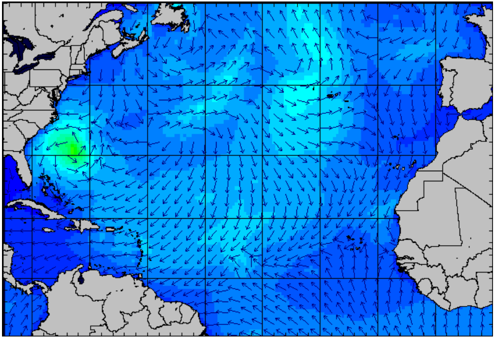
M. Á. Javaloyes, E. Pendás-Recondo and M. Sánchez.
A general model for wildfire propagation with wind and slope.
arXiv:2110.03364 [math.DG] (2021). To appear in SIAGA.

S. Markvorsen and E. Pendás-Recondo.
Snell's law revisited and generalized via Finsler Geometry.
arXiv:2207.13515 [math.DG] (2022). To appear in IJGMMP.

Motivation

Zermelo's navigation problem

- (N, g) Riemannian manifold of dim $n \geq 2$.
- **Goal:** Find the fastest trajectory between two fixed points in the presence of a current.

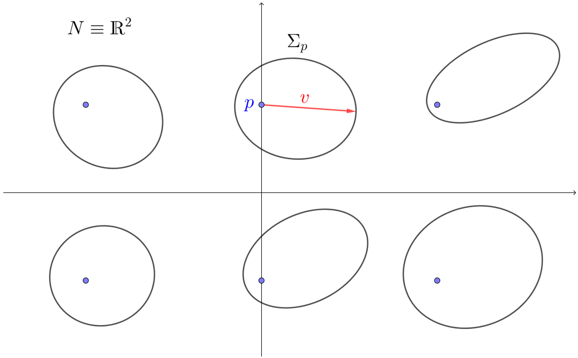


- **Anisotropic** speed profile $V(p, v)$, with

$$V(p, \lambda v) = V(p, v), \quad \forall \lambda > 0.$$

Traveltime

- Velocity vectors: $\Sigma_p := \{v \in T_pN : \|v\|_g = V(p,v)\}$.

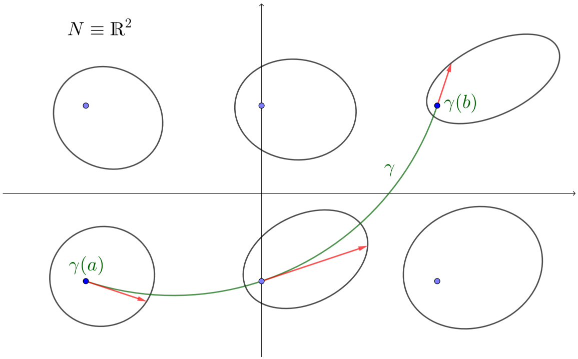


- **Traveltime** along γ:

$$T[\gamma] = \int_a^b \frac{\|\gamma'(t)\|_g}{V(\gamma(t),\gamma'(t))}dt.$$

Traveltime

- Velocity vectors: $\Sigma_p := \{v \in T_pN : \|v\|_g = V(p,v)\}$.



- **Traveltime** along γ:

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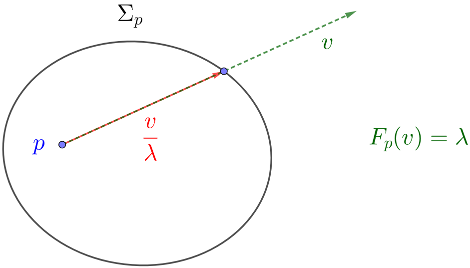
Finsler metrics

- $F_p(v) := \frac{\|v\|_g}{V(p,v)}$ is a Finsler metric iff
  1. $V : TN \setminus \mathbf{0} \to (0, \infty)$ is smooth.
  2. Σ_p is a strongly convex hypersurface of T_pN.



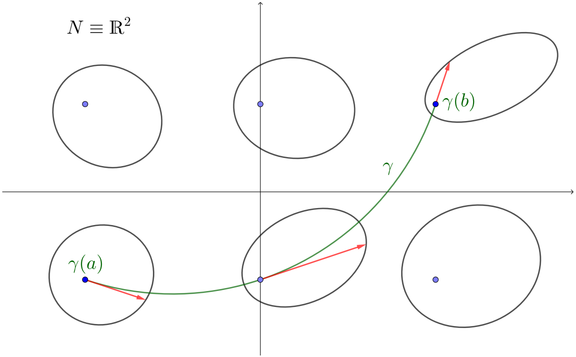
- $\|v\|_g = V(p,v) \iff v \in \Sigma_p \iff F_p(v) = 1.$

Solution to Zermelo's problem

- Traveltime = F-length:

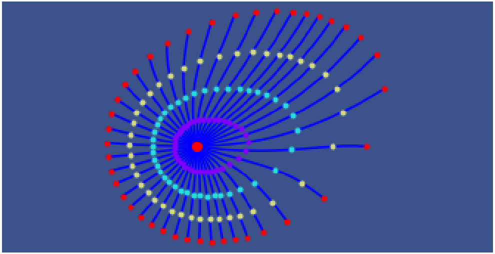
$$T[\gamma] = \int_a^b \frac{\|\gamma'(t)\|_g}{V(\gamma(t), \gamma'(t))} dt = \int_a^b F_{\gamma(t)}(\gamma'(t)) dt = L_F[\gamma].$$

- **Critical points** of $T \iff$ **Pregeodesics** of F.



Wave propagation

- **Wavefront:** All the trajectories that minimize the traveltime from the starting point.

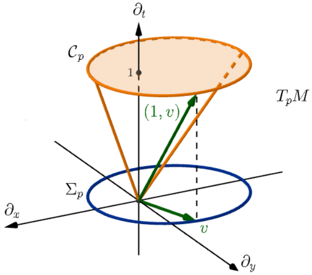


- **Goal:** Find the wavefront when
  1. V is time-dependent $\to$ Finsler spacetimes.
  2. The wave starts at a submanifold $S \subset N$.
  3. V is discontinuous $\to$ Snell's law.

Wave propagation

Finsler spacetimes

- (Non-relativistic) **spacetime** $M := \mathbb{R} \times N$:
  - (N, g) physical space, $t : M \to \mathbb{R}$ absolute time.
- $F_{(t,x)}(v) := \frac{\|v\|_g}{V(t,x,v)}$ **time-dependent** Finsler metric on N.
- $G := dt^2 - F^2$ Lorentz-Finsler metric on M.
- (M, G) Finsler spacetime.



- $v \in \Sigma \iff F(v) = 1 \iff (1, v) \in \mathcal{C} \iff G((1, v)) = 0.$

Causality

Given a Finsler spacetime (M, G):

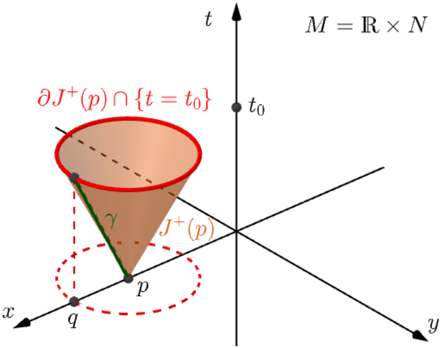
- **Vectors:** $(\tau, v) \in T_pM$ $(\tau \geq 0)$ is
  - spacelike if $G((\tau, v)) < 0$ (outside $\mathcal{C}$),
  - causal if $G((\tau, v)) \geq 0$,
    - timelike if $G((\tau, v)) > 0$ (inside $\mathcal{C}$),
    - lightlike if $G((\tau, v)) = 0$ (on $\mathcal{C}$).
- **Curves:** $\gamma : I \to M$ is spacelike, causal, timelike or lightlike if so is $\gamma'(t)$ $\forall t \in I$.
- **Geodesics:** $\gamma : [a, b] \to M$ is a geodesic of G it is a critical point of the energy functional

$$E_G(\gamma) = \int_a^b G(\gamma'(t))dt.$$

- **Causal future:**
  $J^+(p) := \{q \in M : p = q \text{ or } \exists \text{ causal curve from } p \text{ to } q\}.$

Anisotropic wave propagation

- (M, G) **globally hyperbolic**:
  - $\{t = t_0\} := \{t_0\} \times N$ are Cauchy hypersurfaces.
- If the wave starts at $p \in M$:
  - $J^+(p) \rightarrow$ Propagation of the wave.
  - $\partial J^+(p) \cap \{t = t_0\} \rightarrow$ **Wavefront** at $t = t_0$.

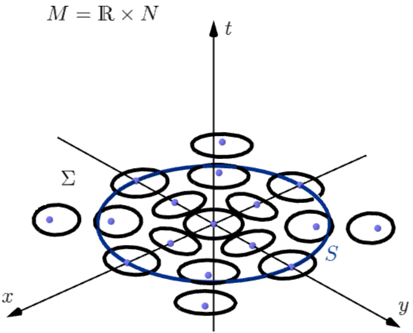


Anisotropic wave propagation

- $S =$ Initial wavefront (compact submanifold of N).
- **Huygens' principle**:

$$\text{front}(t_0) = \partial(\cup_{p\in S} J^+(p)) \cap \{t = t_0\} = \partial J^+(S) \cap \{t = t_0\}.$$

- $\partial J^+(S) \cap \{t = t_0\} \to$ **Wavefront** at $t = t_0$.



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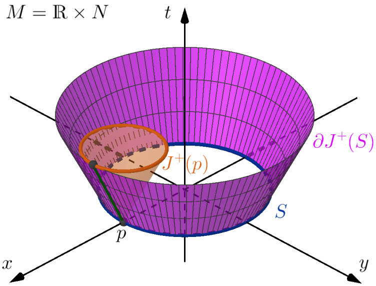

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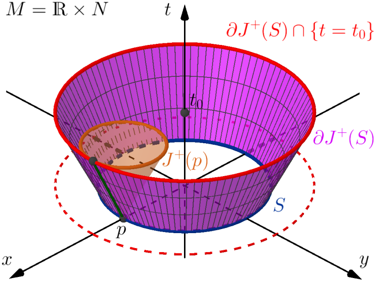
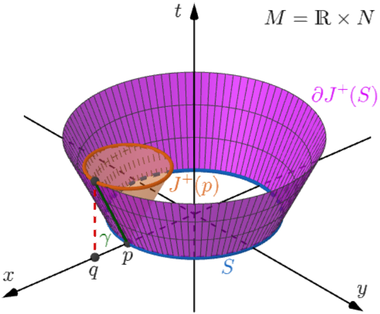


Relation to Zermelo's problem

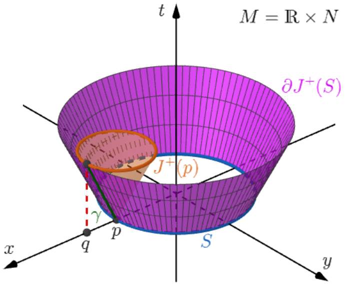
- **Wave trajectories:** Causal curves contained in $\partial J^+(S)$.
- $\gamma \subset \partial J^+(S)$ causal $\iff \gamma$ minimizes the traveltime from S.
- Solutions to the time-dependent **Zermelo's problem**.



Wave trajectories

Proposition

- $\gamma : I \to M$ causal curve contained in $\partial J^+(S) \implies$ γ is a **lightlike pregeodesic** G-orthogonal to S. In fact, $\partial J^+(S)$ admits a unique foliation by such geodesics.
- The converse only holds in $[0, t_0]$, $t_0 \in (0, \infty]$. If $t_0 < \infty$, then $\gamma(t_0)$ is a **cut point** (γ is no longer minimizing).



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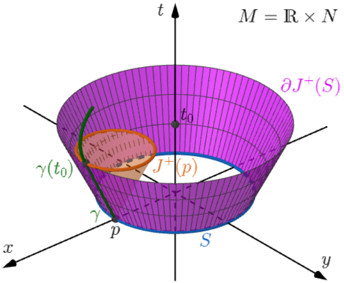


Computation of the wavefront

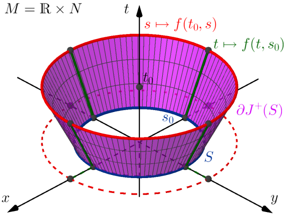
Theorem

For each $s_0 \in S$, the (spacetime) **wave trajectory** $f(t,s_0) = (t, x^1(t,s_0)\dots, x^n(t,s_0))$ is given by the **ODE**:

$$\ddot{x}^k = -\Gamma^k{}_{ij}(\dot{f})\dot{x}^i\dot{x}^j + \Gamma^0{}_{ij}(\dot{f})\dot{x}^i\dot{x}^j\dot{x}^k, \quad k = 1,\dots,n,$$

along with the **initial conditions**:

- $f(0,s_0) = s_0 \in S$,
- $\dot{f}(0,s_0)$ lightlike and G-orthogonal to S.



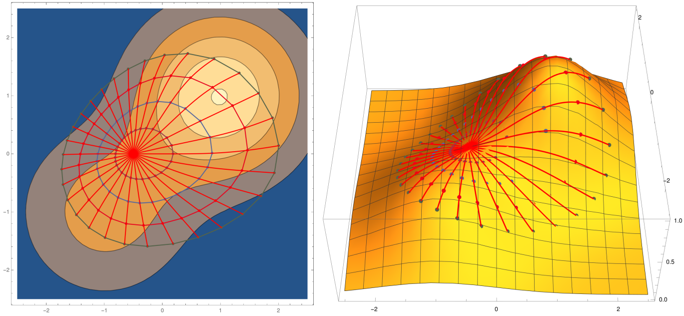
Wildfires

Wildfire spread modeling

- Fire propagation satisfies **Huygens' principle**:

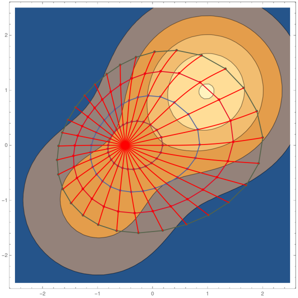
  $$\mathrm{fr}(t_1) = \partial(\cup_{p\in \mathrm{fr}(t_0)} J^+(p)) \cap \{t = t_1\} = \partial J^+(\mathrm{fr}(t_0)) \cap \{t = t_1\}.$$

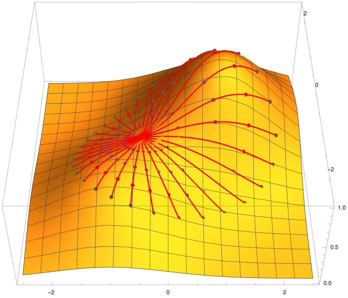
- The propagation is **anisotropic**:
  - Wind: V is greater downwind.
  - Slope: V is greater upwards.



Computation of the firefront

1. Find F such that Σ coincides with the velocities of the fire.
2. Solve the **geodesic equations** of $G = dt^2 - F^2$.
3. Projection of lightlike pregeodesics $\rightarrow$ **Fire trajectories**.
4. Detection of **cut points**:
   - The trajectory is no longer minimizing.
   - Danger zones for firefighters.





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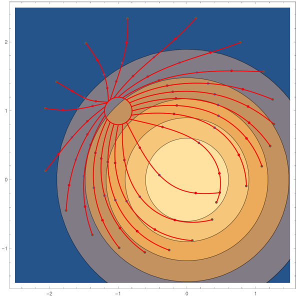
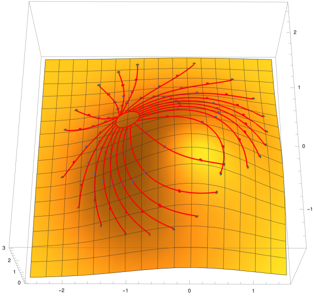


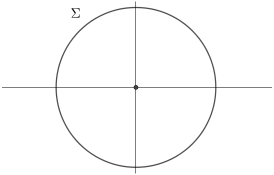


Advantages

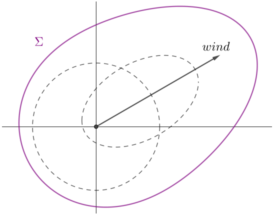
- **Efficiency:**
  - Fire simulators solve a **PDE system**.
  - Cut points are easier to handle.
- **Flexibility:**
  - Fire simulators use an **elliptical** approximation.
  - Σ can adopt any other shape.

Figure: Experimental results by Anderson, 1983.

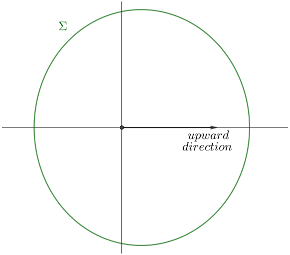
Model with wind and slope



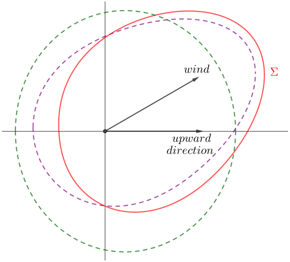
(a) Isotropic case



(b) Wind



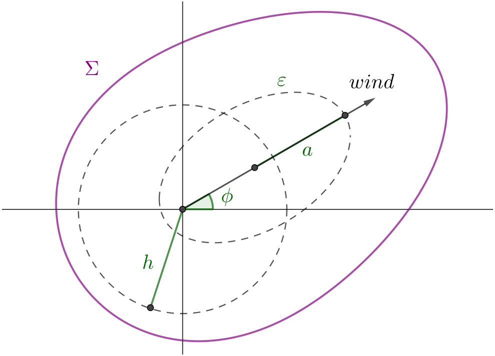
(c) Slope (Matsumoto)



(d) Wind + slope

Real-world application

- **Parameters** of the model $\longleftrightarrow$ **Physical** conditions.
- Wind case:
  - $h, a \longleftrightarrow$ Fuel properties.
  - $\varepsilon \longleftrightarrow$ Strength of the wind.
  - $\phi \longleftrightarrow$ Direction of the wind.

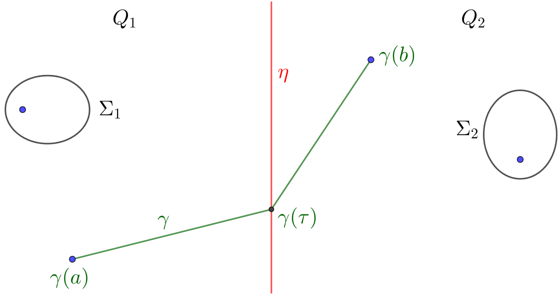


Snell's law

Refracted trajectories

- V **discontinuous** at η and time-independent.
- (Q_1, F_1), (Q_2, F_2) Finsler manifolds.
- **Traveltime** along γ, with $\gamma(\tau) \in \eta$:

$$T[\gamma] = \int_a^\tau F_{1\gamma(t)}(\gamma'(t))dt + \int_\tau^b F_{2\gamma(t)}(\gamma'(t))dt.$$



Generalized Snell's law

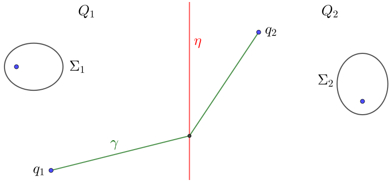
- $\mathcal{N} :=$ set of curves from $q_1 \in Q_1$ to $q_2 \in Q_2$ crossing η once.

Theorem

$\gamma \in \mathcal{N}$ is a **critical point** of $T_{\mathcal{N}}$ iff

- $\gamma|_{(a,\tau)}$ is an F_1-pregeodesic, $\gamma|_{(\tau,b)}$ is an F_2-pregeodesic.
- At $\gamma(\tau) \in \eta$, it satisfies **Snell's law**:

$$\left(\frac{\partial F_1}{\partial \dot{x}^i}(\gamma'(\tau^-)) - \frac{\partial F_2}{\partial \dot{x}^i}(\gamma'(\tau^+)) \right) u^i = 0, \qquad \forall u \in T_{\gamma(\tau)}\eta.$$



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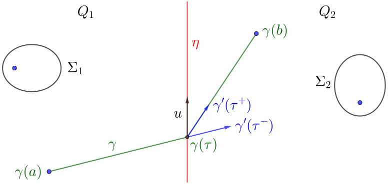


Generalized reflection law

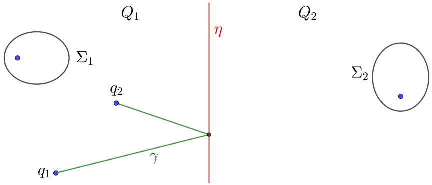
- $\mathcal{N}^* :=$ set of curves from q_1 to q_2 in Q_1, touching η once.

Theorem

$\gamma \in \mathcal{N}^*$ is a **critical point** of $T_{\mathcal{N}^*}$ iff

- $\gamma|_{(a,\tau)}$ and $\gamma|_{(\tau,b)}$ are F_1-pregeodesics.
- At $\gamma(\tau) \in \eta$, it satisfies the **reflection law**:

$$\left(\frac{\partial F_1}{\partial \dot{x}^i}(\gamma'(\tau^-)) - \frac{\partial F_1}{\partial \dot{x}^i}(\gamma'(\tau^+)) \right) u^i = 0, \qquad \forall u \in T_{\gamma(\tau)}\eta.$$



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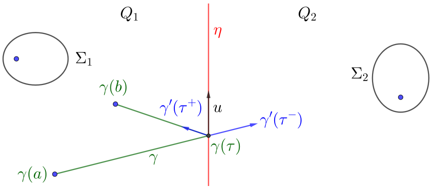


$\mathbb{R}^2$ with Minkowski norms

- $N = \mathbb{R}^2, \quad \eta = \{x = 0\}$.
- $F_i(x, y, \dot{x}, \dot{y}) = F_i(\dot{x}, \dot{y}) = \frac{\sqrt{\dot{x}^2+\dot{y}^2}}{V_i(\dot{x},\dot{y})}, \qquad i = 1, 2.$
- $\theta_1, \theta_2, \theta_3$ denote the angles of **incidence**, **refraction** and **reflection**, respectively (wrt the x-axis).

Proposition

In this setting, **Snell's law** and the **reflection law** become

$$P_1(\theta_1) = P_2(\theta_2), \qquad P_1(\theta_1) = P_1(\theta_3),$$

where P_i is the *raypath parameter*, defined as

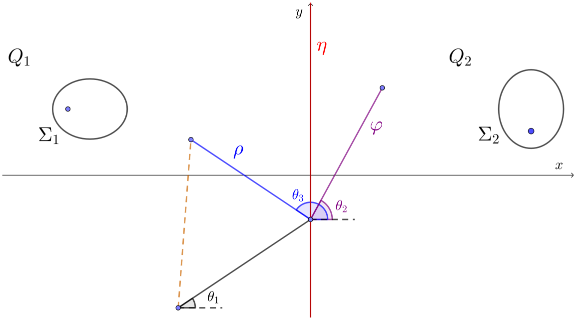
$$P_i(\theta) := \frac{\partial F_i}{\partial \dot{y}}(\theta) = \frac{\sin\theta}{V_i(\theta)} + \cos\theta \frac{\partial}{\partial\theta}\left(\frac{1}{V_i(\theta)}\right), \qquad i = 1, 2.$$

Global minimizers in $\mathbb{R}^2$

Theorem [Part 1]

In $\mathbb{R}^2$ with Minkowski norms and $\eta = \{x = 0\}$:

- Every **refracted trajectory** φ is globally minimizing.
- No **reflected trajectory** ρ is globally minimizing.

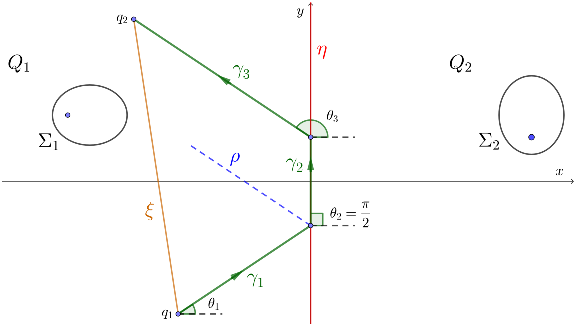


Global minimizers in $\mathbb{R}^2$

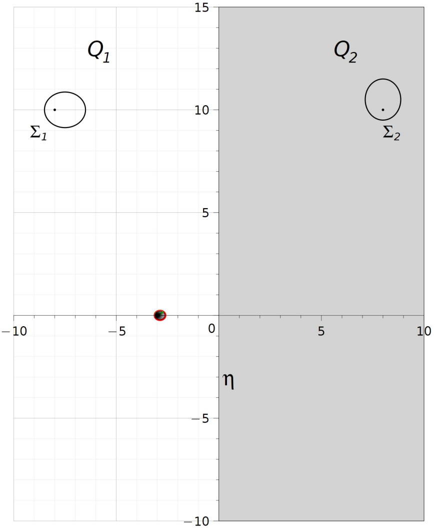
Theorem [Part 2]

Given $q_1, q_2 \in Q_1$, the global minimizer is either

- The **straight line** ξ joining q_1 and q_2, or
- The **three-segment trajectory** γ along η satisfying $P_1(\theta_1) = P_2(\theta_2) = P_1(\theta_3), \quad \theta_2 = \pm\frac{\pi}{2}$.



Wavefronts



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Thank you for your attention!