Causal fermion systems and octonions

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What is a causal fermion system?

- approach to fundamental physics
- novel mathematical model of spacetime
- physical equations are formulated in generalized spacetimes
- Different limiting cases:
  - Continuum limit: Quantized fermionic fields interacting via classical bosonic fields
  - QFT limit: fermionic and bosonic quantum fields (ongoing, more towards the end of the talk)
- For overview, more details (papers, books, videos, online course), applications to cosmology and black holes, . . .

www.causal-fermion-system.com

Which spacetime structures are fundamental?

- Clearly, there are personal preferences.
- Ultimately: Which formulation is capable to describe all known physical phenomena plus quantum gravity?

Which spacetime structures are fundamental?

- Hypothesis: Consider quantum mechanical wave functions as the basic physical objects (QFT later)
  - ψ describes quantum mechanical particle (only wave character, no point particle)
  - is physical reality even without measurements
  - the wave functions have a dynamics as described in the simplest case by Schrödinger equation (or Dirac equation, collapse model, ...)
- Vector ψ in a Hilbert space $(\mathcal{H}, \langle .|. \rangle_{\mathcal{H}})$.
- This is not quite the right description:
  - Phase has no significance: $\psi \to e^{i\Lambda} \psi$ instead of ψ consider ray generated by ψ
  - Local gauge invariance

$$\psi(t, \vec{x}) \to e^{i\Lambda(t,\vec{x})} \, \psi(t, \vec{x})$$

Therefore, only $|\psi(t, \vec{x})|^2$ is of physical significance; interpretation: probability density

Which spacetime structures are fundamental?

- Thus: Consider $|\psi(t,\vec{x})|^2$ of all quantum mechanical wave functions as the basic physical objects.
- General question: Suppose we know $|\psi(t,\vec{x})|^2$ for all the wave functions of the system, what can we say about the spacetime structures (causality, metric, fields, ...)
- Try to probe spacetime by looking at $|\psi(t,\vec{x})|^2$.

  Here "probing" should be thought of as a mathematical operation; no collapse of the wave function involved.

Which spacetime structures are fundamental?

- Begin in Minkowski space (usual spacetime structures)

$$x = (t, \vec{x}), \qquad t \in \mathbb{R}, \; \vec{x} \in \mathbb{R}^3$$

(curved spacetime works similarly)

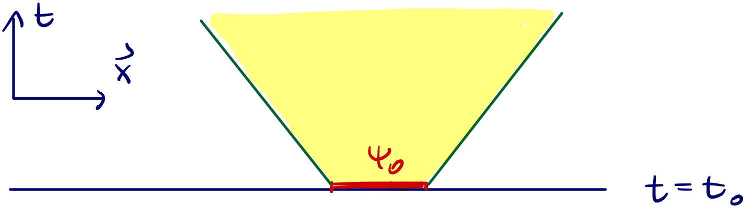
- Consider scalar particle (no spin)

$$|\psi(x)|^2 \qquad \text{(local density)}$$

Which spacetime structures are fundamental?

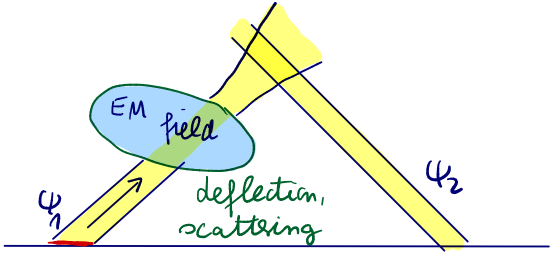
First step: Allow for preparation of the "initial state" at time t.

- Allows for detecting the causal structure of spacetime:



- Allows for detecting an electromagnetic field:

$$|\psi_1 + e^{i\alpha}\psi_2|^2 - |\psi_1|^2 - |\psi_2|^2$$

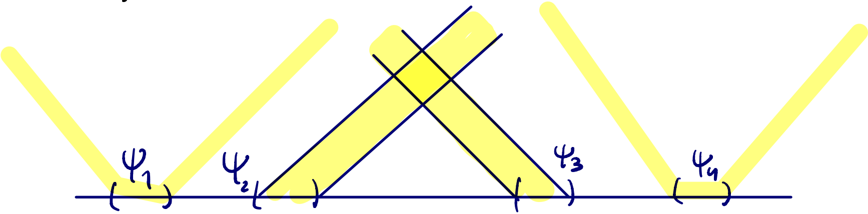


- . . . , . . .

Which spacetime structures are fundamental?

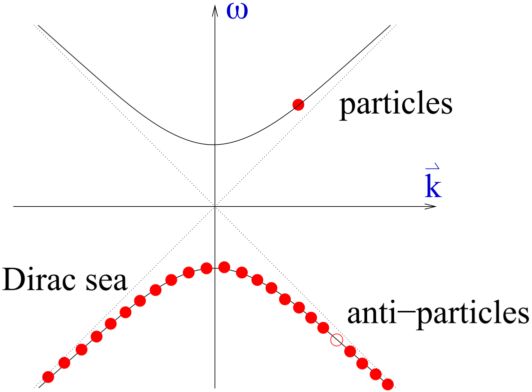
Second step: Do not allow for preparation of the “initial state”.
Instead: Get by with the wave function already present.

- Probing still works, provided that there are “sufficiently many” wave functions around.



- The more wave functions there are, the more information we have on spacetime (spacetime resolution, bandwidth, …)

Physical picture: Dirac's hole theory



Formalize this idea: The local correlation operator

- Consider wave functions $\psi_1, \ldots, \psi_f : \mathscr{M} \to \mathbb{C}$ (with $f < \infty$)
- Are vectors in a Hilbert space, orthonormalize,

$$\langle \psi_k | \psi_l \rangle = \delta_{kl} \,,$$

gives f-dim Hilbert space $(\mathcal{H}, \langle .|. \rangle_{\mathcal{H}})$.

basic object: for any lattice point x introduce

local correlation operator $F(x) : \mathcal{H} \to \mathcal{H}$

- define matrix elements by

$$(F(x))^j_k = \overline{\psi_j(x)} \psi_k(x)$$

basis invariant:

$$\langle \psi, F(x)\, \phi \rangle_{\mathcal{H}} = \overline{\psi(x)} \phi(x) \qquad \text{for all } \psi, \phi \in \mathcal{H}$$

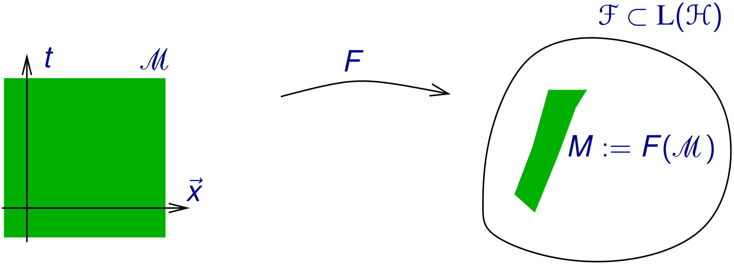
- Hermitian matrix = symmetric operator
- Has rank at most one, is positive semi-definite

$$F(x) = e^* e \qquad \text{with} \qquad e : \mathcal{H} \to \mathbb{C} \,, \quad \psi \mapsto \psi(x)$$

The local correlation map

$$F(x) \;\in\; \mathcal{F} := \big\{ F \text{ rank at most one, positive semi-definite} \big\}$$

We obtain mapping $\qquad x \mapsto F(x) \in \mathcal{F} \subset \mathrm{L}(\mathcal{H})$



- The right side contains all the information which can be retrieved from the wave functions.

The local correlation map

$$F(x) \;\in\; \mathfrak{F} := \big\{ F \text{ rank at most one, positive semi-definite} \big\}$$

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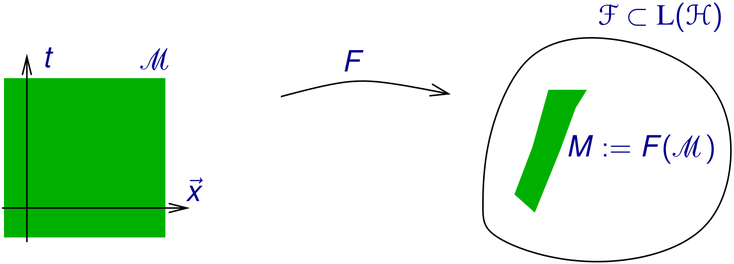
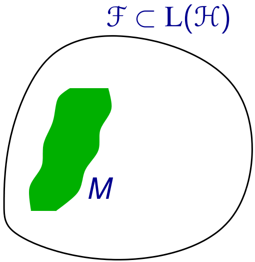


- The right side contains all the information which can be retrieved from the wave functions.
- We consider the objects on the right as the basic physical objects.

Spacetime as the set of all local correlation operators

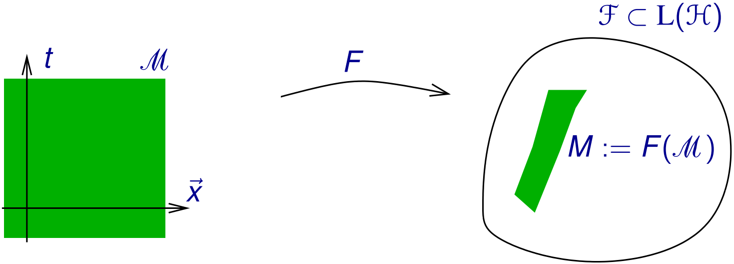
general strategy:

- disregard objects on the left
  (Minkowski space, causal structure, Dirac spinors, . . .)
- work instead exclusively with the objects on the right
  (only local correlation operators)



A volume measure on spacetime

- One important structure is missing: Volume measure on spacetime.

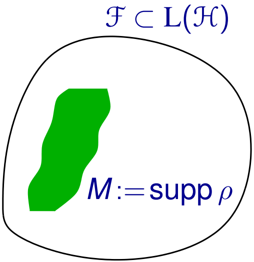


Take push-forward measure of $F : \mathscr{M} \to \mathcal{F}$,

$$\rho := F_*(\mu_{\mathscr{M}}) \qquad \text{(i.e. } \rho(\Omega) := \mu_{\mathscr{M}}\big(F^{-1}(\Omega)\big)\text{)}$$

(where $d\mu_{\mathscr{M}} = d^4x$)

A volume measure on spacetime



- image of F (more precisely, its closure) recovered as the support of the measure,

$$M := \text{supp}\ \rho = \big\{ F \in \mathcal{F} \mid \rho(\Omega) \neq 0 \\ \text{for every open neighborhood } \Omega \text{ of } x \big\}$$

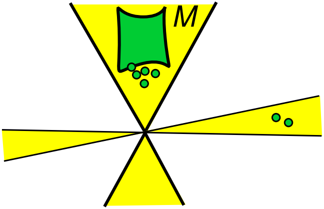
Causal Fermion Systems

Definition (Causal fermion system)

Let $(\mathcal{H}, \langle .|. \rangle_{\mathcal{H}})$ be Hilbert space
Given parameter $n \in \mathbb{N}$ ("spin dimension")
$\mathcal{F} := \Big\{ x \in \mathrm{L}(\mathcal{H})$ with the properties:

- x is symmetric and has finite rank
- x has at most n positive
  and at most n negative eigenvalues $\Big\}$

ρ a measure on $\mathcal{F}$



Example: Dirac spinors in Minkowski space

Space-time is Minkowski space, signature $(+---)$

- free Dirac equation $\quad (i\gamma^k \partial_k - m)\,\psi = 0$
- spin inner product $\prec\phi|\psi\succ = \overline{\phi}\psi$ with $\overline{\phi} := \phi^\dagger \gamma^0$, is indefinite of signature $(2,2)$
- probability density $\psi^\dagger \psi = \prec\psi\,|\,\gamma^0 \psi\succ$,

  gives rise to a scalar product:

$$\langle\psi|\phi\rangle = \int_{t=\text{const}} \prec\psi\,|\,\gamma^0 \phi\succ(t,\vec{x})\,d\vec{x}$$

time independent due to current conservation

Example: Dirac spinors in Minkowski space

- Choose $\mathcal{H}$ as a subspace of the solution space,

$$\mathcal{H} = \overline{\text{span}(\psi_1, \ldots, \psi_f)}$$

- To $x \in \mathbb{R}^4$ associate a local correlation operator

$$F(x)^i_j = -\prec \psi_i(x) | \psi_j(x) \succ \qquad \text{in ONB } (\psi_i)_{i=1,\ldots,f}$$
$$\langle \psi | F(x) \phi \rangle = -\prec \psi(x) | \phi(x) \succ \qquad \forall \psi, \phi \in \mathcal{H}$$

  Is symmetric, rank ≤ 4
  at most two positive and at most two negative eigenvalues

- Here ultraviolet regularization may be necessary:

$$\langle \psi | F(x) \phi \rangle = -\prec (\mathfrak{R}_\varepsilon \psi)(x) | (\mathfrak{R}_\varepsilon \phi)(x) \succ \qquad \forall \psi, \phi \in \mathcal{H}$$

$$\mathfrak{R}_\varepsilon \;:\; \mathcal{H} \to C^0(\mathscr{M}, S\mathscr{M}) \qquad \text{regularization operators}$$

$$\varepsilon > 0 : \text{regularization scale (Planck length)}$$

Example: Dirac spinors in Minkowski space

- Thus $F(x) \in \mathcal{F}$ where

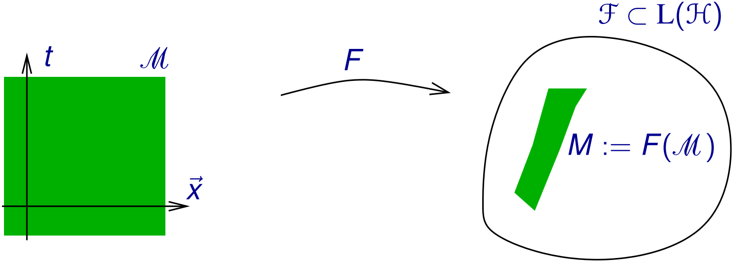
  $$\mathcal{F} := \Big\{ F \in \mathrm{L}(\mathcal{H}) \text{ with the properties:}$$

  - ▷ F is symmetric and has rank ≤ 4
  - ▷ F has at most 2 positive

    and at most 2 negative eigenvalues $\Big\}$

Example: Dirac spinors in Minkowski space

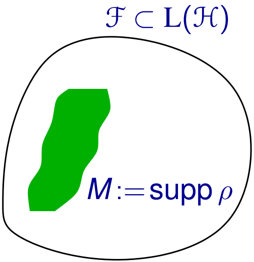
We obtain mapping $\qquad x \mapsto F(x) \in \mathcal{F} \subset \mathrm{L}(\mathcal{H})$



Take push-forward measure

$$\rho := F_*(\mu_{\mathscr{M}}) \qquad \text{(i.e. } \rho(\Omega) := \mu_{\mathscr{M}}\big(F^{-1}(\Omega)\big)\text{)}$$

Example: Dirac spinors in Lorentzian space-time



We thus obtain a causal fermion system of spin dimension two.

- Remark: This construction works similarly in curved spacetimes
  (i.e. globally hyperbolic Lorentzian spin manifold)

The causal action principle

- How can one formulate physical equations in this setting?
  - Intuitive picture: wave functions "organize themselves" in such a way that the Dirac sea configuration is a minimizer.
  - In interacting situation the wave functions organize to solutions of the Dirac equation

$$\big(i\gamma^j \partial_j + e\gamma^j A_j(x) - m\big)\psi = 0$$

This should serve as the definition of A.

Causal action principle

Let $x, y \in \mathcal{F}$. Then x and y are linear operators.

$x \cdot y \in \mathrm{L}(H)$:

- rank $\leq 2n$
- in general not symmetric: $(x \cdot y)^* = y \cdot x \neq x \cdot y$

thus non-trivial complex eigenvalues $\lambda_1^{xy}, \ldots, \lambda_{2n}^{xy}$

Causal action principle

Nontrivial eigenvalues of xy: $\lambda_1^{xy}, \ldots, \lambda_{2n}^{xy} \in \mathbb{C}$

Lagrangian $\quad \mathcal{L}(x,y) = \frac{1}{4n} \sum_{i,j=1}^{2n} \left(|\lambda_i^{xy}| - |\lambda_j^{xy}| \right)^2 \;\geq\; 0$

action $\quad \mathcal{S} = \iint_{\mathcal{F}\times\mathcal{F}} \mathcal{L}(x,y)\, d\rho(x)\, d\rho(y) \;\in\; [0,\infty]$

Minimize $\mathcal{S}$ under variations of ρ, with constraints

volume constraint: $\quad \rho(\mathcal{F}) = \text{const}$

trace constraint: $\quad \int_{\mathcal{F}} \text{tr}(x)\, d\rho(x) = \text{const}$

boundedness constraint: $\quad \iint_{\mathcal{F}\times\mathcal{F}} \sum_{i=1}^{2n} |\lambda_i^{xy}|^2\, d\rho(x)\, d\rho(y) \leq C$

- F.F., "Causal variational principles on measure spaces," *J. Reine Angew. Math.* **646** (2010) 141–194

A few general remarks

One basic object: measure ρ on set $\mathcal{F}$ of linear operators on $\mathcal{H}$, describes spacetime as well as all objects therein

- Underlying structure: family of fermionic wave functions
- Geometric structures encoded in these wave functions

Matter encodes geometry
Quantum spacetime

- Causal action principle describes spacetime as a whole (similar to Einstein-Hilbert action in GR)
- Causal action principle is a nonlinear variational principle (similar to Einstein-Hilbert action or classical field theory)
- Linear dynamics of quantum theory recovered in limiting case

Comparison with non-commutative geometry

- Similarities between NCG and causal fermion systems:
  - spacetime emerges from more fundamental structures
- Differences of causal fermion systems:
  - instead of the Dirac operator, the wave functions are the central objedts (this is more general, because Dirac operator no longer needs to exist)
  - the setup is Lorentzian
  - from technical point of view, causal action resembles a Lorentzian version of the spectral action;
    heat kernel expansion is replaced by light-cone expansion

Interpretation in terms of spacetime events

- operators in $\mathcal{F}$ can be interpreted as "possible local correlation operators"
  or simply as possible events
- operators in M are the events realized in spacetime
- spacetime is made up of all the realized events
- the physical equations relate the events to each other

For details on this connection:

- F.F, J. Fröhlich, C. Paganini, C. and M. Oppio,
  "Causal fermion systems and the ETH approach to quantum theory,"
  arXiv:2004.11785 [math-ph] (2020)

Inherent structures of a causal fermion system

Let $(\rho, \mathcal{F}, \mathcal{H})$ be a causal fermion of spin dimension n, spacetime $M := \text{supp}\,\rho$.

spacetime points are linear operators on $\mathcal{H}$

- For $x \in M$, consider eigenspaces of x.
- For $x, y \in M$,
  - consider operator products xy
  - project eigenspaces of x to eigenspaces of y

Gives rise to:

- quantum objects (spinors, wave functions)
- geometric structures (connection, curvature)
- causal structure, analytic structures

Causal structure

Let $x, y \in M$. Then
$x \cdot y \in \mathrm{L}(H)$ has non-trivial complex eigenvalues $\lambda_1^{xy}, \ldots, \lambda_{2n}^{xy}$

Definition (causal structure)

The points $x, y \in \mathcal{F}$ are called

$$\begin{cases} \text{spacelike separated} & \text{if } |\lambda_j^{xy}| = |\lambda_k^{xy}| \text{ for all } j, k = 1, \ldots, 2n \\ \text{timelike separated} & \text{if } \lambda_1^{xy}, \ldots, \lambda_{2n}^{xy} \text{ are all real} \\ & \text{and } |\lambda_j^{xy}| \neq |\lambda_k^{xy}| \text{ for some } j, k \\ \text{lightlike separated} & \text{otherwise} \end{cases}$$

- Lagrangian is compatible with causal structure:

$$\text{Lagrangian} \quad \mathcal{L}(x, y) = \frac{1}{4n} \sum_{i,j=1}^{2n} \left(|\lambda_i^{xy}| - |\lambda_j^{xy}| \right)^2 \geq 0$$

thus x, y spacelike separated $\quad \Rightarrow \quad \mathcal{L}(x, y) = 0$

"points with spacelike separation do not interact"

A distinguished time direction

$$x(\mathcal{H}) \subset \mathcal{H} \qquad \text{subspace of dimension } \leq 2n$$

Introduce the functional

$$\mathcal{C} \,:\, M \times M \to \mathbb{R}\,, \qquad \mathcal{C}(x,y) := i \,\mathrm{tr}\big(y\,x\,\pi_y\,\pi_x - x\,y\,\pi_x\,\pi_y\big)$$

For timelike separated points $x, y \in M$,

$$\begin{cases} y \text{ likes in the future of } x & \text{if } \mathcal{C}(x,y) > 0 \\ y \text{ likes in the past of } x & \text{if } \mathcal{C}(x,y) < 0 \end{cases}$$

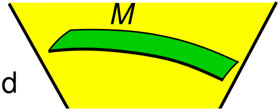
- The resulting relation “lies in the future of” is not necessarily transitive.

Quantum spacetimes

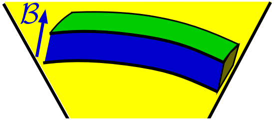
General question: How does an interacting measure look like?

- In more mathematical terms: What is the structure of minimizing measures?

a classical spacetime:
M diffeomorphic to manifold



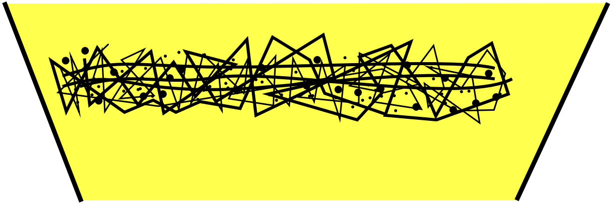
a quantum spacetime:
$M \simeq \mathscr{M} \times \mathcal{B}$



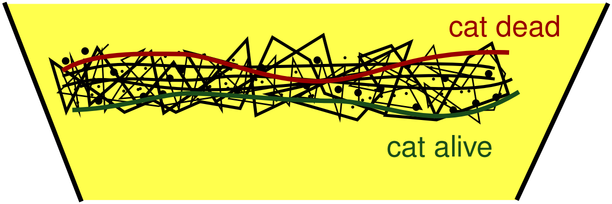
- integrating over additional “degrees of freedom” $\mathcal{B}$ resembles path integral

Quantum spacetimes

Complicated non-smooth structure expected:



This accounts for macroscopic superpositions and entanglement:



Quantum spacetimes

- Quantum field theory limit is work in progress. So far:
  - Construction of quantum state at any time t
  - Proof of general entanglement
  - Next step: Dynamics of quantum state.

- F.F., N. Kamran, “Complex structures on jet spaces and bosonic Fock space dynamics for causal variational principles,” arXiv:1808.03177 [math-ph], *Pure Appl. Math. Q.* **17** (2021) 55–140
- F.F. and Kamran, N., “Fermionic Fock spaces and quantum states for causal fermion systems,” arXiv:2101.10793 [math-ph], *Ann. Henri Poincaré* **23** (2022) 1359–1398
- F.F., Kamran, N. and Reintjes, M., “Entangled quantum states of causal fermion systems and unitary group integrals,” arXiv:2207.13157 [math-ph]

The continuum limit

Causal fermion system

- abstract mathematical framework
- quantum geometry, causal action

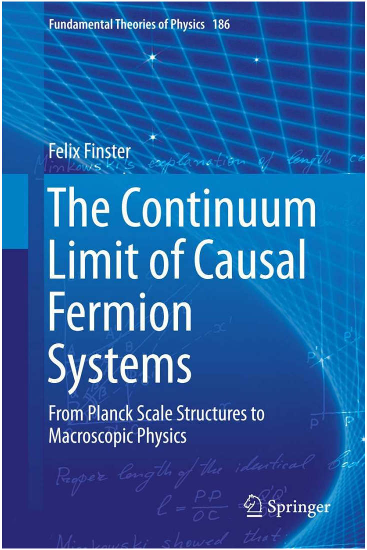
$$\Big\Downarrow \quad \text{continuum limit}$$

description in the continuum limit

- Dirac fields
- strong and electroweak gauge fields
- gravitational field

- fermion field: second-quantized
- bosonic field: classical

The causal action principle in the continuum limit



Fundamental Theories of Physics **186**

Springer, 2016

548+xi pages

arXiv:1605.04742 [math-ph]

The causal action principle in the continuum limit

- specify vacuum as sum of Dirac seas,

$$P(x,y) = \sum_{\beta=1}^{g} P^{\text{sea}}_{m_\beta}(x,y)$$

$$P^{\text{sea}}_{m_\beta}(x,y) = \int \frac{d^4k}{(2\pi)^4}\, (\slashed{k} + m_\beta)\, \delta(k^2 - m_\beta^2)\, \Theta(-k^0)\, e^{-ik(x-y)}$$

β labels “generations” of elementary particles

The causal action principle in the continuum limit

- Model involving neutrinos and quarks:

$$P(x,y) = \sum_{\beta=1}^{3} \underbrace{P^{\text{sea}}_{m_\beta}(x,y) \oplus \cdots \oplus P^{\text{sea}}_{m_\beta}(x,y)}_{\text{7 identical direct summands}} \oplus P^{\text{sea}}_{\tilde{m}_\beta}(x,y)$$

  - again three generations
  - $4 \times 8 = 32$-component wave functions
  - spin dimension 16
- Regularize on the scale ε (Planck scale), regularization of neutrinos breaks chiral symmetry

The causal action principle in the continuum limit

Results:

- The direct summands form pairs, spontaneous block formation
- gauge group $\mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3)$
- coupling to the fermions exactly as in the standard model:
  - U_{CKM}, U_{MNS}-mixing matrices
  - $\mathrm{SU}(2)$ gauge fields left-handed and massive
- Einstein equations

The causal action principle in the continuum limit

Remarks on methods for analyzing the continuum limit:

- Consider the Dirac equation in an external potential

$$(i\partial\!\!\!/ + \mathcal{B} - mY)\psi = 0\,.$$

- Question: Are the EL equations of causal action principle satisfied in the limit $\varepsilon \searrow 0$?
- Answer: Yes, if and only if $\mathcal{B}$ has a certain structure and satisfies the classical field equations.

Connection to octonions

- Octonions: $\mathbb{O} = \{e_0, \ldots, e_7\}$
- form a non-associative algebra; on the other hand: operator algebras are associative
- Form complex octonions

$$\mathbb{C} \otimes \mathbb{O} \ \ni \ a = \sum_{i=0}^{7} c_n e_n$$

- Form chains $a(b(c \cdots)))$.
  This gives an associative algebra, the so-called complex octonionic chain algebra $\mathbb{C} \otimes \overleftarrow{\mathbb{O}}$.
  - is associative
  - $\mathbb{O} \times (\mathbb{C} \otimes \overleftarrow{\mathbb{O}}) \to \mathbb{C} \otimes \overleftarrow{\mathbb{O}}$ action of octonions on algebra

Connection to octonions

- Let the octonions act on the vacuum measure:

$$P(x,y) = \sum_{\beta=1}^{3} \underbrace{P^{\text{sea}}_{m_\beta}(x,y) \oplus \cdots \oplus P^{\text{sea}}_{m_\beta}(x,y)}_{\text{7 identical direct summands}} \oplus P^{\text{sea}}_{\tilde{m}_\beta}(x,y)$$

  - eight direct summands; identify with $\mathbb{C} \otimes \overleftarrow{\mathbb{O}}$
  - use above action $\mathbb{O} \times (\mathbb{C} \otimes \overleftarrow{\mathbb{O}}) \to \mathbb{C} \otimes \overleftarrow{\mathbb{O}}$
- Is an approximate symmetry transformation.

Connection to octonions

- We have three generations. For interacting systems, they give rise to direct sums of sectors,

$$P^{\text{aux}}(x,y) = \bigoplus_{\beta=1}^{3} \underbrace{P^{\text{sea}}_{m_\beta}(x,y) \oplus \cdots \oplus P^{\text{sea}}_{m_\beta}(x,y)}_{\text{7 identical direct summands}} \oplus P^{\text{sea}}_{\tilde{m}_\beta}(x,y)$$

$$\bigoplus_{\beta=1}^{3} P^{\text{sea}}_{m_\beta} = \begin{pmatrix} P^{\text{sea}}_{m_1} & 0 & 0 \\ 0 & P^{\text{sea}}_{m_2} & 0 \\ 0 & 0 & P^{\text{sea}}_{m_3} \end{pmatrix}$$

- This leads to 3×3-matrices with octonionic entries. Exceptional Jordan algebras.
- Possibly much more . . .

www.causal-fermion-system.com

Thank you for your attention!

Underlying physical principles

- local gauge principle:
  freedom to perform local unitary transformations of the spinors
- Pauli exclusion principle:
  Choose orthonormal basis $\psi_1, \ldots, \psi_f$ of $\mathcal{H}$. Set

  $$\Psi = \psi_1 \wedge \cdots \wedge \psi_f \,,$$

  gives equivalent description by Hartree-Fock state.
- the "equivalence principle":
  symmetry under "diffeomorphisms" of M
  (note: M merely is a topological measure space)

Spacetime and causal structure are emergent

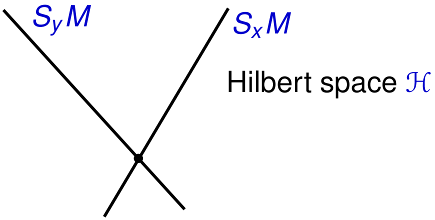
Inherent structures of a causal fermion system

- Spinors

$$S_xM := x(\mathcal{H}) \subset \mathcal{H} \qquad \text{“spin space”}, \;\dim S_xM \leq 2n$$

$$\prec u|v \succ_x := -\langle u \,|\, x\, v\rangle_{\mathcal{H}} \qquad \text{“spin scalar product”},$$

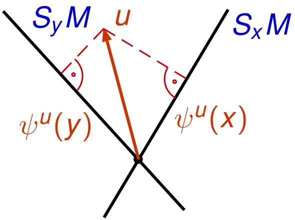
inner product of signature $(\leq n, \leq n)$



- Physical wave functions

$$\psi^u(x) = \pi_x\, u \;\text{ with } u \in \mathcal{H} \qquad \text{physical wave function}$$

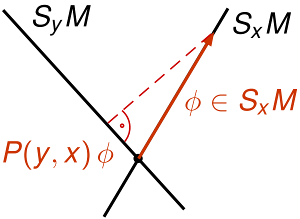
$$\pi_x : \mathcal{H} \to \mathcal{H} \qquad \text{orthogonal projection on } x(\mathcal{H})$$



Inherent structures in spacetime

- The kernel of the fermionic projector:

$$P(y,x) = \pi_y \, x|_{S_xM} \; : \; S_xM \to S_yM$$



$$P(y,x) = -\sum_{i=1}^{f} |\psi^{e_i}(y) \succ \prec \psi^{e_i}(x)| \qquad \text{where } (e_i) \text{ ONB of } \mathcal{H}$$

Inherent structures in spacetime

- Geometric structures
  - $P(x,y) \;:\; S_yM \to S_xM$ yields relations between spin spaces.
    Using a polar decomposition $(\ldots, \ldots)$ one gets:

    $$D_{x,y} \;:\; S_yM \to S_xM \text{ unitary} \qquad \text{“spin connection”}$$

  - tangent space T_x, carries Lorentzian metric,

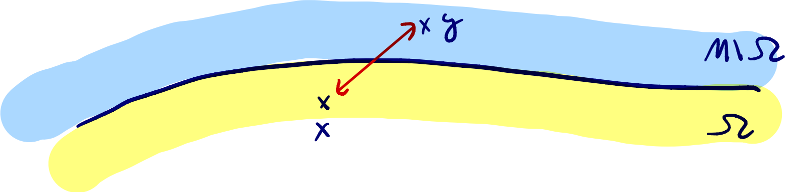
    $$\nabla_{x,y} \;:\; T_y \to T_x \qquad \text{corresponding “metric connection”}$$

  - holonomy of connection gives curvature

    $$R(x,y,z) = \nabla_{x,y} \nabla_{y,z} \nabla_{z,x} \;:\; T_x \to T_x$$

Objects in space

- In our daily experience: objects in space (densities, ...), spatial integrals (integral over densities, ...).
- Arise in causal fermion systems as surface layer integrals,



$$\int_\Omega d\rho(x) \int_{M\setminus\Omega} d\rho(y)\, (\cdots)\, \mathcal{L}(x,y)$$

- F.F., J. Kleiner, "Noether-like theorems for causal variational principles," arXiv:1506.09076 [math-ph], *Calc. Var. Partial Differential Equations* **55:35** (2016)
- F.F., J. Kleiner, "A class of conserved surface layer integrals for causal variational principles," arXiv:1801.08715 [math-ph], *Calc. Var. Partial Differential Equations* **58:38** (2019)