Causal fermion systems as an approach to quantum theory

Felix Finster

Fakultät für Mathematik
Universität Regensburg

Johannes-Kepler-Forschungszentrum
für Mathematik, Regensburg

Lecture at Conference "Quantum Mathematical Physics"
Regensburg, October 1st, 2014

Overview

Causal fermion system

- approach to describe fundamental physics
- candidate for a unified physical theory

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- novel approach to describe space and space-time, as well as structures therein:
  "quantum space-time," "quantum geometry"
- dynamics described by **causal action principle**
  - intrinsic, no space-time presupposed
  - space-time emerges by minimizing the causal action
  - generally covariant

Overview: limiting cases

Causal fermion system

- abstract mathematical framework
- quantum geometry, causal action

$$\Downarrow \quad \text{continuum limit}$$

description in the continuum limit

- Dirac fields
- strong and electroweak gauge fields
- gravitational field

arXiv:1409.2568 [math-ph]

- fermion field: second-quantized
- bosonic field: classical

Overview: limiting cases

Causal fermion system

$\Downarrow$ continuum limit

description in the continuum limit

$\Downarrow$ one particle

Dirac equation: $(i\gamma^j \partial_j + \mathcal{B} - m)\psi = 0$

$\Downarrow$ non-relativistic limit

Schrödinger equation: $i\partial_t \psi = H\psi$

quantum mechanics

Overview: limiting cases

Causal fermion system

$\Downarrow$ continuum limit

description in the continuum limit

$\Downarrow$ microscopic mixing

- also second-quantized bosonic field
- loop diagrams, renormalization, . . . (work in progress)

relativistic quantum field theory

arXiv:1409.2568 [math-ph], J. Math. Phys. **55** (2014) 042301

Overview: limiting cases

Causal fermion system

$\Downarrow$ continuum limit

description in the continuum limit $\Longrightarrow$ classical Yang-Mills theory, general relativity

$\Downarrow$ $\Downarrow$

QM QFT

Overview: limiting cases

Causal fermion system $\Bigg\}$ nonlinear

$\Downarrow$ continuum limit

description in the continuum limit $\Longrightarrow$ YM GR $\Bigg\}$ nonlinear

$\Downarrow$ $\Downarrow$

QM QFT $\Bigg\}$ linear

Towards causal fermion systems

Space-time point is a linear operator on a Hilbert space

Thus we need

- Hilbert space $(\mathcal{H}, \langle .|. \rangle_{\mathcal{H}})$
- a collection of linear operators on $\mathcal{H}$

Example: Dirac spinors in Minkowski space

Space-time is Minkowski space, signature $(+ - - -)$

space-time point $x \in \mathbb{R}^4$, need to associate operator $F(x)$

- free Dirac equation $\quad (i\gamma^k \partial_k - m)\,\psi = 0$
- probability density $\psi^\dagger \psi = \overline{\psi} \gamma^0 \psi$,

  gives rise to a scalar product:

$$\langle \psi | \phi \rangle = \int_{t=\text{const}} (\overline{\psi} \gamma^0 \phi)(t, \vec{x})\, d\vec{x}$$

time independent due to current conservation

Example: Dirac spinors in Minkowski space

- Consider a collection of one-particle wave functions

$$\psi_1, \ldots, \psi_f$$

(Pauli exclusion principle, . . . , later)

- orthonormalize: $\langle \psi_k | \psi_l \rangle = \delta_{kl}$

- For space-time point x introduce

$$F(x)^j_k = -\overline{\psi_j(x)}\psi_k(x) \quad \text{local correlation matrix}$$

  - Hermitian $f \times f$-matrix
  - rank at most four: Gram matrix or

$$F(x) = e_x^* e_x\,, \qquad e_x \,:\, \mathcal{H} \to \mathbb{C}^4, \quad \psi \mapsto \psi(x)$$

  - at most two positive and at most two negative eigenvalues

Example: Dirac spinors in Minkowski space

- similarly basis independent:

$$\mathcal{H} := <\psi_1, \ldots, \psi_f> \qquad \text{Hilbert space}$$

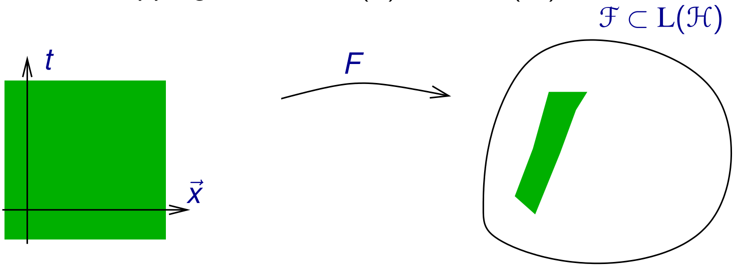
$$\langle \psi | F(x) \phi \rangle = -\overline{\psi(x)} \phi(x) \qquad \forall \psi, \phi \in \mathcal{H}$$

local correlation operator, is self-adjoint operator in $\mathrm{L}(\mathcal{H})$

- Thus $F(x) \in \mathcal{F}$ where
  $\mathcal{F} := \Big\{ F \in \mathrm{L}(\mathcal{H})$ with the properties:
  - ▷ F is self-adjoint and has rank ≤ 4
  - ▷ F has at most 2 positive and at most 2 negative eigenvalues $\Big\}$

Example: Dirac spinors in Minkowski space

We obtain mapping $\quad x \mapsto F(x) \in \mathcal{F} \subset \mathrm{L}(\mathcal{H})$

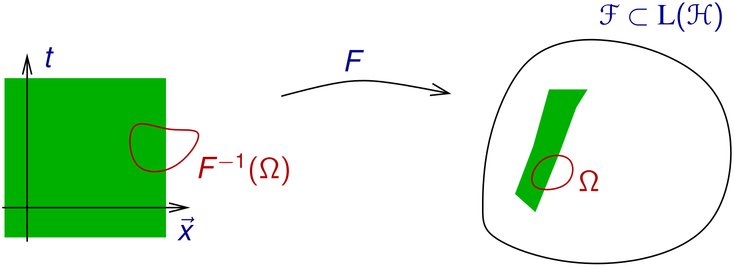


Concept:

- disregard the left side
- work exclusively with objects on the right.

Example: Dirac spinors in Minkowski space

one more thing: The space-time volume

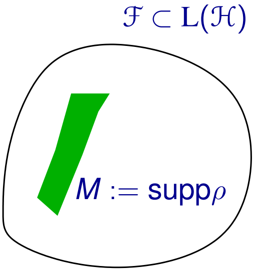


$$\rho(\Omega) := \int_{F^{-1}(\Omega)} d^4x = \mu\big(F^{-1}(\Omega)\big)$$

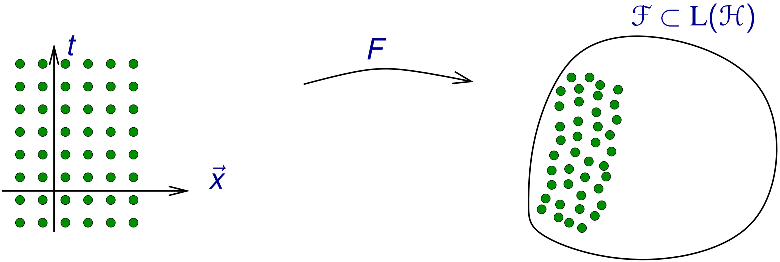
- push-forward measure, is measure on $\mathcal{F}$.
- image of F recovered as the support of the measure,

$$M := \mathrm{supp}\, \rho = \big\{ F \in \mathcal{F} \mid \rho(\Omega) \neq 0 \text{ for every open neighborhood } \Omega \text{ of } x \big\}$$

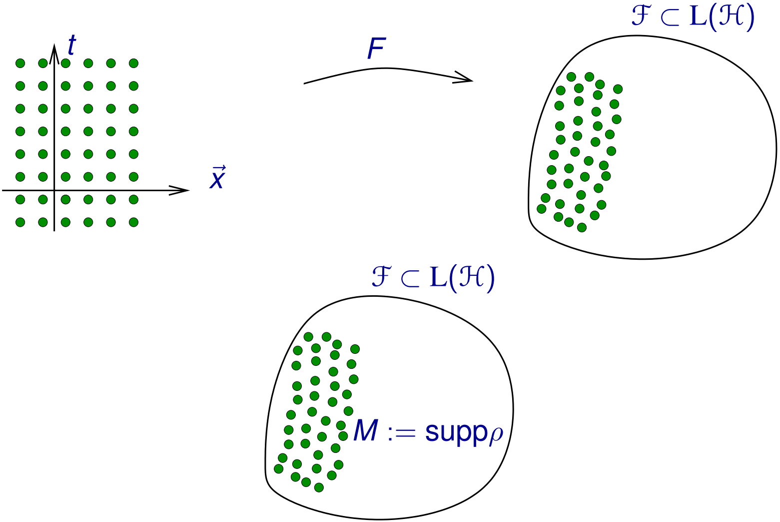
resulting structure: measure ρ on $\mathcal{F} \subset \mathrm{L}(\mathcal{H})$



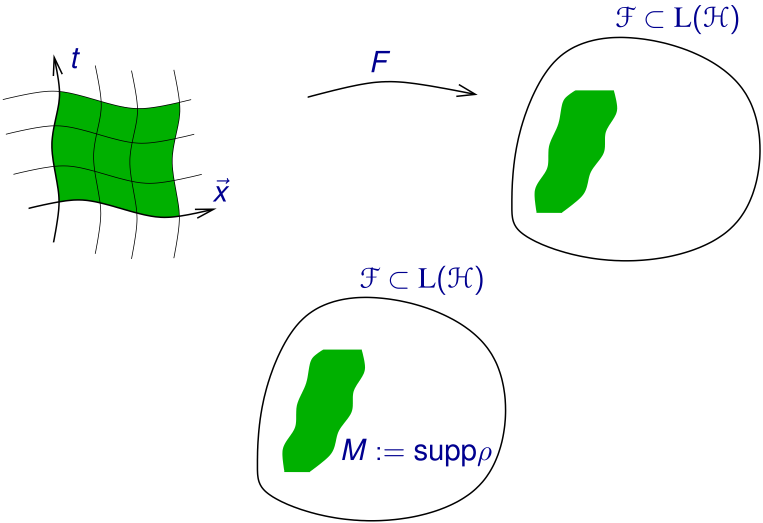
Example: A space-time lattice



Example: A space-time lattice



Example: curved space-time



Definition (Causal fermion system)

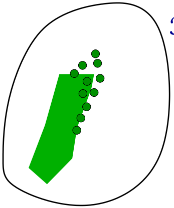
Let $(\mathcal{H}, \langle .|. \rangle_{\mathcal{H}})$ be Hilbert space ("particle space")
Given parameter $n \in \mathbb{N}$ ("spin dimension")

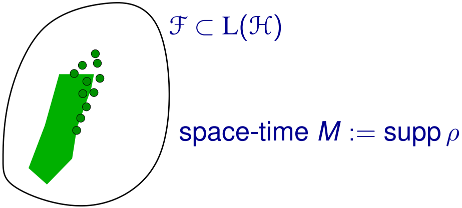
$$\mathcal{F} := \Big\{ x \in \mathrm{L}(\mathcal{H}) \text{ with the properties:}$$

- ▷ x is self-adjoint and has finite rank
- ▷ x has at most n positive and at most n negative eigenvalues $\Big\}$

ρ a measure on $\mathcal{F}$ ("universal measure")

$(\rho, \mathcal{F}, \mathcal{H})$ is a causal fermion system.





Advantage of general framework:

- "Spinors on singular spaces . . . ,"
  F-Kamran, arXiv:1403.7885 [math-ph]
- UV-regularized space-times

Inherent structures in space-time

Let $(\rho, \mathcal{F}, \mathcal{H})$ be a causal fermion system,
space-time defined by $M := \text{supp}\rho$.

Space-time points are linear operators on $\mathcal{H}$

- For $x \in M$, consider eigenspaces of x.
- For $x, y \in M$ consider
  - consider operator products xy
  - project eigenspaces of x to eigenspaces of y

Gives rise to:

- quantum objects (spinors, wave functions)
- geometric structures (connection, curvature)
- causal structure, analytic structures

Causal structure

Let $x, y \in M$. Then

$x \cdot y \in \mathrm{L}(H)$:

- rank $\leq 2n$
- in general not self-adjoint: $(x \cdot y)^* = y \cdot x \neq x \cdot y$

thus non-trivial complex eigenvalues $\lambda_1^{xy}, \ldots, \lambda_{2n}^{xy}$

Definition (causal structure)

The points $x, y \in \mathcal{F}$ are called

$$\left\{ \begin{array}{ll} \text{timelike separated} & \text{if } \lambda_1^{xy}, \ldots, \lambda_{2n}^{xy} \text{ are all real} \\ \text{spacelike separated} & \text{if } \lambda_1^{xy}, \ldots, \lambda_{2n}^{xy} \text{ are all non-real} \\ & \text{and } |\lambda_i^{xy}| = |\lambda_j^{xy}| \quad \forall i, j \\ \text{lightlike separated} & \text{otherwise} \end{array} \right.$$

Causal action principle

$$\text{Lagrangian} \quad \mathcal{L}[A_{xy}] = \frac{1}{4n} \sum_{i,j=1}^{2n} \left(|\lambda_i^{xy}| - |\lambda_j^{xy}| \right)^2 \geq 0$$

$$\text{action} \qquad \mathcal{S} = \iint_{x,y \in M} \mathcal{L}[A_{xy}] \, d\rho(x) \, d\rho(y)$$

Minimize $\mathcal{S}$ under variations of ρ, impose suitable constraints.
Gives mathematically well-defined variational principles.

- Lagrangian is compatible with causal structure, i.e.

$$x, y \text{ spacelike separated} \quad \Rightarrow \quad L(x,y) = 0$$

"points with spacelike separation do not interact"

Inherent structures

- Spinors

$$S_x := x(\mathcal{H}) \subset \mathcal{H} \quad \text{“spin space”}, \ \dim S_x \leq 2n$$

$$\prec u|v \succ_x := -\langle u \,|\, x\, v \rangle_{\mathcal{H}} \quad \text{“spin scalar product”},$$

is indefinite of signature $(\leq n, \leq n)$

- Space of one-particle wave functions

$$\psi : x \in M \mapsto \psi(x) \in S_x \qquad \text{“wave function”}$$

wave functions form Krein space $(\mathcal{K}, \langle .|. \rangle)$:

$$\langle \psi | \phi \rangle := \int_M \prec \psi(x) | \phi(x) \succ_x \, d\rho(x) \quad \text{indefinite inner product}$$

$$|||\, \psi \,|||^2 = \int_M \langle \psi(x) | \, |x| \, \psi(x) \rangle_{\mathcal{H}} \, d\rho(x) \quad \text{norm, induces topology}$$

Inherent structures

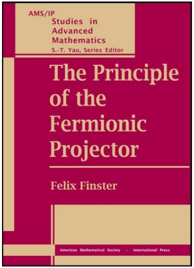
- Physical wave functions and the fermionic operator

$$\psi(x) = \pi_x \, \psi \quad \text{with } \psi \in \mathcal{H} \qquad \text{physical wave function}$$

$$P(x,y) = \pi_x \, y \;:\; S_y \to S_x \qquad \text{“kernel of fermionic operator”}$$

$$= -\sum_{i=1}^{f} |\psi_i(x) \succ \prec \psi_i(y)| \quad \text{where } \psi_i \text{ basis of } \mathcal{H}$$

The fermionic operator was indeed the starting point:



“The Principle of the Fermionic Projector”
AMS/IP Studies in Advanced Math. 35 (2006)

Inherent structures

- Geometric structures
  - $P(x,y) \;:\; S_y \rightarrow S_x$ yields relations between spin spaces. Using a polar decomposition $(\ldots, \ldots)$ one gets:

$$D_{x,y} : \; S_y \rightarrow S_x \text{ unitary} \qquad \text{“spin connection”}$$

  - tangent space T_x, carries Lorentzian metric,

$$\nabla_{x,y} \;:\; T_y \rightarrow T_x \qquad \text{corresponding “metric connection”}$$

  - a distinguished time direction
  - holonomy of connection gives curvature

$$R(x,y,z) = \nabla_{x,y} \, \nabla_{y,z} \, \nabla_{z,x} \;:\; T_x \rightarrow T_x \,.$$

- Local gauge freedom
  Choose pseudo-orthonormal basis $(\mathfrak{e}_\alpha)$ of S_x, i.e.

$$\prec \mathfrak{e}_\alpha | \mathfrak{e}_\beta \succ = s_\alpha \delta_{\alpha\beta} \quad \text{with} \quad s_1, \ldots, s_p = 1,\; s_{p+1}, \ldots, s_{p+q} = -1$$

Then $\Psi \in \mathcal{K}$ can be written in components as

$$\psi(x) = \sum_{\alpha=1}^{p+q} \psi^\alpha(x)\, \mathfrak{e}_\alpha(x)$$

The basis $(\mathfrak{e}_\alpha)$ can be chosen freely at every x:

$$\mathfrak{e}_\alpha \to \sum_{\beta=1}^{p+q} (U^{-1})^\beta_\alpha \, \mathfrak{e}_\beta \quad \text{mit } U \in \mathrm{U}(p,q)$$

$$\psi^\alpha(x) \;\to \sum_{\beta=1}^{p+q} U(x)^\alpha_\beta \, \psi^\beta(x)$$

interpretation: local gauge freedom with group $\mathrm{U}(p,q)$

Underlying physical principles

- Pauli exclusion principle:
  Choose orthonormal basis $\psi_1, \ldots, \psi_f$ of $\mathcal{H}$. Set

$$\Psi = \psi_1 \wedge \cdots \wedge \psi_f \,,$$

  gives equivalent description by Hartree-Fock state.
- local gauge principle:
  freedom to perform local unitary transformations.
- the "equivalence principle":
  symmetry under "diffeomorphisms" of *M*
  (note: *M* merely is a topological measure space)

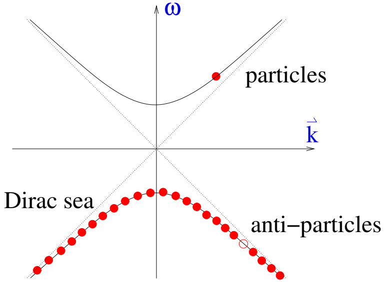
but no locality

causality and time direction not preassumed but emergent

The causal action principle in the continuum limit

Specify vacuum:

- Choose $\mathcal{H}$ as the space of all negative-energy solutions, hence "Dirac sea"



Fixes length scale ("Compton length")

- Introduce ultraviolet regularization, "quantum geometry"

  Fixes length scale ("Planck length")

The causal action principle in the continuum limit

- specify vacuum as sum of Dirac seas,

$$P(x,y) = \sum_{\beta=1}^{g} P^{\text{sea}}_{m_\beta}(x,y)$$

$$P^{\text{sea}}_{m_\beta}(x,y) = \int \frac{d^4k}{(2\pi)^4}\, (\not{k} + m_\beta)\, \delta(k^2 - m_\beta^2)\, \Theta(-k^0)\, e^{-ik(x-y)}$$

β labels "generations" of elementary particles

$\Longrightarrow$ Dynamical equations only if three generations ($g = 3$)

The causal action principle in the continuum limit

- Model involving neutrinos and quarks:

$$P(x,y) = \sum_{\beta=1}^{3} \underbrace{P^{\text{sea}}_{m_\beta}(x,y) \oplus \cdots \oplus P^{\text{sea}}_{m_\beta}(x,y)}_{\text{7 identical direct summands}} \oplus P^{\text{sea}}_{\tilde{m}_\beta}(x,y)$$

  (thus $4 \times 8 = 32$-component wave functions)
  again three generations
- Regularize the neutrinos suitably
  (shear and general surface states), break chiral symmetry

The causal action principle in the continuum limit

Results:

- The direct summands form pairs, spontaneous block formation
- gauge group $\mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3)$
- coupling to the fermions exactly as in the standard model:
  - U_{CKM}, U_{MNS}-mixing matrices
  - $\mathrm{SU}(2)$ gauge fields left-handed and massive
- Einstein equations

Remarks on methods for analyzing the continuum limit:

- The Dirac sea vacuum is a critical point of the causal action (in a well-defined mathematical sense)
- Vary the fermionic projector:
  Consider the Dirac equation in an external potential

$$(i\partial\!\!\!/ + \mathcal{B} - mY)\psi = 0\,.$$

  - Singularities of $P(x, y)$ drop out of EL equations, no counter terms needed!
  - Non-perturbative method for constructing $P(x, y)$: mass oscillation property (F-Reintjes 2013).

A few references

Survey articles:

- F.F., A. Grotz, D. Schiefeneder, "Causal Fermion Systems: A quantum space-time emerging from an action principle," arXiv:1102.2585 [math-ph], in "Quantum Field Theory and Gravity," Birkhäuser (2012) 157-182
- F.F., "A formulation of quantum field theory realizing a sea of interacting Dirac particles," arXiv:0911.2102 [hep-th], *Lett. Math. Phys.* **97** (2011) 165-183

For further reading:

- F.F., A. Grotz, "A Lorentzian quantum geometry," arXiv:1107.2026 [math-ph], *Adv. Theor. Math. Phys.* **16** (2012) 1197–1290
- F.F., "Perturbative quantum field theory in the framework of the fermionic projector," arXiv:1310.4121 [math-ph], *J. Math. Phys.* **55** (2014) 042301
- F.F., "Causal variational principles on measure spaces," arXiv:0811.2666 [math-ph], *J. Reine Angew. Math.* **646** (2010) 141–194
- F.F., "The Principle of the Fermionic Projector," hep-th/0001048, hep-th/0202059, hep-th/0210121, *AMS/IP Studies in Advanced Mathematics* **35** (2006)
- F.F., "An action principle for an interacting fermion system and its analysis in the continuum limit," arXiv:0908.1542 [math-ph]
- F.F., "The continuum limit of a fermion system involving neutrinos: weak and gravitational interactions," arXiv:1211.3351 [math-ph]
- F.F., "The continuum limit of a fermion system involving leptons and quarks: strong, electroweak and gravitational interactions," arXiv:1409.2568 [math-ph]

Thank you for your attention!

Example: the Minkowski vacuum

Theorem (F-Grotz, 2011)

There are regularizations $P^\varepsilon(x,y)$ such that in the limit $\varepsilon \searrow 0$:

- *The causal structure goes over to that of Minkowski space*
- $(S_x, \prec .|. \succ_x)$ *can be identified with the usual spinor space,*

$$\prec \psi | \phi \succ_x = \overline{\psi(x)} \phi(x)$$

- *The spin connection $D_{x,y}$ becomes trivial.*

$$x \cdot y \simeq a\,(y-x)^j \gamma_j + b \qquad \text{with } a, b \in \mathbb{R} \quad \text{(Lorentz symmetry)}$$

$$\lambda_i^{xy} = b \pm |a| \sqrt{(y-x)_j (y-x)^j}$$

$$\begin{cases} \text{are real} & \text{if } y-x \text{ timelike} \\ \text{form complex conjugate pair} & \text{if } y-x \text{ spacelike} \end{cases}$$

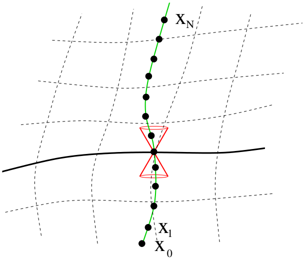
Example: a globally hyperbolic space-time

Analogous result in presence of gravitational field:

Theorem (F-Grotz, 2011)

Let (M, g) be a globally hyperbolic Lorentzian manifold. There are regularizations $P^\varepsilon(x, y)$ such that in the limit $\varepsilon \searrow 0$:

- *$D_{x,y}$ goes over to the metric spin connection.*
  Curvatures gives the Riemann curvature tensor.



$$\lim_{N\to\infty} \lim_{\varepsilon\searrow 0} D_{x_N,x_{N-1}} D_{x_{N-1},x_{N-2}} \cdots D_{x_1,x_0}$$
$$= D^{\text{LC}}_{x,y} + \mathcal{O}\left(L(\gamma)\, \frac{\|\nabla R\|}{m^2} \right) \left(1 + \mathcal{O}\left(\frac{\text{scal}}{m^2} \right) \right)$$

The mechanism of microscopic mixing

Decompose the universal measure as

$$\tilde{\rho} = \frac{1}{L} \sum_{\mathfrak{a}=1}^{L} \rho_{\mathfrak{a}} \,.$$

The action becomes

$$\begin{aligned} \mathcal{S}(\tilde{\rho}) &= \frac{1}{L^2} \sum_{\mathfrak{a},\mathfrak{b}=1}^{L} \iint_{\mathcal{F}\times\mathcal{F}} \mathcal{L}(x,y)\, d(V_{\mathfrak{a}}\rho)(x)\, d(V_{\mathfrak{b}}\rho)(y) \\ &= \frac{\mathcal{S}(\rho)}{L} + \frac{1}{L^2} \sum_{\mathfrak{a}\neq\mathfrak{b}} \iint_{\mathcal{F}\times\mathcal{F}} \mathcal{L}(x,y)\, d(V_{\mathfrak{a}}\rho)(x)\, d(V_{\mathfrak{b}}\rho)(y) \,. \end{aligned}$$

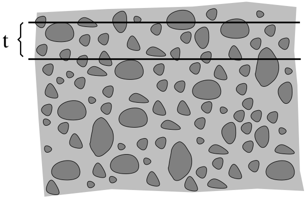
“Mixed terms” can be made small by dephasing (decoherence).

The mechanism of microscopic mixing

- Leads to decompositino of space-time

$$M = M_1 \cup M_2 \qquad \text{with} \qquad M_1 \cap M_2 = \varnothing$$

"fine-grained on microscopic scale"



- Effective description using Fock spaces