Quantum field theory without quantisation – from standard subspaces to observable algebras

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joint work with Ricardo Correa da Silva

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FAU

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Plan of the talk

1. Free QFT without quantisation
2. Standard subspaces

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1. Free QFT without quantisation
2. Standard subspaces
3. Twisted Araki-Woods algebras: Derivation of crossing symmetry and Yang-Baxter equation

Free QFT

Typical setup of free QFT:

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$$\varphi_{\mathsf{cl}}(x)$$

classical free field

Free QFT

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$$\varphi_{\mathsf{cl}}(x) \rightsquigarrow \phi_{\mathsf{qu}}(x)$$

quantum free field (distribution)

Free QFT

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$$\varphi_{\mathsf{cl}}(x) \rightsquigarrow \phi_{\mathsf{qu}}(x) \rightsquigarrow \phi_{\mathsf{qu}}(f) = \int dx\, f(x)\phi(x)$$

smeared quantum free field (operator)

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Weyl operators (bounded functions of quantum free field)

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- Classical limit $\phi_{\mathsf{qu}} \to \varphi_{\mathsf{cl}}$ more meaningful than $\varphi_{\mathsf{cl}} \to \phi_{\mathsf{qu}}$
- In free QFT, quantisation can be avoided completely.

Local Subspaces of a massive Klein-Gordon field

Two main ideas [Brunetti/Guido/Longo '02]:

1. Base the construction on description of particle content (representation of Poincaré group → masses, spins)
2. Focus on "local subspaces". For $\mathcal{O} \subset \mathbb{R}^d$ (Minkowski space) consider

$$H(\mathcal{O}) = \{\hat{f} = \phi_{\mathsf{qu}}(f)\Omega \,:\, \mathrm{supp}(f) \subset \mathcal{O}, \quad f \text{ real}\}^-$$

$$\text{Ex.: } (\boldsymbol{p} \mapsto \hat{f}(\boldsymbol{p}) = \tilde{f}(\sqrt{\|\boldsymbol{p}\|^2 + m^2}, \boldsymbol{p})) \in L^2(\mathbb{R}^{d-1}, \tfrac{d\boldsymbol{p}}{\sqrt{\|\boldsymbol{p}\|^2+m^2}})$$

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  That is, $H(\mathcal{O}_1) \subset H(\mathcal{O}_2)'$ (symplectic complement).
- $H(\mathcal{O}) + iH(\mathcal{O}) \subset \mathcal{H}$ is dense (Reeh-Schlieder property, **cyclicity**)
- $H(\mathcal{O}) \cap iH(\mathcal{O}) = \{0\}$ **separating**

Standard subspaces

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- Standard subspaces H are also mathematically interesting. Come with Tomita operator

$$S_H : H + iH \to H + iH, \qquad h_1 + ih_2 \mapsto h_1 - ih_2.$$

Polar decomposition $S_H = J_H \Delta_H^{1/2}$ yields an "internal dynamics" (unit. 1-par.grp Δ_H^{it}) and a "TCP operator" J_H,

$$\Delta_H^{it} H = H, \qquad J_H H = H', \qquad H = \ker(1 - J_H \Delta_H^{1/2})$$

H determines J_H, Δ_H and vice versa.

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- **Define:** $H(W) = \ker(1 - U(j_W)U(\lambda_W(i\pi)))$ by reflection j_W at edge of W and boost λ_W in direction of W (with W a **wedge region**)
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- Rest of the construction: Second quantisation:

$$\mathcal{H} \supset H \longrightarrow \mathcal{A}(H) = \{\text{Weyl}(h) : h \in H\}''$$

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Remark: Fock vacuum cyclic ($\mathcal{A}(H)\Omega$ dense) and separating ($\mathcal{A}(H)$ contains no annihilators)

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"Standard subspaces = localisation regions"
Not restricted to Minkowski space.

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- Rest of talk: Sketch a particular approach. Others exist (e.g. [Buchholz/L/Summers '11])

General Fock spaces

[Bożejko/Speicher '94; Jørgensen/Schmitt/Werner '95]

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$$P_{T,1} = 1, \quad P_{T,2} = 1 + T, \quad P_{T,3} = 1 + T_1 + T_2 + T_1 T_2 + T_2 T_1 + T_2 T_1 T_2,$$
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T-twisted Fock space

$$\mathcal{F}_T(\mathcal{H}) := \bigoplus_{n \geq 0} \overline{\mathcal{H}^{\otimes n} / \ker P_{T,n}}^{\langle \cdot , \cdot \rangle_{T,n}}$$

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An example from QFT

$\mathcal{H} = L^2(\mathbb{R}, d\theta)$, $s : \mathbb{R} \to S^1$, $s(-\theta) = \overline{s(\theta)}$. Then

$$(Tf)(\theta_1, \theta_2) = s(\theta_1 - \theta_2) \cdot f(\theta_2, \theta_1) \qquad \text{is a unitary twist.}$$

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Interpretation:

- Think of θ as rapidity and s as elastic two-body S-matrix.

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$$a^{\star}_{L,T}(\xi)\Omega = \xi, \qquad a_{L,T}(\xi)\Omega = 0, \qquad \Omega : \text{Fock vacuum}$$

$$a^{\star}_{L,T}(\xi)[\Psi_n] = [\xi \otimes \Psi_n], \qquad \Psi_n \in \mathcal{H}^{\otimes n},$$

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- **"Left field operators:"**

$$\phi_{L,T}(\xi) := a^\star_{L,T}(\xi) + a_{L,T}(\xi).$$

(Left) twisted Araki-Woods Algebra

$$\mathcal{L}_T(H) := \{\phi_{L,T}(h) : h \in H\}'' \subset \mathcal{B}(\mathcal{F}_T(\mathcal{H}))$$

with $H \subset \mathcal{H}$ a standard subspace.

This coincides with the local observable algebras of the Klein-Gordon field for suitable $H = H(\mathcal{O})$ and $T = F$.

Questions

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In the following: $H \subset \mathcal{H}$ an arbitrary standard subspace (i.e. arbitrary modular group Δ_H^{it}), and T a twist.

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Basic assumption: T and H are **compatible** in the sense $[T, \Delta_H^{it} \otimes \Delta_H^{it}] = 0$.

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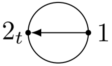
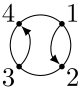
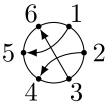
- Graphical notation



$\langle J_H h_1, \Delta_H^{it} h_2 \rangle$,

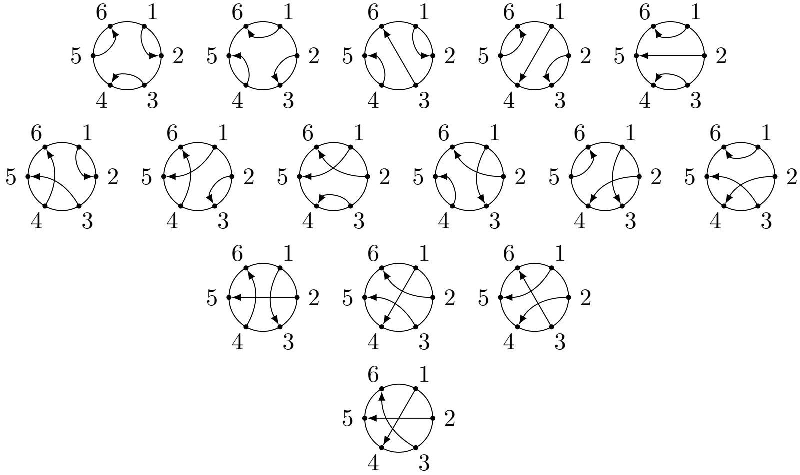


$\langle \bar{1}, 2 \rangle \cdot \langle \bar{3}, \Delta_H^{it} 4 \rangle$,



$\langle \bar{3} \otimes T(\bar{2} \otimes \bar{1}), T(4 \otimes 5) \otimes 6_t \rangle$

Six-point function $\langle 1\,2\,\ldots\,6_t \rangle$



By imposing the KMS condition, one can extract two properties of T:

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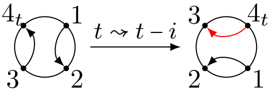

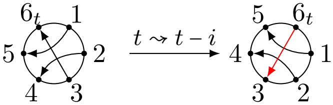

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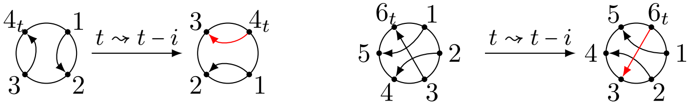
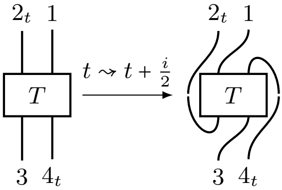


$\langle 2_t \otimes 1, T(3 \otimes 4_t) \rangle =$



$= \langle 1 \otimes \bar{4}_t, T(\bar{2}_t \otimes 3) \rangle$

This is a condition on T.

Definition

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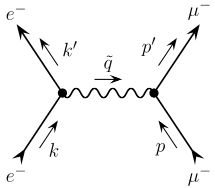

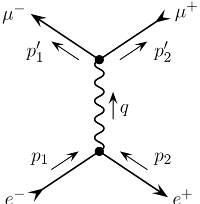

2 Yang-Baxter equation

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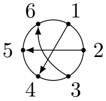
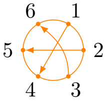
The two possible triple crossing terms in the 6-point function differ by a Reidemeister move of type III ($\rightsquigarrow$ knot theory).





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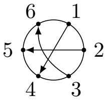

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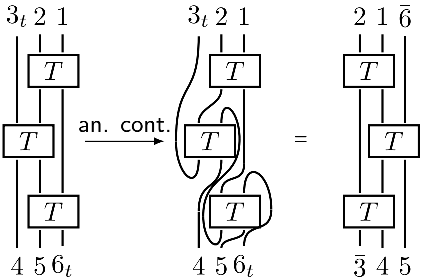

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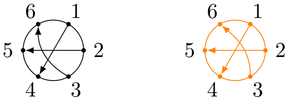

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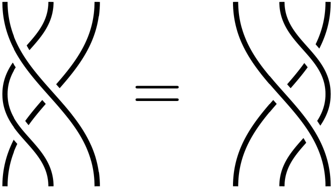

By exploiting KMS condition, one can show that one must have RHS = LHS

$$T_1 T_2 T_1 = T_2 T_1 T_2 \qquad \text{Yang-Baxter equation}$$

Theorem ([Correa da Silva/L '22])

Let $H \subset \mathcal{H}$ be a standard subspace and T a compatible twist. The following are equivalent:

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- In situation of theorem, also have *right* fields/algebras, and **left-right duality**

$$\mathcal{L}_T(H)' = \mathcal{R}_T(H').$$

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- Deformations with $\|T\| < 1$ are better accessible with operator algebra methods but more non-local (extreme case is $T = 0$ — serve as non-local counterexamples)