An Introduction to Causal Fermion Systems and the Causal Action Principle

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What is a causal fermion system?

- approach to fundamental physics
- novel mathematical model of spacetime
- physical equations are formulated in generalized spacetimes
- Different limiting cases:
 - Continuum limit: Quantized fermionic fields interacting via classical bosonic fields
 - QFT limit: fermionic and bosonic quantum fields (ongoing, more towards the end of the talk)
- For overview, more details (papers, books, videos, online course), applications to cosmology and black holes, ...

www.causal-fermion-system.com

- ► Clearly, there are personal preferences.
- Ultimately: Which formulation is capable to describe all known physical phenomena plus quantum gravity?

- Hypothesis: Consider quantum mechanical wave functions as the basic physical objects (QFT later)
 - ψ describes quantum mechanical particle (only wave character, no point particle)
 - is physical reality even without measurements
 - dynamics as described in the simplest case by Schrödinger equation (or Dirac equation, collapse model, ...)
- Vector ψ in a Hilbert space $(\mathcal{H}, \langle . | . \rangle_{\mathcal{H}})$.
- This is not quite the right description:
 - Phase has no significance: ψ → e^{iΛ}ψ instead of ψ consider ray generated by ψ
 - Local gauge invariance

$$\psi(t,\vec{x}) \rightarrow e^{i\Lambda(t,\vec{x})} \psi(t,\vec{x})$$

Therefore, only $|\psi(t, \vec{x})|^2$ is of physical significance; interpretation: probability density

- ► Thus: Consider $|\psi(t, \vec{x})|^2$ of all quantum mechanical wave functions as the basic physical objects.
- ► General question: Suppose we know |ψ(t, x)|² for all the wave functions of the system, what can we say about the spacetime structures (causality, metric, fields, ...)
- Try to probe spacetime by looking at $|\psi(t, \vec{x})|^2$.

Here "probing" should be thought of as a mathematical operation; no collapse of the wave function involved.

• Begin in Minkowski space (usual spacetime structures)

$$x = (t, \vec{x}), \qquad t \in \mathbb{R}, \ \vec{x} \in \mathbb{R}^3.$$

(later curved spacetime)

• Consider scalar particle (no spin)

 $|\psi(x)|^2$ (local density)

First step: Allow for preparation of the "initial state" at time t.

Allows for detecting the causal structure of spacetime:



Allows for recovering the full metric information: consider massive situation



Allows for detecting an electromagnetic field:





Second step: Do not allow for preparation of the "initial state". Instead: Get by with the wave function already present.

 Probing still works, provided that there are "sufficiently many" wave functions around.



 The more wave functions there are, the more information we have on spacetime (spacetime resolution, bandwidth, ...)

Physical picture: Dirac's hole theory



Formalize this idea: The local correlation operator

- ▶ Consider wave functions $\psi_1, \ldots, \psi_f : \mathcal{M} \to \mathbb{C}$ (with $f < \infty$)
- Are vectors in a Hilbert space, orthonormalize,

 $\langle \psi_{\mathbf{k}} | \psi_{\mathbf{l}} \rangle = \delta_{\mathbf{k} \mathbf{l}} \,,$

gives *f*-dim Hilbert space $(\mathcal{H}, \langle . | . \rangle_{\mathcal{H}})$.

basic object: for any lattice point x introduce

local correlation operator F(x) : $\mathcal{H} \to \mathcal{H}$

define matrix elements by

$$(F(x))_k^j = \overline{\psi_j(x)}\psi_k(x)$$

basis invariant:

 $\langle \psi, F(x) \phi \rangle_{\mathcal{H}} = \overline{\psi(x)} \phi(x) \quad \text{ for all } \psi, \phi \in \mathcal{H}$

- Hermitian matrix = symmetric operator
- Has rank at most one, is positive semi-definite

$$F(x) = e^*e$$
 with $e: \mathcal{H} \to \mathbb{C}, \quad \psi \mapsto \psi(x)$



- The right side contains all the information which can be retrieved from the wave functions.
- We consider the objects on the right as the basic physical objects.

general strategy:

- disregard objects on the left (Minkowski space, causal structure, Dirac spinors, ...)
- work instead exclusively with the objects on the right (only local correlation operators)



A volume measure on spacetime

 One important structure is missing: Volume measure on spacetime.



Take push-forward measure of $F : \mathcal{M} \to \mathfrak{F}$,

$$\rho := F_*(\mu_{\mathscr{M}}) \quad (\text{i.e. } \rho(\Omega) := \mu_{\mathscr{M}}(F^{-1}(\Omega)))$$

Example: Dirac spinors in Minkowski space



▶ image of *F* recovered as the support of the measure,

$$\boldsymbol{M} := \operatorname{supp} \rho = \big\{ \boldsymbol{F} \in \mathcal{F} \mid \rho(\Omega) \neq \boldsymbol{0} \big\}$$

for every open neighborhood Ω of x

Causal Fermion Systems

Definition (Causal fermion system)

Let $(\mathcal{H}, \langle . | . \rangle_{\mathcal{H}})$ be Hilbert space Given parameter $n \in \mathbb{N}$ ("spin dimension") $\mathcal{F} := \Big\{ x \in L(\mathcal{H}) \text{ with the properties:} \Big\}$

- x is symmetric and has finite rank
- x has at most n positive

and at most *n* negative eigenvalues }

 ρ a measure on \mathcal{F}



Let (\mathcal{M}, g) be a Lorentzian space-time, for simplicity 4-dimensional, globally hyperbolic, then automatically spin,

 $(SM, \prec . | . \succ)$ spinor bundle

- $S_p \mathcal{M} \simeq \mathbb{C}^4$
- spin scalar product

$$\prec . | . \succ_{p} : S_{p}\mathcal{M} \times S_{p}\mathcal{M} \to \mathbb{C}$$

is indefinite of signature (2,2)

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 $(\mathcal{D} - m)\psi_m = 0$ Dirac equation

- Cauchy problem well-posed, global smooth solutions (for example symmetric hyperbolic systems)
- finite propagation speed

 $C^{\infty}_{sc}(\mathcal{M}, S\mathcal{M})$ spatially compact solutions

$$(\psi_m | \phi_m)_m := 2\pi \int_{\mathcal{N}} \prec \psi_m | \psi \phi_m \succ_x d\mu_{\mathcal{N}}(x)$$
 scalar product

completion gives Hilbert space $(\mathcal{H}_m, (.|.)_m)$

 \blacktriangleright Choose ${\mathcal H}$ as a subspace of the solution space,

$$\mathcal{H} = \overline{\mathrm{span}(\psi_1, \ldots, \psi_f)}$$

▶ To $x \in \mathbb{R}^4$ associate a local correlation operator

$$\langle \psi | F(\mathbf{x}) \phi \rangle = - \prec \psi(\mathbf{x}) | \phi(\mathbf{x}) \succ_{\mathbf{x}} \qquad \forall \psi, \phi \in \mathcal{H}$$

Is symmetric, rank ≤ 4 at most two positive and at most two negative eigenvalues
Here ultraviolet regularization may be necessary:

$$\langle \psi | F(\mathbf{x}) \phi \rangle = - \prec (\mathfrak{R}_{\varepsilon} \psi)(\mathbf{x}) | (\mathfrak{R}_{\varepsilon} \phi)(\mathbf{x}) \succ_{\mathbf{x}} \qquad \forall \psi, \phi \in \mathfrak{H}$$

 $\mathfrak{R}_{\varepsilon} : \mathfrak{H} \to C^{0}(\mathfrak{M}, S\mathfrak{M})$ regularization operators

 $\varepsilon > 0$: regularization scale (Planck length)

Thus F(x) ∈ 𝔅 where
 𝔅 := {F ∈ L(𝔅) with the properties:
 ▷ F is symmetric and has rank ≤ 4
 ▷ F has at most 2 positive
 and at most 2 negative eigenvalues }



Take push-forward measure

 $\rho := F_*(\mu_{\mathcal{M}}) \quad (\text{i.e. } \rho(\Omega) := \mu_{\mathcal{M}}(F^{-1}(\Omega)))$



We thus obtain a causal fermion system of spin dimension two.

How can one formulate physical equations in this setting?

- Intuitive picture: wave functions "organize themselves" in such a way that the Dirac sea configuration is a minimizer.
- In interacting situation the wave functions organize to solutions of the Dirac equation

$$(i\gamma^j\partial_j + e\gamma^j A(x) - m)\psi = 0$$

This should serve as the definition of A.

- Let $x, y \in \mathcal{F}$. Then x and y are linear operators.
 - $\mathbf{x} \cdot \mathbf{y} \in L(H)$:
 - rank ≤ 2*n*

• in general not symmetric: $(x \cdot y)^* = y \cdot x \neq x \cdot y$ thus non-trivial complex eigenvalues $\lambda_1^{xy}, \ldots, \lambda_{2n}^{xy}$

Causal action principle

Nontrivial eigenvalues of *xy*: $\lambda_1^{xy}, \ldots, \lambda_{2n}^{xy} \in \mathbb{C}$

$$\begin{array}{ll} \text{-agrangian} \quad \mathcal{L}(x,y) = \frac{1}{4n} \sum_{i,j=1}^{2n} \left(|\lambda_i^{xy}| - |\lambda_j^{xy}| \right)^2 \geq 0 \\ \text{action} \qquad \mathcal{S} = \iint_{\mathcal{F} \times \mathcal{F}} \mathcal{L}(x,y) \, d\rho(x) \, d\rho(y) \in [0,\infty] \end{array}$$

Minimize S under variations of ρ , with constraints

volume constraint: $\rho(\mathcal{F}) = \text{const}$ trace constraint: $\int_{\mathcal{F}} \text{tr}(x) d\rho(x) = \text{const}$ boundedness constraint: $\iint_{\mathcal{F} \times \mathcal{F}} \sum_{i=1}^{2n} |\lambda_i^{xy}|^2 d\rho(x) d\rho(y) \leq C$

F.F., "Causal variational principles on measure spaces,"
 J. Reine Angew. Math. 646 (2010) 141–194

One basic object: measure ρ on set \mathcal{F} of linear operators on \mathcal{H} , describes spacetime as well as all objects therein

- Underlying structure: family of fermionic wave functions
- Geometric structures encoded in these wave functions

Matter encodes geometry Quantum spacetime

- Causal action principle describes spacetime as a whole (similar to Einstein-Hilbert action in GR)
- Causal action principle is a nonlinear variational principle (similar to Einstein-Hilbert action or classical field theory)
- Linear dynamics of quantum theory recovered in limiting case (more details later)

local gauge principle:

freedom to perform local unitary transformations of the spinors

► Pauli exclusion principle:

Choose orthonormal basis ψ_1, \ldots, ψ_f of \mathcal{H} . Set

$$\Psi = \psi_1 \wedge \cdots \wedge \psi_f \,,$$

gives equivalent description by Hartree-Fock state.

 the "equivalence principle": symmetry under "diffeomorphisms" of *M* (note: *M* merely is a topological measure space)

Spacetime and causal structure are emergent

Interpretation in terms of spacetime events

- operators in *F* can be interpreted as "possible local correlation operators" or simply as possible events
- ▶ operators in *M* are the events realized in spacetime
- spacetime is made up of all the realized events
- ▶ the physical equations relate the events to each other

For details on this connection:

 F.F, J. Fröhlich, C. Paganini, C. and M. Oppio, "Causal fermion systems and the ETH approach to quantum theory," arXiv:2004.11785 [math-ph] (2020) Let $(\rho, \mathcal{F}, \mathcal{H})$ be a causal fermion of spin dimension *n*, spacetime $M := \operatorname{supp} \rho$.

spacetime points are linear operators on $\mathcal H$

- For $x \in M$, consider eigenspaces of x.
- ► For *x*, *y* ∈ *M*,
 - consider operator products xy
 - project eigenspaces of x to eigenspaces of y

Gives rise to:

- quantum objects (spinors, wave functions)
- geometric structures (connection, curvature)
- causal structure, analytic structures

Spinors

$$S_{x}M := x(\mathcal{H}) \subset \mathcal{H}$$
$$\prec u | v \succ_{x} := -\langle u | x v \rangle_{\mathcal{H}}$$

"spin space", dim $S_x M \le 2n$ ("spin scalar product", inner product of signature ($\le n, \le n$)



Inherent structures in spacetime

Physical wave functions

 $\psi^{u}(x) = \pi_{x} u$ with $u \in \mathcal{H}$ physical wave function $\pi_{x} : \mathcal{H} \to \mathcal{H}$ orthogonal projection on $x(\mathcal{H})$



Inherent structures in spacetime

► The kernel of the fermionic projector:

$$P(y, x) = \pi_y x|_{S_x M} : S_x M \to S_y M$$



$$\mathcal{P}(\mathbf{y}, \mathbf{x}) = -\sum_{i=1}^{f} |\psi^{e_i}(\mathbf{y}) \succ \prec \psi^{e_i}(\mathbf{x})|$$
 where (e_i) ONB of \mathcal{H}

Geometric structures

P(*x*, *y*) : *S_yM* → *S_xM* yields relations between spin spaces.

Using a polar decomposition (\ldots, \ldots) one gets:

 $D_{x,y}$: $S_y M \to S_x M$ unitary "spin connection"

• tangent space T_x , carries Lorentzian metric,

 $\nabla_{x,y}$: $T_y \to T_x$ corresponding "metric connection"

holonomy of connection gives curvature

$$R(x, y, z) = \nabla_{x, y} \nabla_{y, z} \nabla_{z, x} : T_x \to T_x$$

Causal structure

Let $x, y \in M$. Then $x \cdot y \in L(H)$ has non-trivial complex eigenvalues $\lambda_1^{xy}, \ldots, \lambda_{2n}^{xy}$ Definition (causal structure) The points $x, y \in \mathcal{F}$ are called spacelike separated if $|\lambda_i^{xy}| = |\lambda_k^{xy}|$ for all $j, k = 1, \dots, 2n$ if $\lambda_1^{xy}, \ldots, \lambda_{2n}^{xy}$ are all real timelike separated and $|\lambda_i^{xy}| \neq |\lambda_k^{xy}|$ for some j, klightlike separated otherwise

Lagrangian is compatible with causal structure:

Lagrangian
$$\mathcal{L}(\mathbf{x}, \mathbf{y}) = \frac{1}{4n} \sum_{i,j=1}^{2n} \left(|\lambda_i^{xy}| - |\lambda_j^{xy}| \right)^2 \ge 0$$

thus x, y spacelike separated $\Rightarrow \mathcal{L}(x, y) = 0$

"points with spacelike separation do not interact"

 $x(\mathcal{H}) \subset \mathcal{H}$ subspace of dimension $\leq 2n$

Introduce the functional

 $\mathcal{C} : \boldsymbol{M} \times \boldsymbol{M} \to \mathbb{R}, \qquad \mathcal{C}(\boldsymbol{x}, \boldsymbol{y}) := i \operatorname{tr} (\boldsymbol{y} \, \boldsymbol{x} \, \pi_{\boldsymbol{y}} \, \pi_{\boldsymbol{x}} - \boldsymbol{x} \, \boldsymbol{y} \, \pi_{\boldsymbol{x}} \, \pi_{\boldsymbol{y}})$

For timelike separated points $x, y \in M$,

- $\begin{cases} y \text{ likes in the future of } x & \text{ if } \mathbb{C}(x, y) > 0 \\ y \text{ likes in the past of } x & \text{ if } \mathbb{C}(x, y) < 0 \end{cases}$
- The resulting relation "lies in the future of" is not necessarily transitive.

Objects in space

- In our daily experience: objects in space (densities, ...), spatial integrals (integral over densities, ...).
- Arise in causal fermion systems as surface layer integrals,



- F.F., J. Kleiner, "Noether-like theorems for causal variational principles," arXiv:1506.09076 [math-ph], *Calc. Var. Partial Differential Equations* 55:35 (2016)
- F.F., J. Kleiner, "A class of conserved surface layer integrals for causal variational principles," arXiv:1801.08715 [math-ph], Calc. Var. Partial Differential Equations 58:38 (2019)

Causal fermion system

- abstract mathematical framework
- quantum geometry, causal action

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continuum limit

description in the continuum limit

- Dirac fields
- strong and electroweak gauge fields
- gravitational field

Causal fermion system

- abstract mathematical framework
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continuum limit

description in the continuum limit

- Dirac fields
- strong and electroweak gauge fields
- gravitational field
- fermion field: second-quantized
- bosonic field: classical

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The Continuum Limit of Causal Fermion Systems

D Springer

From Planck Scale Structures to Macroscopic Physics Fundamental Theories of Physics **186** Springer, 2016 548+xi pages

arXiv:1605.04742 [math-ph]

The quantum field theory limit

- ▶ Work in progress. So far:
 - Construction of quantum state at any time t
 - Proved: General entanglement
 - Next step: Dynamics of quantum state.
- F.F., N. Kamran, "Complex structures on jet spaces and bosonic Fock space dynamics for causal variational principles," arXiv:1808.03177 [math-ph], *Pure Appl. Math. Q.* 17 (2021) 55–140
- F.F. and Kamran, N., "Fermionic Fock spaces and quantum states for causal fermion systems," arXiv:2101.10793 [math-ph], Ann. Henri Poincaré 23 (2022) 1359–1398
- F.F., Kamran, N. and Reintjes, M., "Entangled quantum states of causal fermion systems and unitary group integrals," arXiv:2207.13157 [math-ph]

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Thank you for your attention!

Felix Finster Causal fermion systems

The causal action principle in the continuum limit

specify vacuum as sum of Dirac seas,

$$\begin{split} P(x,y) &= \sum_{\beta=1}^{g} P_{m_{\beta}}^{\text{sea}}(x,y) \\ P_{m_{\beta}}^{\text{sea}}(x,y) &= \int \frac{d^{4}k}{(2\pi)^{4}} \left(\not\!\!\! k + m_{\beta} \right) \delta(k^{2} - m_{\beta}^{2}) \,\Theta(-k^{0}) \, e^{-ik(x-y)} \end{split}$$

 β labels "generations" of elementary particles

 \implies Dynamical equations only if three generations (g = 3)

The causal action principle in the continuum limit

Model involving neutrinos and quarks:

$$P(x,y) = \sum_{\beta=1}^{3} \underbrace{P_{m_{\beta}}^{\text{sea}}(x,y) \oplus \cdots \oplus P_{m_{\beta}}^{\text{sea}}(x,y)}_{7 \text{ identical direct summands}} \oplus P_{\tilde{m}_{\beta}}^{\text{sea}}(x,y)$$

- again three generations
- $4 \times 8 = 32$ -component wave functions
- spin dimension 16
- Regularize on the scale ε (Planck scale), regularization of neutrinos breaks chiral symmetry

Results:

- The direct summands form pairs, spontaneous block formation
- gauge group $U(1) \times SU(2) \times SU(3)$
- coupling to the fermions exactly as in the standard model:
 - U_{CKM}, U_{MNS}-mixing matrices
 - SU(2) gauge fields left-handed and massive
- Einstein equations

Remarks on methods for analyzing the continuum limit:

Consider the Dirac equation in an external potential

 $(i\partial + \mathcal{B} - mY)\psi = 0.$

- ► Question: Are the EL equations of causal action principle satisfied in the limit *ε* ↘ 0?
- Answer: Yes if and only if B has a certain structure and satisfies the classical field equations.

Going beyond the continuum limit

Basic question: What about objects in space (densities, probabilities, etc.)

- \blacktriangleright The scalar product $\langle .|.\rangle_{\mathcal{H}}$ is given abstractly
- A-priori missing: Representation as spatial integral. Is there an analog of the relation

$$\langle \psi | \phi
angle_{\mathcal{H}} = \int_{\mathcal{N}} \prec \psi | \phi \succ_{\mathbf{X}} d\mu_{\mathcal{N}}(\mathbf{X})$$

for Dirac spinors?

- Related question: What about current conservation? Are there conservation laws?
- Ultimately: What is the quantum state? What are quantum probabilities?

Outlook: Dynamics of the quantum state

- Construction so far gives ω^t for all t
- Next steps:
 - Construct time evolution for the density operator

$$\mathfrak{L}_{t_0}^t : \sigma^{t_0} \to \sigma^t$$

• Is there a unitary time evolution on the Fock space?

$$\omega^t = U_{t_0}^t \omega^{t_0} \left(U_{t_0}^t
ight)^{-1}$$

- Answer: Yes. In this limiting case one gets QFT including loop diagrams, but with intrinsic regularization on scale ε.
- There are nonlinear corrections. Connection to collapse phenomena?
- Is ongoing work with N. Kamran and M. Reintjes

Outlook: a quantum spacetime

a classical space-time: *M* diffeomorphic to manifold



a quantum space-time: $M \simeq \mathcal{M} \times \mathcal{B}$



General structure:

- ▶ Nonlinear dynamics of $\tilde{\rho}$ (from causal action principle)
- Conservation laws hold (current conservation, conserved symplectic form, ...)
- Causality holds in the sense
 "pairs of points with spacelike separation do not interact" in particular: no superluminal signalling
- In approximation ("approximation of inhomogeneous fluctuating fields") one gets linear and unitary time evolution

 $U_{t_0}^t$: $\mathcal{F} \to \mathcal{F}$

As observed by J. Kleiner, this seems to indicate that causal fermion systems are an effective collapse theory.

A. Bassi, D. Dürr, G. Hinrichs, *"Uniqueness of the equation for quantum state vector collapse,"* Phys. Rev. Lett. **111**, 210401 (2013)

- No faster-than-light signalling
- Time evolution Markovian and homogeneous in time
- \implies collapse theory

Can this be adapted to causal fermion systems?

Summary and Outlook

No deterministic laws:

not possible to proceed in time steps

- No strong causation
- But causal propagation on macroscopic scales
 - based on positivity properties of surface layer integrals (energy estimates, ...)
- Causal fermion system approach is background-free
- correct limiting cases:
 - classical field theory: strong, electroweak forces and gravity
 - quantum field theory (work in progress)

What does causality mean?

There are causal relations:

- distinction spacelike, timelike
- direction of time
- Locality holds:

Spacetime regions with spacelike separation have independent dynamics

BUT

- relation "lies in the future of" not necessarily transitive
- no causation