Quasi-Local Mass, Scalar Curvature and a Positive Mass Theorem for Causal Variational Principles

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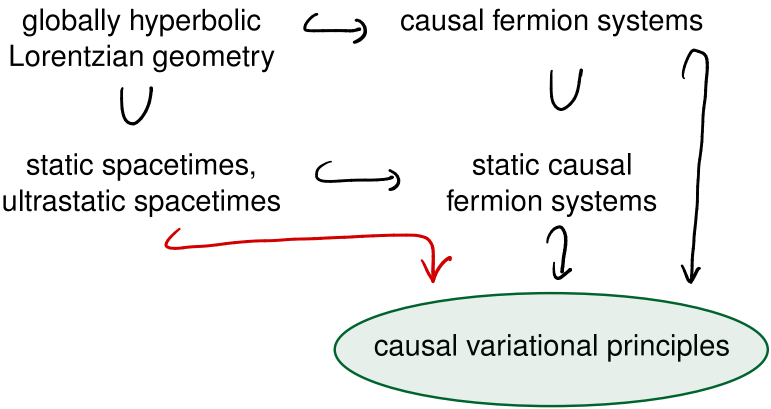
Overview



- Begin with setting of causal variational principles
- The red arrow will be explained at the end of the talk

General setup of causal variational principles

- Let $\mathcal{G}$ be a manifold of $\dim \mathcal{G} \gg 4$
  (or locally compact topological space, no smoothness!)
- Lagrangian $\mathcal{L} : \mathcal{G} \times \mathcal{G} \to \mathbb{R}_0^+$
  - continuous (or lower semi-continuous)
  - $\mathcal{L}(x, y) = \mathcal{L}(y, x)$ (symmetric)
  - $\mathcal{L}(x, x) > 0$ for all $x \in \mathcal{G}$ (strictly positive on diagonal)
- Let μ be a regular Borel measure (= Radon measure) on $\mathcal{G}$

action $$\mathcal{S}(\mu) := \int_{\mathcal{G}} d\mu(x) \int_{\mathcal{G}} d\mu(y)\, \mathcal{L}(x, y)$$

Causal variational principle

Minimize $\mathcal{S}$ under variations of μ,
keeping the total volume $\mu(\mathcal{G})$ fixed.

How to keep the total volume fixed if $\mu(\mathcal{G}) = \infty$?

General setup of causal variational principles

Definition (minimizer under variations of finite volume)

The measure μ is a *minimizer under variations of finite volume* if for all $\hat{\mu}$ with

$$|\mu - \hat{\mu}| < \infty \quad \text{and} \quad (\mu - \hat{\mu})(\mathcal{G}) = 0 \,,$$

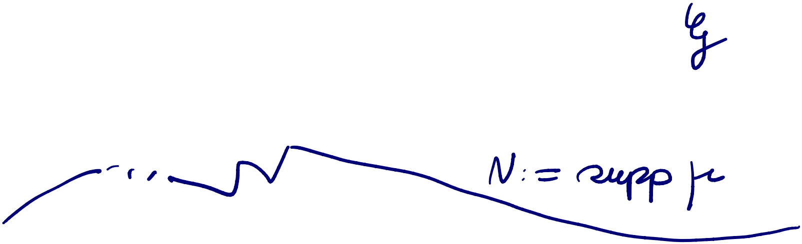
the difference of actions is non-negative, $\mathcal{S}(\hat{\mu}) - \mathcal{S}(\mu) \geq 0$.

$$\begin{aligned} 0 &= \mathcal{S}(\hat{\mu}) - \mathcal{S}(\mu) = \mathcal{S}\big(\mu + (\hat{\mu} - \mu)\big) - \mathcal{S}(\mu) \\ &:= 2 \int_{\mathcal{G}} d(\hat{\mu} - \mu)(x) \int_{\mathcal{G}} d\mu(y)\, \mathcal{L}(x,y) \\ &\quad + \int_{\mathcal{G}} d(\hat{\mu} - \mu)(x) \int_{\mathcal{G}} d(\hat{\mu} - \mu)(y)\, \mathcal{L}(x,y) \end{aligned}$$

- Minimizers exist under general assumptions
  F.F, C. Langer, "Causal Variational Principles in the σ-Locally Compact Setting: Existence of Minimizers," arXiv:2002.04412 [math-ph], Adv. Calc. Var. **15** (2022) 551–575

General setup of causal variational principles

Typical example:



- $N := \operatorname{supp} \mu$ space (no smoothness!)
  Typically, N is contained in a low-dimensional subset of $\mathcal{G}$.

General setup of causal variational principles

What does minimality mean?

- Let μ be a minimizing measure, $N := \text{supp}\, \mu$
- Choose $\Omega \subset N$ of finite volume ($\mu(\Omega) < \infty$).
- Choose $x \in \mathcal{G}$.
- For $\tau \in [0, 1]$ consider family

$$\hat{\mu}_\tau := \chi_{N \setminus \Omega}\, \mu + (1 - \tau)\, \chi_\Omega\, \mu + \tau\, \mu(\Omega)\, \delta_x$$

  satisfies the volume constraint
- Therefore,

$$\begin{aligned} & 0 \leq \mathcal{S}(\hat{\mu}_\tau) - \mathcal{S}(\mu) \qquad \text{for all } \tau \\ \Longrightarrow \quad & 0 \leq \frac{d}{d\tau} \Big(\mathcal{S}(\hat{\mu}_\tau) - \mathcal{S}(\mu) \Big) \Big|_{\tau=0} \end{aligned}$$

General setup of causal variational principles

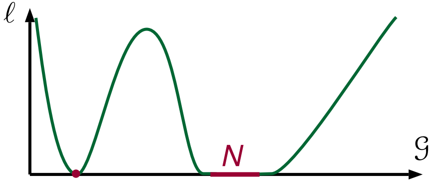
Working this out gives the following result:

$$\ell(x) := \int_{\mathcal{G}} \mathcal{L}(x,y)\, d\mu(y) - \mathfrak{s}$$

Lemma (The Euler-Lagrange equations)

Let μ be a minimizer of the causal action. Then for suitable $\mathfrak{s} > 0$,

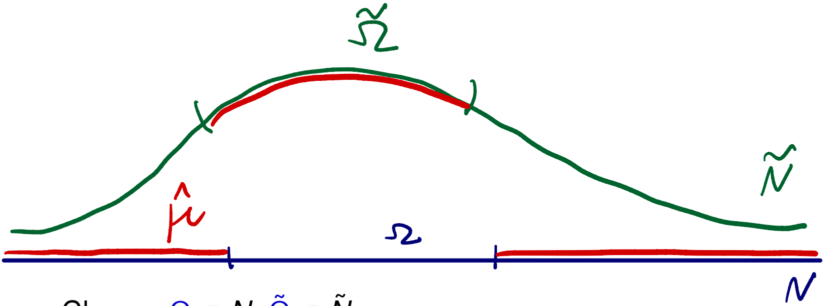
$$\ell|_M \equiv \inf_{\mathcal{F}} \ell = 0$$



A nonlinear positive functional

Ongoing work with Niky Kamran (McGill, Montréal)

- Let μ again be minimizer (vacuum), $N := \text{supp}\, \mu$
- Let $\tilde{\mu}$ be critical (interacting or curved space), $\tilde{N} := \text{supp}\, \tilde{\mu}$



- Choose $\Omega \subset N$, $\tilde{\Omega} \subset \tilde{N}$, of the same volume: $\mu(\Omega) = \tilde{\mu}(\tilde{\Omega})$
- "Take out" Ω and "glue in" $\tilde{\Omega}$ into N,

$$\hat{\mu} := \chi_{N \setminus \Omega}\, \mu + \chi_{\tilde{\Omega}}\, \tilde{\mu}$$

A nonlinear positive functional

$$\hat{\mu} := \chi_{N \setminus \Omega} \mu + \chi_{\tilde{\Omega}} \tilde{\mu}$$

- satisfies volume constraint

Therefore,

$$\begin{aligned}
0 &\leq \mathcal{S}(\hat{\mu}) - \mathcal{S}(\mu) = \cdots = \\
&= 2 \int_{\tilde{\Omega}} d\tilde{\mu}(x) \int_{M \setminus \Omega} d\mu(y)\, \mathcal{L}(x,y) \\
&\quad - \int_{\tilde{\Omega}} d\tilde{\mu}(x) \int_{\tilde{M} \setminus \tilde{\Omega}} d\tilde{\mu}(y)\, \mathcal{L}(x,y) - \int_{\Omega} d\mu(x) \int_{M \setminus \Omega} d\mu(y)\, \mathcal{L}(x,y)
\end{aligned}$$

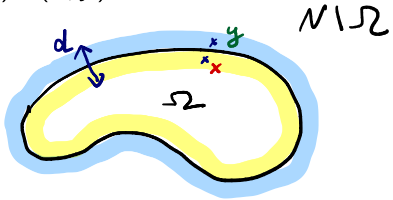
What does this mean?

A nonlinear positive functional

- $V := \mu(\Omega)$ volume
- $A := \int_\Omega d\mu(x) \int_{N \setminus \Omega} d\mu(y)\, \mathcal{L}(x,y)$ area

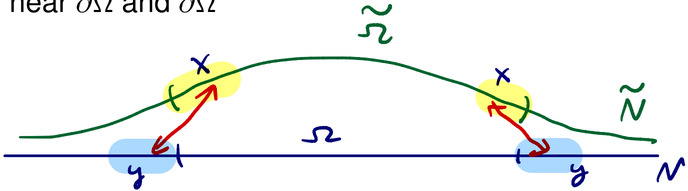
  surface layer integral

  $d \sim \frac{1}{m}$



- $\int_{\tilde{\Omega}} d\tilde{\mu}(x) \int_{M \setminus \Omega} d\mu(y)\, \mathcal{L}(x,y)$

  is a nonlinear surface layer integral comparing spacetimes near $\partial\Omega$ and $\partial\tilde{\Omega}$



More general positive functionals

$$V = \int_\Omega d\mu \qquad \text{volume}$$

$$A = \int_\Omega d\mu(x) \int_{N\setminus\Omega} d\mu(y)\, \mathcal{L}(x,y) \qquad \text{area}$$

- With volume constraint and $\tilde{\mu}$ critical:

$$2 \int_{\tilde{\Omega}} d\tilde{\mu}(x) \int_{N\setminus\Omega} d\mu(y)\, \mathcal{L}(x,y) - \tilde{A} - A \geq 0$$

- Remove volume constraint:

$$\mathfrak{M}(\tilde{\Omega},\Omega) := 2 \int_{\tilde{\Omega}} d\tilde{\mu}(x) \int_{N\setminus\Omega} d\mu(y) \mathcal{L}(x,y) - \tilde{A} - A - \mathfrak{s}\, \big(\tilde{V} - V\big) \geq 0$$

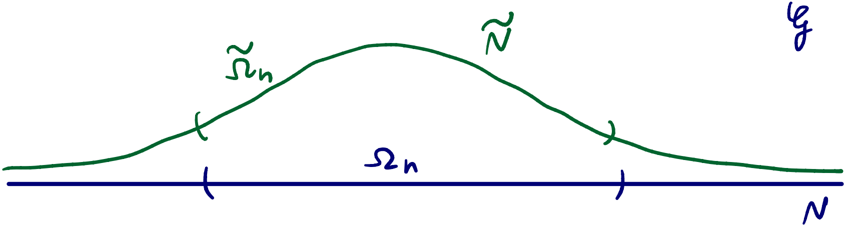
- Remove criticality of $\tilde{\mu}$:

$$\mathfrak{N}(\tilde{\Omega},\Omega) := \mathfrak{M}(\tilde{\Omega},\Omega) - \mathfrak{s}\, \tilde{V} + \int_{\tilde{\Omega}} d\tilde{\mu}(x) \int_{\tilde{N}} d\tilde{\mu}(y)\, \mathcal{L}(x,y) \geq 0$$

The total mass

- Let $(\Omega_n)_{n\in\mathbb{N}}$ be exhaustion of N by compact sets, $(\tilde{\Omega}_n)_{n\in\mathbb{N}}$ exhaustion of $\tilde{N}$ with

$$\mu(\Omega_n) = \tilde{\mu}(\tilde{\Omega}_n) \quad \text{for all } n$$



$$\mathfrak{M} := \lim_{n\to\infty} \mathfrak{M}(\tilde{\Omega}, \Omega)$$

This gives back the total mass introduced in

- F.F, A. Platzer, "A positive mass theorem for static causal fermion systems," arXiv:1912.12995 [math-ph], Adv. Theor. Math. Phys. **25** (2021) 1735–1818

The total mass

Theorem

$$\mathfrak{M} \geq 0$$

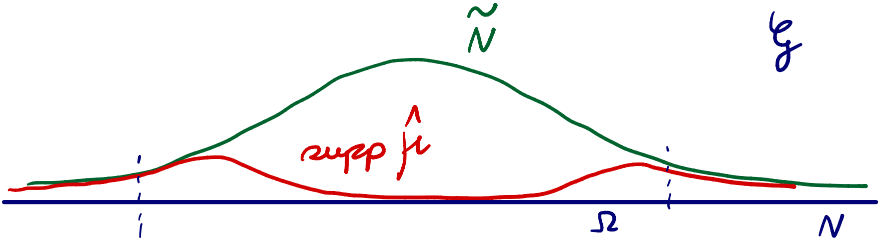
Thus we get the positive mass theorem without a local energy condition!

A positive mass theorem without volume constraint

- The condition

$$\mu(\Omega_n) = \tilde{\mu}(\tilde{\Omega}_n) \quad \text{for all } n$$

seems somewhat artificial. It can indeed be removed! Instead, Ω and $\tilde{\Omega}$ must be asymptotically aligned



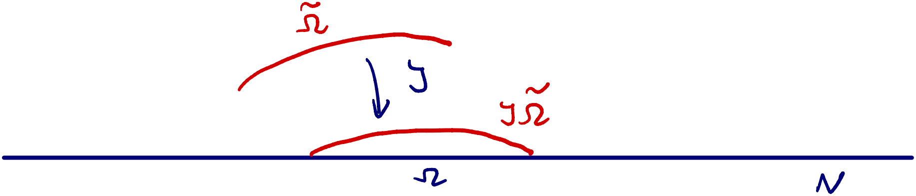
A quasi-local mass

Given $\tilde{\Omega} \subset \tilde{N}$ define

$$\mathfrak{M}(\tilde{\Omega}) := \inf_{\Omega \subset M} \inf_{I} \mathfrak{M}(\tilde{\Omega}, I\Omega)$$

with $I : \mathcal{G} \to \mathcal{G}$ isometry of the Lagrangian, i.e.

$$\mathcal{L}(x,y) = \mathcal{L}(Ix, Iy) \qquad \text{for all } x, y \in \mathcal{G}$$



Gives close connection to Brown-York mass

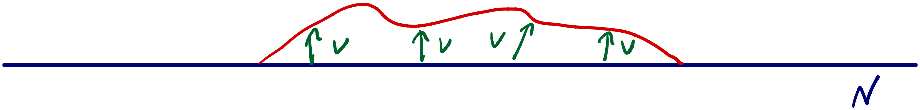
$$\text{choose} \quad \iota : \partial\tilde{\Omega} \hookrightarrow \partial\Omega \subset \mathbb{R}^3 \quad \text{isometric}$$

$$\mathfrak{M}_{\text{BY}} = \int_{\partial\tilde{\Omega}} \tilde{H}\, d\mu_{\partial\tilde{\Omega}} - \int_{\partial\Omega} H\, d\mu_{\partial\Omega}$$

- Brown-York (1993), Shi-Tam (2002), Liu-Wang-Yau, ...

Synthetic scalar curvature

- If $\tilde{\Omega}$ and Ω are "sufficiently close", we can linearize:



$$\int_{\tilde{\Omega}} g \, d\tilde{\mu}(x) \approx \int_{\Omega} (1 + \nabla_{\mathfrak{v}} g) \, d\mu(x)$$

$$\nabla_{\mathfrak{v}} g(x) = a(x)\, g(x) + \big(D_v g\big)$$

Then the so-called linearized field operator Δ comes into play

$$\mathfrak{M}(\tilde{\Omega}) = -\int_N \big(\Delta \mathfrak{v}\big)(x) \, d\mu(x)$$

$$\big(\Delta \mathfrak{v}\big)(x) := \int_N d\mu(y) \, \big(\nabla_{1,\mathfrak{v}} + \nabla_{2,\mathfrak{v}}\big)\mathcal{L}(x,y) - \nabla_{\mathfrak{v}} \, \mathfrak{s}$$

Therefore, define scalar curvature by

$$\mathrm{scal}(x) := -(\Delta \mathfrak{v})(x)\,.$$

Then, in the linearized setting, the quasi-local mass is given as in Newtonian gravity by

$$\mathfrak{M}(\tilde{\Omega}) = \int_N \mathrm{scal}(x)\, d\mu(x)$$

- The so-defined scalar curvature can be positive or negative.

From spin geometry to causal variational principles

Let $\mathscr{M}$ be a Lorentzian space-time,
for simplicity 4-dimensional, globally hyperbolic,
ultrastatic, i.e.

$$\mathscr{M} = \mathbb{R} \times \mathscr{N}$$

$$ds^2 = dt^2 - \sum_{\alpha,\beta=1}^{3} g_{\alpha\beta}\, dx^\alpha\, dx^\beta$$

with complete Riemannian metric g on $\mathscr{N}$
Then automatically spin,

$$(S\mathscr{M}, \prec.|.\succ) \qquad \text{spinor bundle}$$

- $S_p\mathscr{M} \simeq \mathbb{C}^4$
- spin inner product

$$\prec.|.\succ_p \;:\; S_p\mathscr{M} \times S_p\mathscr{M} \to \mathbb{C}$$

is indefinite of signature (2,2)

$$(\mathcal{D} - m)\,\psi_m = 0 \qquad \text{Dirac equation}$$

Dirac equation in Hamiltonian formulation,

$$i\partial_t \psi = H\psi$$

H self-adjoint operator on $\mathcal{H}_m := L^2(\mathcal{N}, S\mathcal{M})$

$$\psi(t,x) = e^{-itH}\,\psi(0,x)\,.$$

From spin geometry to causal variational principles

- Choose $\mathcal{H}$ as a subspace of all negative-frequency solutions,

$$\mathcal{H} = \mathrm{rg}\,\chi_{-\infty,0}(H) \subset \mathcal{H}_m$$

- introduce an ultraviolet regularization on scale ε (Planck length)

$$\mathfrak{R}_\varepsilon \;:\; \mathcal{H} \to C^0(\mathscr{M}, S\mathscr{M}) \qquad \text{regularization operators}$$

$$\text{for example} \qquad \mathfrak{R}_\varepsilon = e^{\varepsilon H}$$

- Define local correlation operator $F(t,x) \in \mathrm{L}(\mathcal{H})$ by

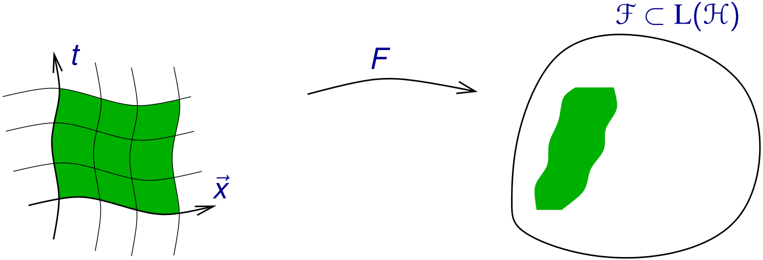
$$\langle \psi | F(t,x) \phi \rangle = - \prec (\mathfrak{R}_\varepsilon \psi)(t,x) | (\mathfrak{R}_\varepsilon \phi)(t,x) \succ \qquad \forall \psi, \phi \in \mathcal{H}$$

Is self-adjoint, rank ≤ 4
at most two positive and at most two negative eigenvalues

From spin geometry to causal variational principles

- Thus $F(t,x) \in \mathcal{F}$ where
  $\mathcal{F} := \Big\{ F \in \mathrm{L}(\mathcal{H})$ with the properties:
  - ▷ F is self-adjoint and has rank ≤ 4
  - ▷ F has at most 2 positive
    and at most 2 negative eigenvalues $\Big\}$

We obtain mapping $(t, x) \mapsto F(t, x) \in \mathcal{F} \subset \mathrm{L}(\mathcal{H})$



- push-forward measure $\rho := F_*(\mu_{\mathcal{M}})$, is measure on $\mathcal{F}$,

$$\rho(\Omega) := \mu_{\mathcal{M}}\big(F^{-1}(\Omega)\big)$$

- support of the measure is closure of image of F.

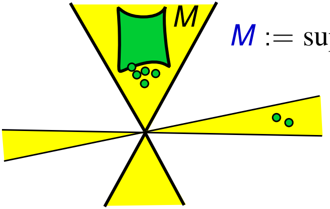
From spin geometry to causal variational principles

Definition (Causal fermion system)

Let $(\mathcal{H}, \langle .|. \rangle_{\mathcal{H}})$ be Hilbert space
Given parameter $n \in \mathbb{N}$ ("spin dimension")
$\mathcal{F} := \Big\{ x \in \mathrm{L}(\mathcal{H})$ with the properties:

- x is self-adjoint and has finite rank
- x has at most n positive and at most n negative eigenvalues $\Big\}$

ρ a measure on $\mathcal{F}$



$M := \operatorname{supp} \rho$ space-time

Starting from an ultrastatic spacetime,

$$\mathcal{U}_t := e^{-itH} \qquad \text{one-parameter group of symmetries}$$

$$F(t,x) = \mathcal{U}_t \, F(0,x) \, \mathcal{U}_t^{-1}$$

$$\rho\big(\mathcal{U}_t \, \Omega \, \mathcal{U}_t^{-1}\big) = \rho(\Omega)$$

$$M := \text{supp}\ \rho = \mathbb{R} \times N$$

$$d\rho = dt\, d\mu\,, \qquad N = \text{supp}\ \mu\,.$$

- define $\mathcal{G} := \mathcal{F}/\mathbb{R}$
- there is an explicitly given static Lagrangian

$$\mathcal{L} : \mathcal{G} \times \mathcal{G} \to \mathbb{R}_0^+$$

Let $X, Y \in \mathcal{F}$. Then X and Y are linear operators.

$X \cdot Y \in \mathrm{L}(H)$:

- rank ≤ 4
- in general not self-adjoint: $(X \cdot Y)^* = Y \cdot X \neq X \cdot Y$

thus non-trivial complex eigenvalues $\lambda_1, \ldots, \lambda_4$

$$\text{Lagrangian} \quad \mathcal{L}(t,x;t,y) = \frac{1}{4n} \sum_{i,j=1}^{4} \big(|\lambda_i| - |\lambda_j| \big)^2 \;\geq\; 0$$

$$\text{static Lagrangian} \quad \mathcal{L}(x,y) := \int_{-\infty}^{\infty} \mathcal{L}(0,x;t,y)\, dt$$

(constraints of causal action principle built in by fixing the trace and adding a Lagrange multiplier term)

Summary

ultrastatic spactime $\mathbb{R} \times \mathscr{N}$ with $(\mathscr{N}, g)$ Riemannian manifold

$\downarrow$

family of spinorial wave functions
& integration measure $d\mu = \sqrt{\det g}\, d^3x$

$\downarrow$

encode information in measure μ on $\mathcal{G}$

Lagrangian $\mathcal{L}$ on $\mathcal{G} \times \mathcal{G}$ via eigenvalues of operator products

$$\mathcal{S}(\mu) = \int_{\mathcal{G}} d\mu(x) \int_{\mathcal{G}} d\mu(y)\, \mathcal{L}(x, y)$$

www.causal-fermion-system.com

Thank you for your attention!