Quantum Mechanics as a Complete Theory[1]

"One is [thus] led to conclude that the description of reality as given by a wave function is not complete."

(A. Einstein, B. Podolsky, N. Rosen)

Three heros of quantum theory:

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Contents and credits

Contents:

Credits:

I wish to thank my collaborators on projects related to this lecture, including my last PhD student *Baptiste Schubnel* and my friend *Alessandro Pizzo*, for the joy of joint efforts, and numerous colleagues in many places for useful discussions on *Quantum Mechanics*. – I thank *Felix Finster* and his colleagues for giving me an opportunity to present ideas and results (as of now largely ignored) on **problems** that, I think, are truly **fundamental**.

Summary

I propose ideas about how to complete (non-relativistic) *Quantum Mechanics (QM)* to a theory that makes sense. My proposal is called

ETH - **Approach** to *QM*

"*E*" standing for Events, "*T*" for Trees, and "*H*" for Histories.[2] This approach supplies the last one of **three pillars** *QM* can be constructed upon, which are:

(i) Physical quantities characteristic of a system are represented by selfadjoint operators. Their time evolution is given by the *Heisenberg equations*.

(ii) Meaningful notions of *states* and of *potential* and *actual events*.

(iii) A general *Law for the Time Evolution of states* of a system.

After explaining some general ideas underlying the *ETH*-Approach I will discuss an application to the quantum theory of fluorescence of an atom coupled to the radiation field.

My general goal is to help removing some of the **enormous confusion** befuddling many people who claim to work on the foundations of *QM*.

[2] The *ETH*-Approach to *QM* can be reconciled with *Relativity Theory*.

1. What's missing in text-book *QM*?

"It seems clear that the present quantum mechanics is not in its final form." (Paul Adrien Maurice Dirac)

Text-book Quantum Mechanics is a theory of (*ensembles* of identical) physical systems and of their time evolution – alas, incomplete – based on the following two pillars:

(i) A system, S, is characterized by a list

$$\mathcal{O}_S = \left\{ \widehat{X}_\iota = \widehat{X}_\iota^* \middle| \iota \in \mathfrak{I}_S \right\}$$

of abstract bounded self-adjoint operators, where $\mathfrak{I}_S$ is a continuum set of indices, and where every operator $\widehat{X} \in \mathcal{O}_S$ represents a *physical quantity* characteristic of S, such as the total momentum, energy or spin of all particles localized in a specified bounded region of physical space and belonging to an ensemble of (possibly ∞ many) particles constituting the system S. [Different operators in $\mathcal{O}_S$ do in general *not* commute with one another. One assumes that if $\widehat{X} \in \mathcal{O}_S$ and F is a real-valued, bounded continuous function on $\mathbb{R}$ then $F(\widehat{X}) \in \mathcal{O}_S$, too; in general $\mathcal{O}_S$ does not have any additional structure (it is usually not a real linear space, let alone an algebra).]

The two pillars text-book *QM* is based upon

At every time t, there is a representation of $\mathcal{O}_S$ by bounded self-adjoint operators acting on a separable Hilbert space $\mathcal{H}$:

$$\mathcal{O}_S \ni \widehat{X} \mapsto X(t) = X(t)^* \in B(\mathcal{H}) \tag{1}$$

Heisenberg picture: If S is an **isolated** system then the operators $X(t)$ and $X(t')$ representing a physical quantity $\widehat{X} \in \mathcal{O}_S$ at two arbitrary times t and t' are unitarily conjugated to one another,

$$X(t') = U_S(t', t)^*\, X(t)\, U_S(t', t), \tag{2}$$

where $\{U_S(t', t) \big| t, t' \in \mathbb{R}\}$ is the unitary propagator of S. The system S is *autonomous* iff $U_S(t', t) = \exp[-i(t' - t)H_S/\hbar]$, for arbitrary t and t', where H_S is the Hamiltonian of S.

(ii) "States" of S are given by density matrices Ω, i.e., by non-negative trace-class operators on $\mathcal{H}$ of trace one. The expectation at time t of an operator $\widehat{X} \in \mathcal{O}_S$ in the "state" Ω of S is given by

$$\omega\big(X(t)\big) := \mathrm{Tr}\big(\Omega\, X(t)\big).$$

Ω is pure iff it is given by a rank-1 projection $P = P^* = P^2$.

The "blunder" of the Schrödinger picture and -equation

In text-book *QM*, it is usually assumed (following *Schrödinger*) that, in the *Heisenberg picture*, "states" of an isolated physical system are *independent* of time t, and, hence, that the Heisenberg picture is equivalent to the *Schrödinger picture*:

$$\omega(X(t)) = \mathrm{Tr}\big(\Omega\, X(t)\big) = \mathrm{Tr}\big(\Omega(t)\, X\big), \quad X := X(t_0),\, \Omega := \Omega(t_0),$$

where

$$\Omega(t) = U_S(t,t')\,\Omega(t')\,U_S(t,t')^*\,. \tag{3}$$

In the Schrödinger picture, physical quantities of a system S are thus represented by *time-independent* bounded operators X on $\mathcal{H}$, while the states of S depend on time in a way described by Eq. (3), or, equivalently, by the *Schrödinger-Liouville(-von Neumann)* equation

$$\dot{\Omega}(t) = -\frac{i}{\hbar}\big[H(t), \Omega(t)\big]\,.$$

More generally, the time-dependence of states of a system S interacting with its environment is described by *completely positive linear* maps, $\big\{\Gamma(t,t')\big|\, t \geq t'\big\}$, with $\Gamma(t,t') = \Gamma(t,t'')\cdot\Gamma(t'',t')$ and $\Gamma(t,t) = \mathbf{1}$:

$$\Omega(t) = \Gamma(t,t')\big[\Omega(t')\big]\,, \qquad \forall t' \geq t\,. \tag{4}$$

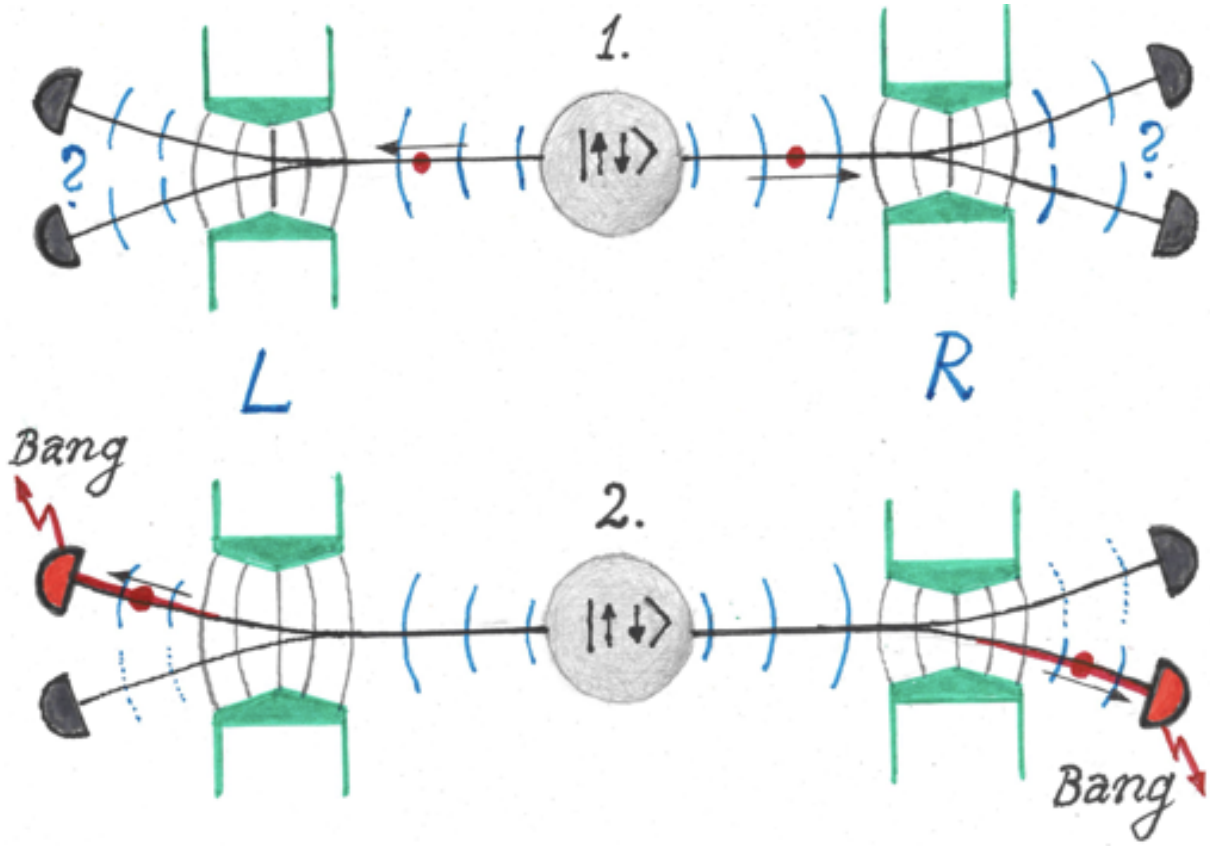
But what about the probabilistic nature of *QM*?

One thus observes that, in text-book *QM*, the *time evolution of states* in the Schrödinger picture (see Eqs. (3), (4)) is **linear** and **deterministic**!

Of course, this **cannot be the full story**, as already discovered by *Einstein* in 1916 (*A*- and *B*-coefficients). According to the *Copenhagen interpretation* of *QM*, the deterministic evolution of the state of S is *"interrupted"* at all times t when an *"event"* happens, such as the emission or absorption of a photon by an atom, or the completion of a measurement of the value of a physical quantity $\widehat{X} \in \mathcal{O}_S$. In the second case, the "state" of S exhibits a *"quantum jump"* to a state in the range of the spectral projection of $X(t)$ corresponding to the eigenvalue of $X(t)$ equal to the value of $\widehat{X}$ measured at time t. *QM* is claimed to give the *probabilities* of quantum jumps to eigenstates corresponding to *different possible values* of $\widehat{X}$: These probabilities are given by *Born's Rule*.

If the machinery used to measure the value of $\widehat{X}$ is *included* in what constitutes the *total* system S (assumed to be *isolated*) one might expect (erroneously) that the event corresponding to a measurement of the value of $\widehat{X}$ could be viewed as the result of the *Schrödinger-Liouville evolution of the state of the* **total** *system*. This would imply that *QM* is deterministic – **which it obviously isn't!** So, what's going on?

The example of a Stern-Gerlach experiment



1. $\leftrightarrow$ Schrödinger-Liouville evolution. Cluster properties $\Rightarrow$ "**no** correlations between L and R" – if whatever is done in L to generate an *event* could be described by *Schrödinger evolution of the total system* then that event would **not** bias what will be observed in R! $\Rightarrow$ discredits *Everett*!

2. $\leftrightarrow$ Illustration of evolution of system as seen in experiments.

Analogy with Brownian motion

Only the *events* "$\underbrace{\uparrow - \downarrow}_{L \quad R}$" and "$\underbrace{\downarrow - \uparrow}_{L \quad R}$" are observed in experiments.

Schrödinger evolution plus *Lüders' collapse postulate* (Copenhagen "mumbo-jumbo") $\Rightarrow$ probabilities of these two events are each $= \frac{1}{2}$.

How does a completion of QM describe this coherently?

A useful analogy: Consider a system consisting of very many identical "test particles" suspended in a liquid and exhibiting *Brownian motion*. Let $\rho_t(x)$ be the normalized particle density in $x \in \mathbb{R}^3$, i.e., the "state" of the system, at time t; its time-dependence is governed by the **diffusion equation**, viz. by a *deterministic linear* law of evolution:

$$\dot{\rho}_t(x) = D\,(\Delta\rho_t)(x), \qquad D : \text{diffusion constant.} \tag{5}$$

Solution:

$$\rho_t(x) = \int_{\mathbb{R}^3} d^3x'\, \Gamma_{t-t'}(x-x')\rho_{t'}(x'), \quad \Gamma_t(x) := (2\pi Dt)^{-\frac{3}{2}} e^{-\frac{|x|^2}{2Dt}},$$

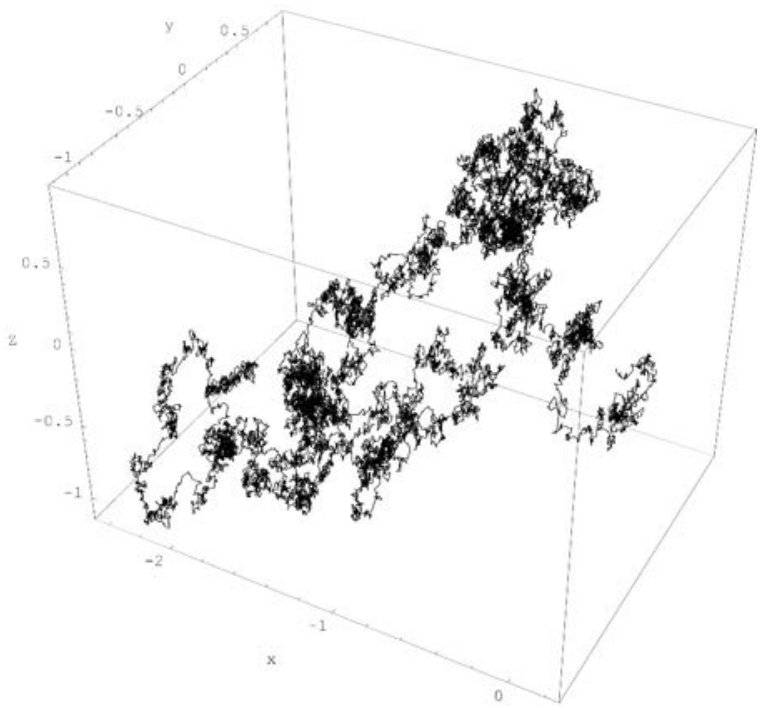
The operator kernels Γ_t satisfy the *Chapman-Kolmogorov* equation.

Ontology of Brownian motion according to Einstein, Smoluchowski and Wiener

Ontology: Consider a system consisting of a single test particle suspended in a fluid. Its Brownian motion arises from its random collisions with compounds of fluid molecules ($\Rightarrow D = \frac{k_B T}{6\pi\eta r}$, ...); and

(i) the test particle is localized in a point $x_\xi(t) \in \mathbb{R}^3$ at every time t;

(ii) its trajectory $\xi := \left\{x_\xi(t)\right\}_{t \geq t_0}$ is a random continuous curve in physical space $\mathbb{R}^3$, as indicated in the figure below.



A measure on the "space of histories"

As shown by *N. Wiener*, $\exists$ a **probability measure**, $dW_{x_0}(\xi)$, on the space, Ξ, of particle trajectories, $\xi := \left\{ x_\xi(t) \in \mathbb{R}^3 \middle| t \geq t_0, x_\xi(t_0) = x_0 \right\}$, starting at x_0 at time t_0; dW_{x_0} is supported on trajectories ξ that are *Hölder continuous of index* $\frac{1}{2}$, etc.

An *"event"* at time t is the position, $x_\xi(t)$, of the test particle. The trajectory ξ can thus be viewed as a *"history of events,"* a **random object**, and Ξ is the *"space of histories."* Wiener measure allows us to predict probabilities of *measurable sets of histories*. Example:

$$\begin{aligned} &\text{prob}\left\{ \xi \in \Xi \middle| x_\xi(t_i) \in \mathcal{O}_i,\ i = 1, 2, \ldots, n,\ t_0 < t_1 < \cdots < t_n \right\} \\ &\qquad = \int_\Xi dW_{x_0}(\xi) \prod_{i=1}^{n} \chi\big(\xi \big| x_\xi(t_i) \in \mathcal{O}_i\big)\,. \end{aligned} \tag{6}$$

Using Wiener measure to take an average over an ensemble of *very many identical systems of test particles*, one recovers the **deterministic law** in Eq. (5) for the evolution of the "state" ρ_t:

$$\begin{aligned} \rho_t(x) &= \int d^3x_0\, \Gamma_{t-t_0}(x - x_0)\rho_{t_0}(x_0) \\ &= \int d^3x_0\, \rho_{t_0}(x_0) \int_\Xi dW_{x_0}(\xi)\, \chi\big(\xi \big| x_\xi(t) = x\big). \end{aligned} \tag{7}$$

2. "Unraveling" the Schrödinger-Liouville equation

"If you are receptive and humble, mathematics will lead you by the hand." (P.A.M. Dirac)

Chapman-Kolmogorov for $\Gamma_t \Rightarrow$ Markov property for dW_{x_0}. One says that the Wiener measure *"unravels"* the diffusion equation (5).

In the following I propose a completion of *QM* involving an *"unraveling"* of the Schrödinger-Liouville equation for the propagation of "states." The **upshot of my analysis** will be that the Schrödinger-Liouville equation can be understood to arise from taking an *average over the random histories of many identical, identically prepared systems* (= average over stochastic evolutions of their states).

The **ontology of** *QM* lies in *"random histories of events"*; and *QM* equips the (non-commutative) space of such histories with a *"quantum probability measure"*, in analogy with the Wiener measure for Brownian motion. Our task is to *find* this probability measure, or, put differently, *to find an appropriate notion of states of physical systems and to describe their stochastic time evolution.* – The *ETH- Approach to QM*, developed during the past decade, accomplishes this task! It will be sketched for non-relativistic *QM*; (but there exists a *relativistic version* of it).

Elements of the *ETH*- Approach to *QM*

It is convenient to introduce some natural algebras.

$$\mathcal{E}_I := \big\langle X(t) \big| \widehat{X} \in \mathcal{O}_S, t \in I \subset \mathbb{R} \big\rangle, \qquad \mathcal{E}_{\geq t} := \overline{\bigvee_{I \subset [t,\infty)} \mathcal{E}_I}\,, \tag{8}$$

[where we take a closure in the weak topology of $B(\mathcal{H})$, ...]. Clearly

$$\mathcal{E}_{\geq t} \supseteq \mathcal{E}_{\geq t'}, \quad \text{whenever } t' > t\,.$$

Definition 1: Let S be an isolated (...) physical system. **Potential (future) events** in S (*"potentialities"*) at times $\geq t$ are described by partitions of unity,

$$\big\{\pi_\xi \,\big|\, \xi \in \mathfrak{X}\big\} \subset \mathcal{E}_{\geq t}, \quad \mathfrak{X} \text{ countable}\,, \sum_{\xi \in \mathfrak{X}} \pi_\xi = \mathbf{1}\,, \tag{9}$$

by *orthogonal projections*, $\pi_\xi = \pi_\xi^* = \pi_\xi^2 \in \mathcal{E}_{\geq t}$.

In the following, the symbol π will always stand for an orthogonal projection $\pi = \pi^* = \pi^2$ acting on $\mathcal{H}$.

The Principle of Diminishing Potentialities

"Indeed, it is evident that the mere passage of time itself is destructive rather than generative ..., because change is primarily a 'passing away."

(Aristotle, Physics)

An *isolated* system S is characterized by a co-filtration, $\left\{\mathcal{E}_{\geq t} \,\middle|\, t \in \mathbb{R}\right\}$, of *algebras of potential future events; S* is *autonomous* iff

$$\mathcal{E}_{\geq t'} = e^{i(t'-t)H_S}\, \mathcal{E}_{\geq t}\, e^{i(t-t')H_S}\,, \quad (H_S \text{ Hamiltonian of } S).$$

The *Principle of Diminishing Potentialities (PDP)* is the statement that

$$\boxed{\mathcal{E}_{\geq t} \underset{\neq}{\supset} \mathcal{E}_{\geq t'}\,, \text{ whenever } t' > t \geq t_0.} \tag{10}$$

This principle characterizes *isolated* open systems, namely systems that can release *"events"* to the outside world. It can be proven to hold in (axiomatic QFT and in) simple models discussed below.

The analogue of the initial position x_0 of a Brownian test particle at time t_0 is a projection $\pi_0 \in \mathcal{E}_{\geq t_0}$ giving rise to a state $\Omega_{t_0} = \frac{\pi_0}{\mathrm{tr}[\pi_0]}$. A **state**, Ω_t, at time $t > t_0$ is a *quantum probability measure* on the *lattice of all potential events in* $\mathcal{E}_{\geq t}$, which (by theorems of *Gleason* and *Maeda*) is the same as a normal state on $\mathcal{E}_{\geq t}$.

Actual events

Given a state Ω_t at time t, its *centralizer*, $\mathcal{C}_{\Omega_t}$, is the subalgebra of $\mathcal{E}_{\geq t}$ generated by all potential events $\left\{\pi_\xi \,\middle|\, \xi \in \mathfrak{X}\right\} \subset \mathcal{E}_{\geq t}$ commuting with Ω_t; i.e., $\Omega_t = \sum_{\xi\in\mathfrak{X}} \pi_\xi \Omega_t \pi_\xi$.

Let $\mathcal{Z}_{\Omega_t}$ be the abelian subalgebra of $\mathcal{C}_{\Omega_t}$ generated by all potential events $\left\{\pi_\xi \,\middle|\, \xi \in \mathfrak{X}\right\} \in \mathcal{C}_{\Omega_t}$ commuting with *all* the operators in the centralizer $\mathcal{C}_{\Omega_t}$. It is easy to see that $\mathcal{Z}_{\Omega_t}$ is generated by the projections of *one* potential event, $\left\{\pi_\xi \,\middle|\, \xi \in \mathfrak{X}_{\Omega_t}\right\} \in \mathcal{C}_{\Omega_t}$.

Definition 2 (Actualities): $\left\{\pi_\xi \,\middle|\, \xi \in \mathfrak{X}_{\Omega_t}\right\}$ is the potential event **actualizing** at time t, given that the state of S at time t is Ω_t.

We are now prepared to introduce a **Law** governing the **stochastic time evolution** of the state of an individual isolated system S. In order to be precise, mathematically, we temporarily suppose that *time is discrete*, i.e., $t \in \mathbb{Z}_\tau$, where $\tau > 0$ is an elementary time step.
Thanks to *PDP* and the phenomenon of **entanglement**, the following *state-reduction postulate* is meaningful and non-trivial:
Suppose that Ω_t is the state of S at time t; we define $\overline{\Omega}_{t+\tau}$ to be the state on $\mathcal{E}_{\geq t+\tau}$ obtained by *restriction* of Ω_t to $\mathcal{E}_{\geq t+\tau} \underset{PDP}{\subsetneq} \mathcal{E}_{\geq t}$.

The state-reduction postulate

Axiom CP: *Let* $\{\pi_\xi \,|\, \xi \in \mathfrak{X}_{\overline{\Omega}_{t+\tau}}\}$ *be the potential event actualizing at time* $t+\tau$, *given the state* $\overline{\Omega}_{t+\tau}$.

Then **'Nature'** *replaces the state* $\overline{\Omega}_{t+\tau}$ *on* $\mathcal{E}_{\geq t+\tau}$ *by a state*

$$\Omega_{t+\tau} \equiv \Omega_{t+\tau,\xi} := \left[\mathrm{tr}(\overline{\Omega}_{t+\tau}\,\pi_\xi)\right]^{-1} \cdot \pi_\xi \overline{\Omega}_{t+\tau} \pi_\xi\,, \tag{12}$$

for some $\xi \in \mathfrak{X}_{\overline{\Omega}_{t+\tau}}$, *with* $\mathrm{tr}\big(\overline{\Omega}_{t+\tau}\,\pi_\xi\big) \neq 0$.

The probability, $prob_{t+\tau}(\xi)$, *for the state* $\Omega_{t+\tau,\xi}$ *to be selected by* **'Nature'** *as the state of* S *at time* $t+\tau$ *is given by*

$$prob_{t+\tau}(\xi) = \mathrm{tr}[\overline{\Omega}_{t+\tau}\,\pi_\xi] \qquad (\textit{generalized Born Rule}) \tag{13}$$

The projection $\pi(t+\tau) := \pi_\xi \in \mathcal{Z}_{\overline{\Omega}_{t+\tau}}$ appearing in (12) is called **actual event**, or *"actuality"*, at time $t+\tau$.
With a **history**, $\{\pi(t_0+\tau), \ldots, \pi(t)\}$ of actual events (given an initial event π_0 at time t_0) we associate a *"history operator"*

$$H_{\pi_0}(t_0, t) := \pi_0 \prod_{t_0 < t' \leq t} \pi(t'), \qquad \text{with} \quad t' \in \mathbb{Z}_\tau\,.$$

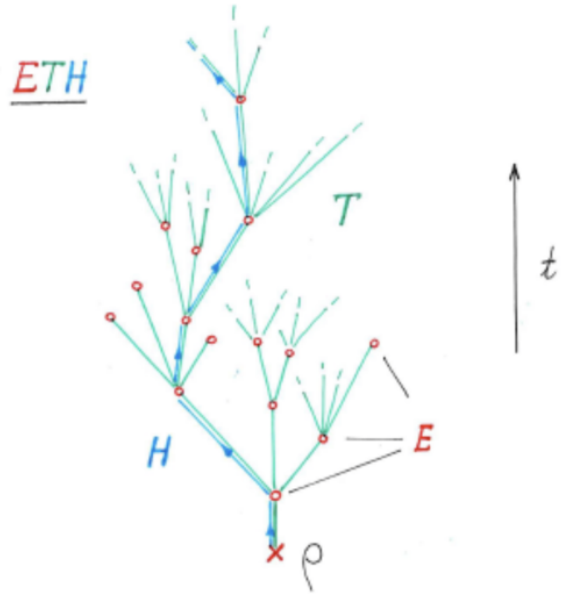
Predicting the probabilities of histories

The analogue of the Wiener measure, dW, is given by

$$prob_{\pi_0}\{\pi(t_0+\tau),\ldots,\pi(t)\} := \frac{\mathrm{tr}\left[H_{\pi_0}(t_0,t)^* \cdot H_{\pi_0}(t_0,t)\right]}{\mathrm{tr}[\pi_0]} \qquad (14)$$

Apparently, the time-evolution of the *state* of a physical system S is described by a *stochastic branching process* (a **"quantum Poisson process"**), with branching rules as determined by **Axiom CP**. This is meaningful, mathematically, if $\tau > 0$; but, for the time being, the limiting theory, as $\tau \searrow 0$, is only well understood in examples (Sects. 3, 4).

The usual Schrödinger-Liouville evolution arises when one takes an average over all possible histories. The following is metaphorical:



3. Huygens' Principle and *PDP*

"... principles are tested by inferences which are derivable from them."

(Christiaan Huygens)

Fact: *Huygens' Principle* for massless modes (photons, gravitons, ...) in isolated physical systems

$$\Rightarrow \quad \underline{\textit{Principle of Diminishing Potentialities}}\,!$$

Example: S an isolated system consisting of a *static atom* located near $\mathbf{x} = 0$, coupled to the *electromagnetic field*.

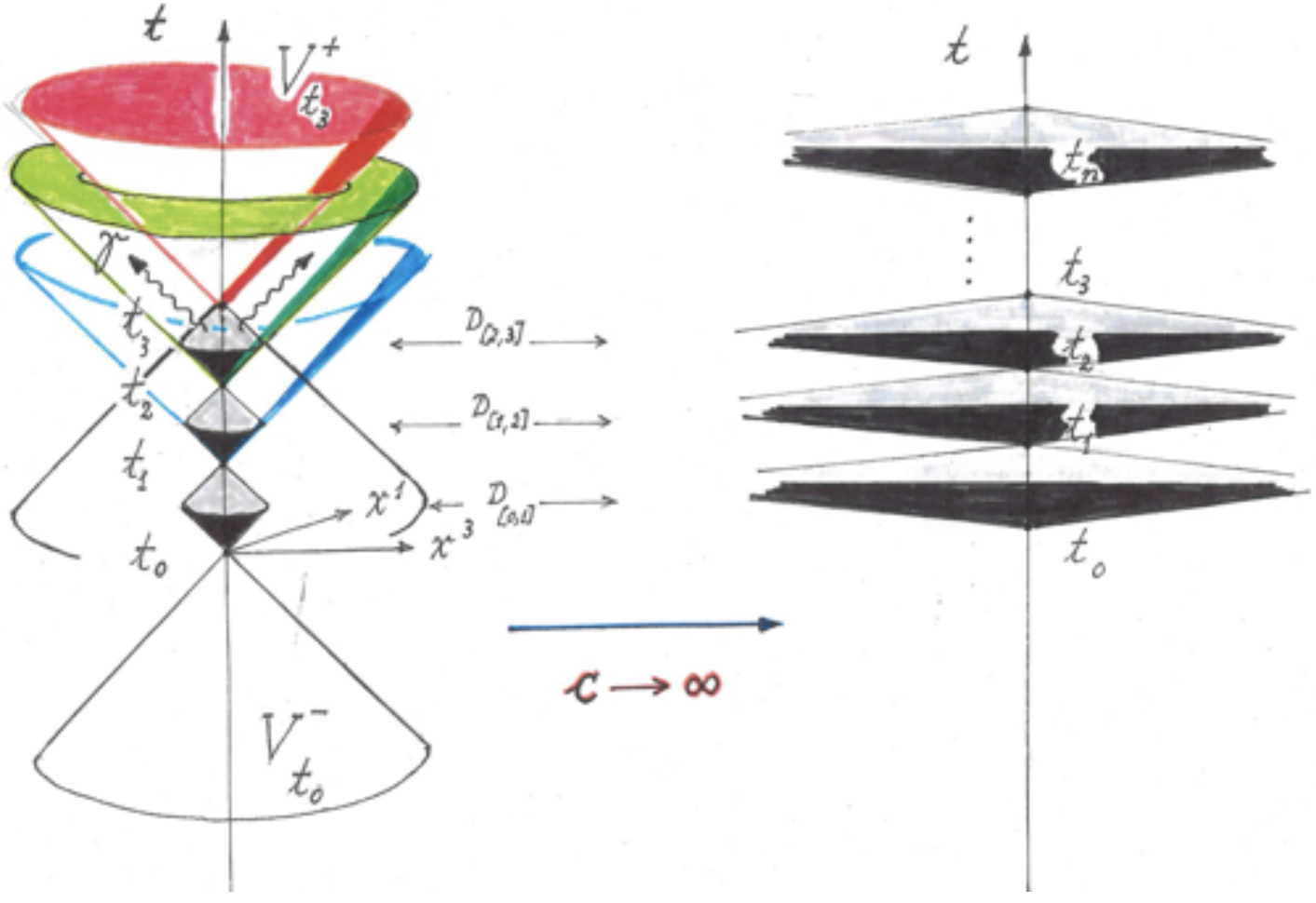
- Atom has M energy levels, Hilbert space $\mathfrak{h}_A \simeq \mathbb{C}^M$.
- Hilbert space of free e.m. field = *Fock space*, $\mathfrak{F}$, of photons; the e.m. field is described by field tensor, $F_{\mu\nu}(\tau, \mathbf{x})$, with the property that, for real-valued test fus. $\{h^{\mu\nu}\}$ on space-time,

$$F(h) := \int_{\mathbb{R}\times\mathbb{R}^3} d\tau\, d\mathbf{x}\, F_{\mu\nu}(\tau, \mathbf{x})\, h^{\mu\nu}(\tau, \mathbf{x})$$

is a self-adjoint op. on $\mathfrak{F}$ and satisfies **locality**. The usual Hamiltonian of the free e.m. field is denoted by H_f; with $H_f = H_f^* \geq 0$ on $\mathfrak{F}$.

Light cones, space-time diamonds, time slices

We consider space-time diamonds $D_{[t,t']} := V_t^+ \cap V_{t'}^-$, $t' > t$, centered on time axis ($\mathbf{x} = 0$), and time slices arising as $c \to \infty$:



space-time diamonds c= velocity of light time slices in NR limit, $c \to \infty$

A concrete model of S

Hilbert space of S:

$$\mathcal{H} := \mathfrak{F} \otimes \mathfrak{h}_A \,.$$

Bounded functions of field operators $F(h)$, $\mathrm{supp}(h^{\mu\nu}) \subseteq D_{[t,t']}$, generate a von Neumann algebra $\mathcal{A}_{I=[t,t']}$. We define algebras

$$\mathcal{D}_I^{(0)} := \mathcal{A}_I \otimes \mathbf{1}\big|_{\mathfrak{h}_A}\,, \qquad \mathcal{E}_I^{(0)} := \mathcal{A}_I \otimes B(\mathfrak{h}_A)\,,$$
$$\mathcal{E}_{\geq t}^{(0)} := \overline{\bigvee_{I \subset [t,\infty)} \mathcal{E}_I^{(0)}}\,. \tag{15}$$

PDP holds for the non-interacting system: Setting $I := [t, t']$, one has

$$\boxed{\left[\mathcal{E}_{\geq t'}^{(0)}\right]' \cap \mathcal{E}_{\geq t}^{(0)} = \mathcal{D}_I^{(0)} \quad (\text{an } \infty - \text{dim. algebra})\,.} \tag{16}$$

Remark: This follows from *"Huygens' Principle"*, namely from

$$[F_{\mu\nu}(x), F_{\rho,\sigma}(y)] = 0, \;\; \text{unless } x - y \text{ is } \textbf{light-like},$$

(see figure). From now on, we make use of an ultraviolet regularization of QED arising from discretizing time: $t_n := n\tau, n \in \mathbb{Z}, \tau > 0$.

PDP for an interacting model

To describe interactions, we pick a unitary op. $U \in \mathcal{E}^{(0)}_{[0,1]}$ and define

$$U_k := e^{i(k-1)\tau H_f}\, U\, e^{-i(k-1)\tau H_f}, \quad k = 1, 2, \ldots, \quad U(n) := \prod_{k=1}^{n} U_k\,,$$

$$\Gamma := e^{-i\tau H_f} U \;\Rightarrow\; \Gamma^n = e^{-in\tau H_f}\, U(n),\; (\Gamma^n)^* = \Gamma^{-n},\; n = 0, 1, 2, \quad (17)$$

$\{\Gamma^n\}_{n\in\mathbb{Z}}$: propagator of interacting systems with discrete time.

It suffices to consider time evolution for times $t \geq t_0 := 0$. Define

$$\mathcal{E} := \mathcal{E}^{(0)}_{\geq 0}, \quad \mathcal{E}_{\geq n} := \left\{ \Gamma^{-n}\, X\, \Gamma^n \,\middle|\, X \in \mathcal{E} \right\}. \quad (18)$$

Verification of PDP for interacting model: Using (17) and (18), one readily shows that

$$\left[\mathcal{E}_{\geq n'}\right]' \cap \mathcal{E}_{\geq n} \simeq \mathcal{D}_{[n,n']}\,, \;\text{ for }\; n' > n, \quad (19)$$

where $\mathcal{D}_{[n,n']} := \left\{ U(n')^*\, X\, U(n') \,\middle|\, X \in \mathcal{D}^{(0)}_{[n,n']} \right\}$.

Preparing the system in an initial state Ω_0 at time $n = 0$, one determines the time evolution of its state according to the *ETH*- Approach, as prescribed in *Definition 2* – actualities – and **Axiom CP** of Sect. 2. Explicit results are hard to derive because of memory effects. ...

4. Fluorescence of two-level atoms coupled to radiation

Matters simplify drastically in the non-relativistic limit, $c \to \infty$. The space-time diamonds then open up to time slices, $\{k\tau \leq t < (k+1)\tau\}$, and functionals of the radiation field localized in different time slices **commute**. The field Hamiltonian H_f is replaced by the generator, $\mathfrak{P}$, of translations in the direction of the time axis, and the algebras $\mathcal{D}^{(0)}_{[k,k+1]}$ "collapse" to

$$\mathcal{D}^{(0)}_{[k,k+1]} \simeq B(\mathcal{H}_k), \text{ with } \mathcal{H}_k \stackrel{e.g.}{=} \mathbb{C}^N, \text{ for some } N \leq \infty, \forall\, k. \qquad (20)$$

The time evolution of suitably chosen states becomes Markovian, and very explicit results can be obtained.

An explicit example: Model of **fluorescence** of a 2-level atom. Every T seconds, an atom source releases an atom prepared in a superposition of a ground state, $|\downarrow\rangle$, and an excited state, $|\uparrow\rangle$. In less than T seconds, the atom propagates to an atom-detector where, e.g., the "observable" $X := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, acting on the Hilbert space, $\mathbb{C}^2$, of internal states of the atom, is measured. During the trip from source to detector, the atom may jump from $|\uparrow\rangle$ to $|\downarrow\rangle$ and emit a photon, γ (spontaneous emission: *Einstein*, 1916). Different atoms are treated as statistically independent.

Two different experimental setups

1. Photons possibly emitted by an atom between source and detector escape the experimental setup and are not detected.
2. Photons possibly emitted by an atom "immediately" hit a photo-multiplier that clicks before atom ends its trip to its own detector.

Hilbert space of system: $\mathcal{H}_S := \mathbb{C}^2 \otimes \mathfrak{F} \otimes \mathfrak{H}_\gamma$, where $\mathfrak{F}$ is the Fock space of the "radiation field," and $\mathfrak{H}_\gamma$ is the Hilbert space of the photo-multiplier. In setup 1, the photomultiplier is turned off.

States of an atom: Let $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ be the Pauli matrices. An atomic state is a density matrix, $\rho(\vec{n})$, acting on $\mathbb{C}^2$ given by

$$\rho(\vec{n}) := \frac{1}{2}\big(\mathbf{1}_2 + \vec{n} \cdot \vec{\sigma}\big), \quad \vec{n} \in \mathbb{R}^3, \text{ with } |\vec{n}| \leq 1\,. \tag{21}$$

$\rho(\vec{n})$ is *pure* iff $|\vec{n}| = 1 \Leftrightarrow \rho(\vec{n})$ is a rank-1 orthogonal projector. Moreover $\rho(\vec{n}) + \rho(-\vec{n}) = \mathbf{1}$, and $\mathrm{tr}\big(\vec{\sigma} \cdot \rho(\vec{n})\big) = \vec{n}$.

The matrix $\rho(\vec{n})$ has eigenvalues $\frac{1+|\vec{n}|}{2}$ and $\frac{1-|\vec{n}|}{2}$, with eigenspaces given by $\mathrm{Range}\big(\rho(\pm\vec{n})\big)$, respectively.

States and Hamiltonians of atom and radiation field

Atomic Hamiltonian: $H_A := (1/2)\vec{\omega} \cdot \vec{\sigma}$, with $\vec{\omega} = (0, 0, \Omega)$.

$$e^{itH_A} \rho(\vec{n}_0) e^{-itH_A} = \rho\big(\vec{n}(t)\big), \quad \vec{n}_0 = (\sin\theta_0 \cos\varphi_0, \sin\theta_0 \sin\varphi_0, \cos\theta_0),$$

where $\vec{n}(t) = (\sin\theta_0 \cos\varphi(t), \sin\theta_0 \sin\varphi(t), \cos\theta_0)$ and $\varphi(t) = \varphi_0 + \Omega \cdot t$.

The **radiation field** is treated in the non-relativistic limit $c \to \infty$.
States of radiation field: We introduce a "vacuum vector":
$|\emptyset\rangle := \bigotimes_{k \in \mathbb{Z}} |0\rangle_k$, where $|0\rangle_k$ is a specific ("no-photon") state in $\mathcal{H}_k$;
$\mathfrak{F} :=$ completion (in the natural norm) of the span of vectors
$\Phi = \bigotimes_{k \in \mathbb{Z}} \phi_k$, with $\phi_k = |0\rangle_k$, except for finitely many values of k.

$|\gamma\rangle$: a state of ≥ 1 photons (i.e., $\phi_k \neq |0\rangle_k$, for some k); and $\langle\gamma|\emptyset\rangle = 0$.

General states of the radiation field are density matrices on $\mathfrak{F}$.
Field Hamiltonian: Given by the operator $\mathfrak{P}$. The time evolution of the radiation field won't enter explicitly in the following sketch of results.

States of the photomultiplier: Won't appear explicitly in what follows. The only important feature is that the state of the "dormant" photomultiplier is **orthogonal** to all its states when hit by some photons.

Time evolution of atom coupled to the radiation field

We assume that the interaction between atom and radiation field is weak, corresponding to a coupling constant whose square is denoted by $\alpha \ll 1$.

Effective time evolution of an atom coupled to the radiation field:

1. Photomultiplier turned off: We choose an initial state of the radiation field not entangling/correlating modes at different times. The effective time evolution of an atom is then "Markovian" and can be determined *explicitly*: In notations inspired by those used in Section 2, with $\tau \mapsto dt$, $\overline{\Omega}_{t+\tau} \mapsto \overline{\vec{n}}(t+dt)$, we find that

$$\begin{aligned} \overline{\vec{n}}(t+dt) &= \vec{n}(t) + d\vec{n}(t)\,, \qquad \text{where} \\ d\vec{n}(t) &= \vec{\omega} \times \vec{n}(t)\, dt + dK\big[\vec{n}(t)\big], \quad \vec{\omega} := (0,0,\Omega) \end{aligned} \tag{22}$$

where dK is a "dissipative" linear map $\propto dt$ with the property that

$$|\overline{\vec{n}}(t+dt)| = 1 - \mathcal{O}(\alpha)\, dt < 1, \quad \text{except if } \vec{n}(t) = -\vec{e}_3.$$

Results

Applying **Axiom CP** of Sect. 2, the evolution of the state of an atom is found to be given by a **Poisson jump process** on the Bloch sphere:

$$+) \quad \vec{n}(t) \mapsto \vec{n}(t+dt) := \frac{\bar{\vec{n}}(t+dt)}{|\bar{\vec{n}}(t+dt)|} \quad \text{with prob. } 1-\mathcal{O}(\alpha)\,,$$

$$-) \quad \vec{n}(t) \mapsto \vec{n}(t+dt) := -\frac{\bar{\vec{n}}(t+dt)}{|\bar{\vec{n}}(t+dt)|} \quad \text{with prob. } \mathcal{O}(\alpha)\,. \tag{23}$$

The rate of this jump process is proportional to α, i.e., there is roughly one random jump from $\vec{n}(t)$ to the *antipode* of $\bar{\vec{n}}(t+dt)$ every $\mathcal{O}(\alpha^{-1})\,T$ seconds.
When entering the atom detector the state of the atom is given by

$$\rho(\vec{n}_{out}), \quad \text{with} \quad \vec{n}_{out} \approx (\sin\theta'\cos\varphi_{out}, \sin\theta'\sin\varphi_{out}, \cos\theta')\,, \tag{24}$$

where $|\theta' - \theta_0| = \mathcal{O}(\alpha)$ (no jump), $|\theta' - \pi + \theta_0| = \mathcal{O}(\alpha)$ (1 jump).

Results - ctd.

2. If the photomultiplier is turned on then, according to the *ETH*-Approach, the time evolution of the initial state, $\Psi_{in} := \rho(\vec{n}_0) \otimes P_0$ to the final state when the atom enters the atom detector is given by

$$\Psi_{in} \mapsto \Psi_{out}^{(0)} = \rho(\vec{n}_{out}) \otimes P_0, \quad \text{with prob.} \quad 1 - \mathcal{O}(\alpha), \tag{18}$$

where $\vec{n}_{out} = (\sin\theta_0 \cos\varphi_{out}, \sin\theta_0 \sin\varphi_{out}, \cos\theta_0)$; and

$$\Psi_{in} \mapsto \Psi_{out}^{(1)} = \rho(-\vec{e}_3) \otimes P_\gamma, \quad \text{with prob.} \quad \mathcal{O}(\alpha), \tag{19}$$

with $\rho(-\vec{e}_3) = |\downarrow\rangle\langle\downarrow|$.

If α is not very small the difference between the atomic out-states in the *absence* of the photomultiplier and in its *presence*, respectively, can be detected by measurements of suitable atomic "observables" in the atom detector (also correctly describable within the *ETH*- Approach (!)).

The *ETH*-Approach provides a mathematically rigorous treatment of the model considered here (where $c \to \infty$); and one can pass to the limit $\tau \equiv dt \to 0$! *Nothing like this appears to have been accomplished before.*

Summary and Conclusions ...

- The *ETH-Approach to Quantum Mechanics* provides a completion of *QM* leading to a logically coherent description of the **stochastic time evolution of states** of **individual** systems in *QM* (unravelling Schrödinger evolution) and of *events* and their recordings. It has resemblences with "Many Worlds," "GRW," ... , but supersedes these imprecise formalisms. It describes only **One World**: hopefully ours! Of course, it will have to stand the test of experiments, such as those in Sect. 4 and other experiments in quantum optics.
- As in the foundation of Special Relativity, *fields describing massless modes* (photons, gravitons, ...) and the even-dimensionality of space-time appear to play key roles in the foundation of the *Principle of Diminishing Potentialities (PDP)*, which is a corner stone of our completion of *QM* leading to a solution of the "measurement problem." (This has **not** been properly appreciated, so far!)
- In the *ETH*-Approach to *QM*, *PDP* introduces a fundamental "arrow of time," i.e., a distinction between past and future.

I thank you for your attention!