The Dynamics of the Hubbard Model

through

Stochastic Calculus

and

Girsanov Transformation

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Content:

45/91 slides are pictures or title/cover slides or appendix, motivation has 21 pictures

1. Setup and Motivation

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Bose-Hubbard Hamiltonian in d Dimensions (n.n. = nearest neighbors):

$$H = -J \sum_{\substack{i,j \\ \text{n.n.}}} (a_i^+ a_j + a_j^+ a_i) + \frac{U}{2} \sum_j a_j^+ a_j^+ a_j a_j + \sum_j \epsilon_j a_j^+ a_j$$

Bosonic Annihilation and Creation operators:

$$[a_i , a_j^+] = \delta_{i,j}$$

Cubic Lattice Γ:

$$j = (j_1, \cdots, j_d) \in \{1, 2, \cdots, L\}^d =: \Gamma$$

1. Setup and Motivation: Quantities of Interest

Initial state: N particles at some lattice site j_0

$$\psi_0 \quad := \quad \frac{(a^+_{j_0})^N}{\sqrt{N!}}\,|0\rangle$$

Time evolved state:

$$\psi_t \quad = \quad e^{\,-\,itH}\,\psi_0$$

Number of particles $\langle n_{j,t}\rangle$ at lattice site j at some time $t > 0$:

$$\langle n_{j,t}\rangle \quad := \quad \langle \psi_t, a_j^+ a_j\;\psi_t\rangle$$

Already for just two lattice sites 1 and 2, quite complex behaviour, Example:

1. Setup and Motivation: Example Two Site Bose-Hubbard Model

Two Site Bose-Hubbard model:

$$H \;=\; \varepsilon\,(\,a_1^+ a_2 \;+\; a_2^+ a_1\,) \;+\; u\,(\,a_1^+ a_1^+ a_1\, a_1 \;+\; a_2^+ a_2^+ a_2\, a_2\,)$$

$N = 20$ particles at lattice site $j_0 = 1$:

$$\psi_0 \;:=\; \frac{(a_1^+)^N}{\sqrt{N!}}\,|0\rangle$$

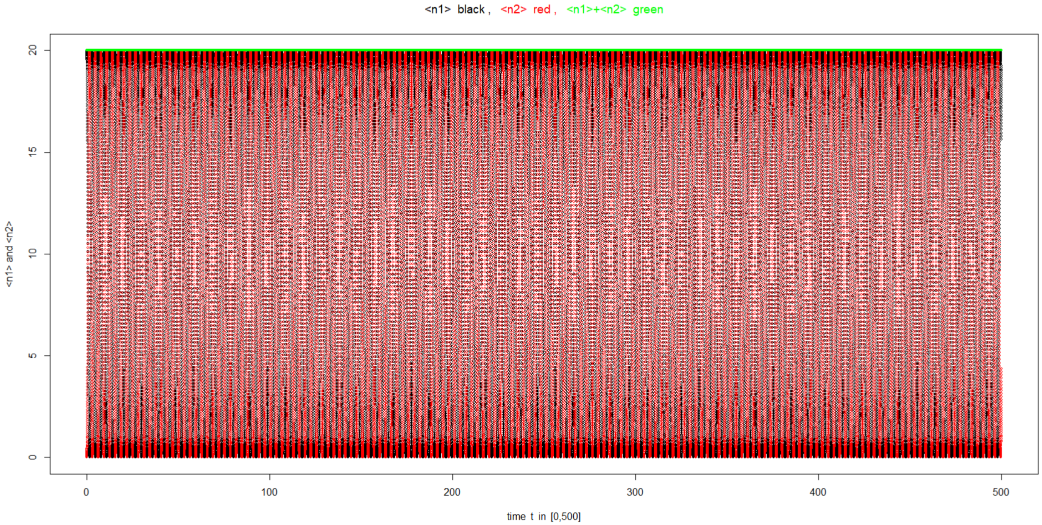
Plot $\langle n_{1,t}\rangle = \langle \psi_t, a_1^+ a_1\, \psi_t\rangle$ for

$$\begin{aligned} \varepsilon &= 1 \\ t &\in [0, 500] \\ N &= 20 \\ u &= g/N \end{aligned}$$

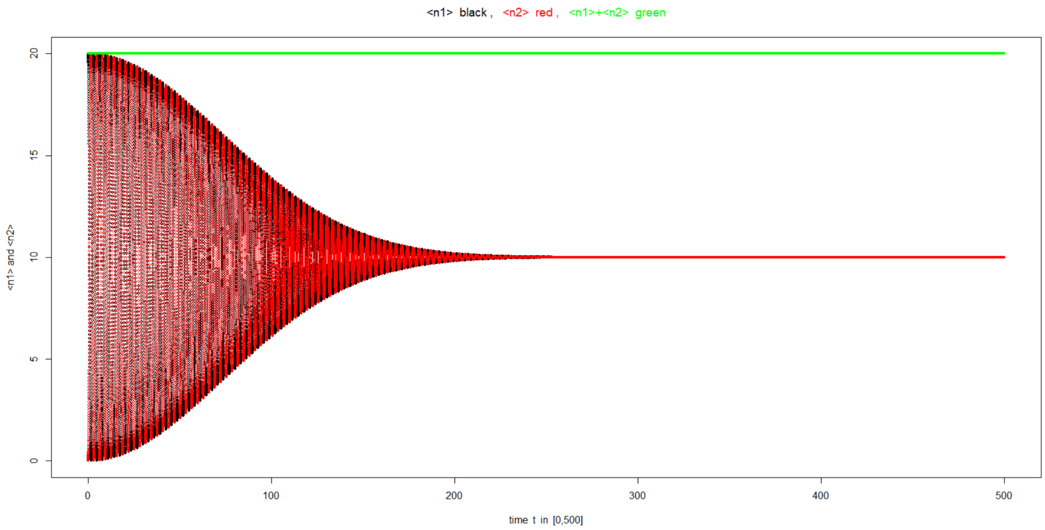
with interaction strength

$$g \;\in\; \{\,0\,,\, \tfrac{1}{16}\,,\, \tfrac{1}{8}\,,\, \tfrac{1}{4}\,,\, \tfrac{1}{2}\,,\, 1\,,\, 1.5\,,\, 2.0\,,\, 2.5\,,\, 3\,,\, 4\,,\, 5\,,\, 25\,\}$$

$g = 0$, non interacting case, $\langle n_{1,t} \rangle = N \cos^2 \varepsilon t$



$$g = 1/16$$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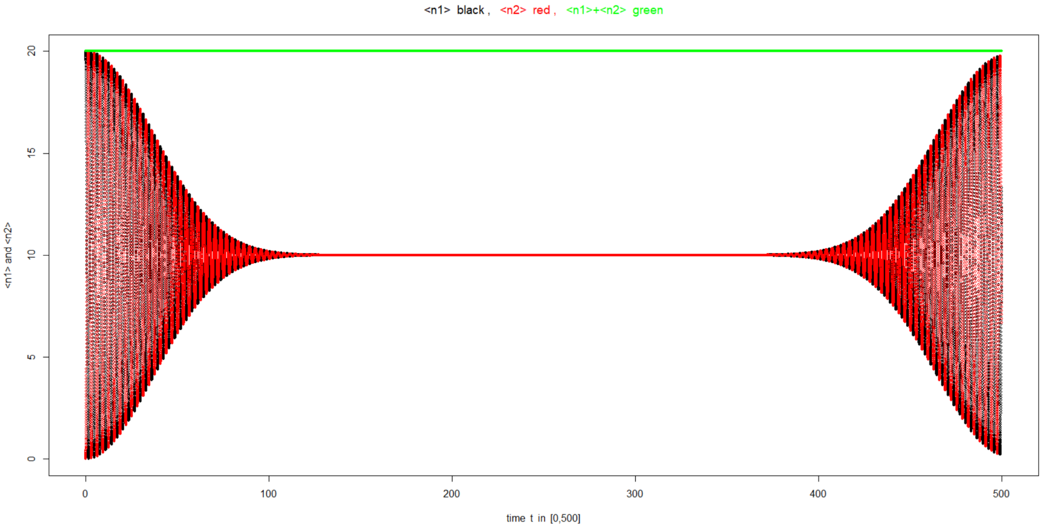
time t in [0,500]

$g = 1/8$



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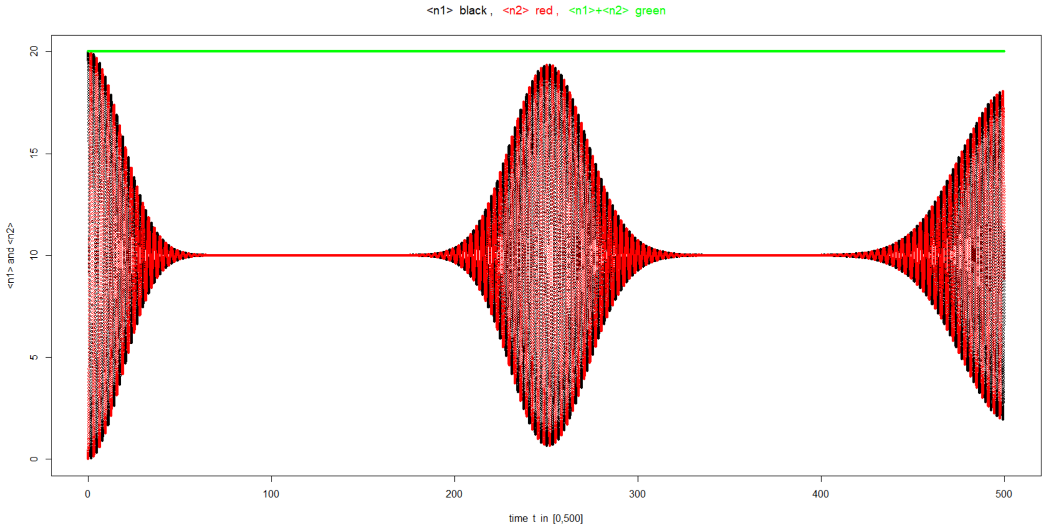
$$g = 1/4$$



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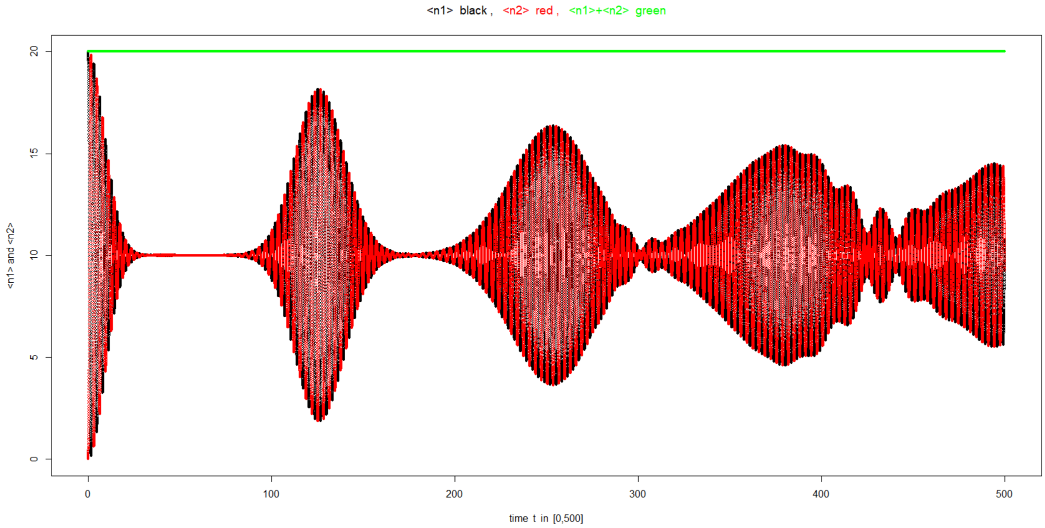
$g = 1/2$



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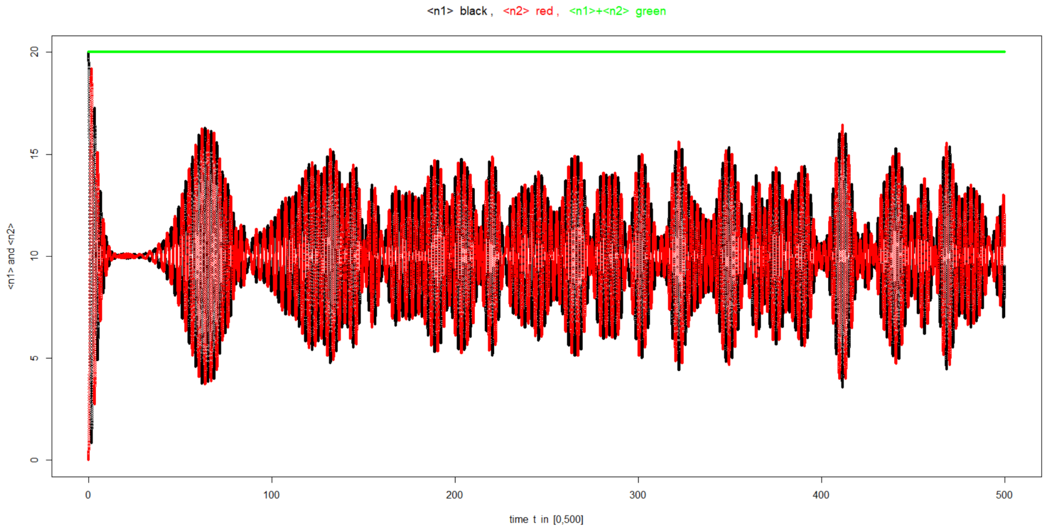
$g = 1$



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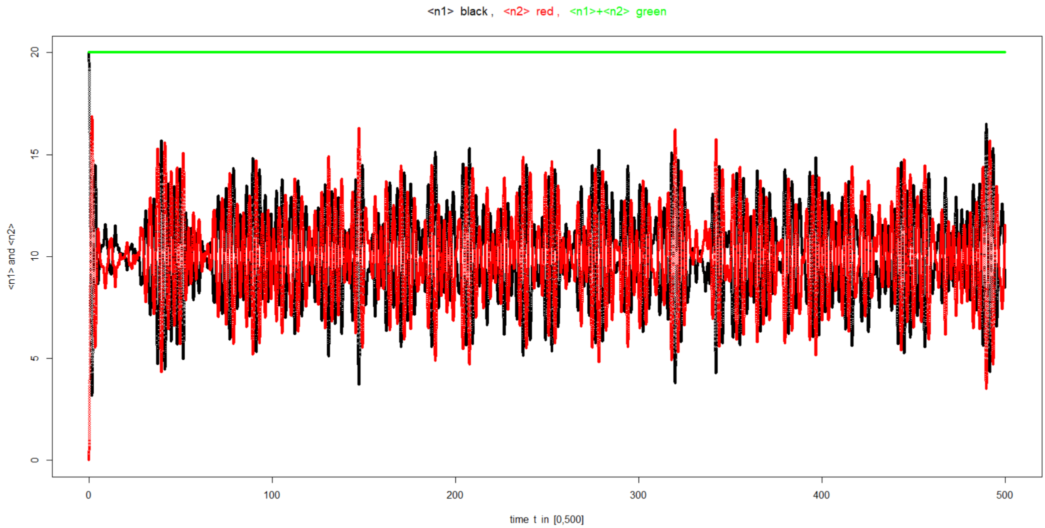
$g = 1.5$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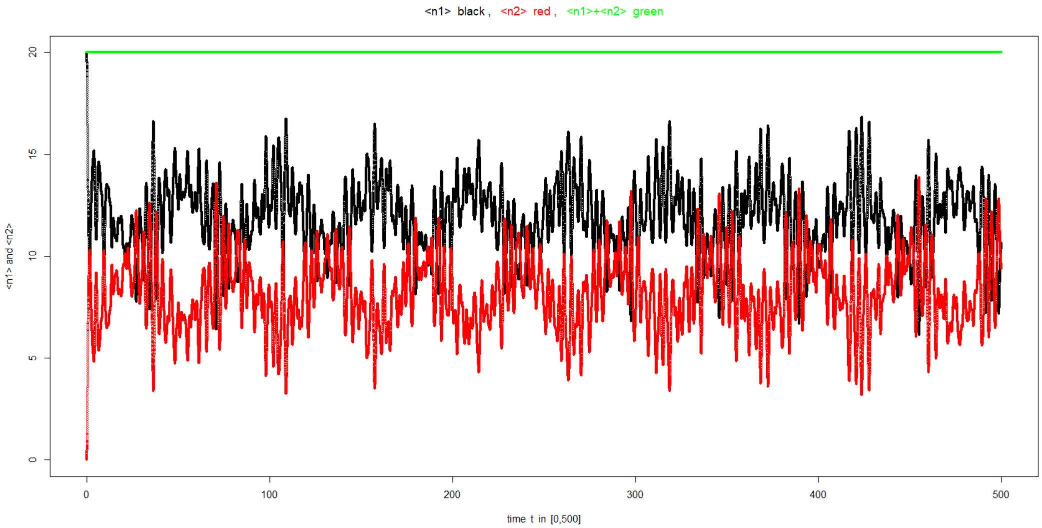
$g = 2.0$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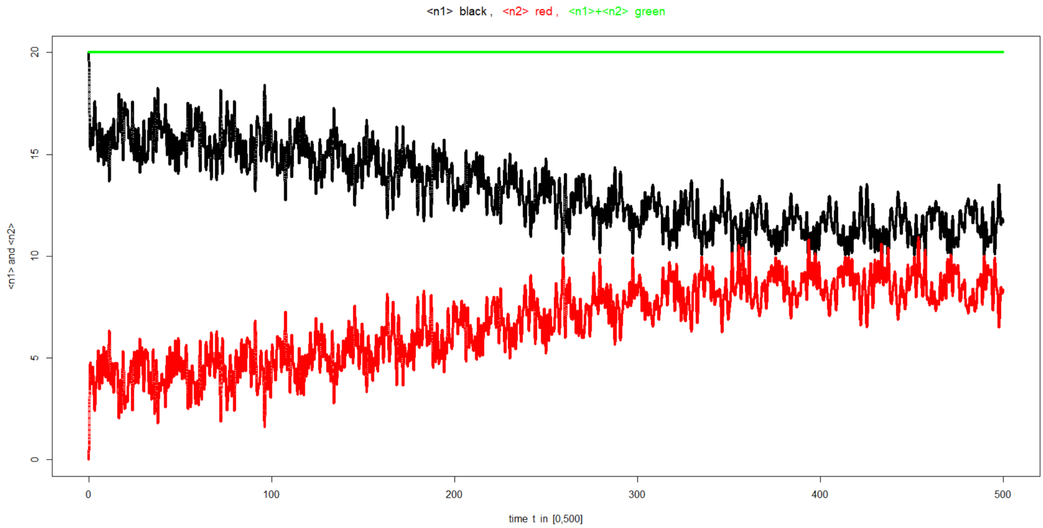
$$g = 2.5$$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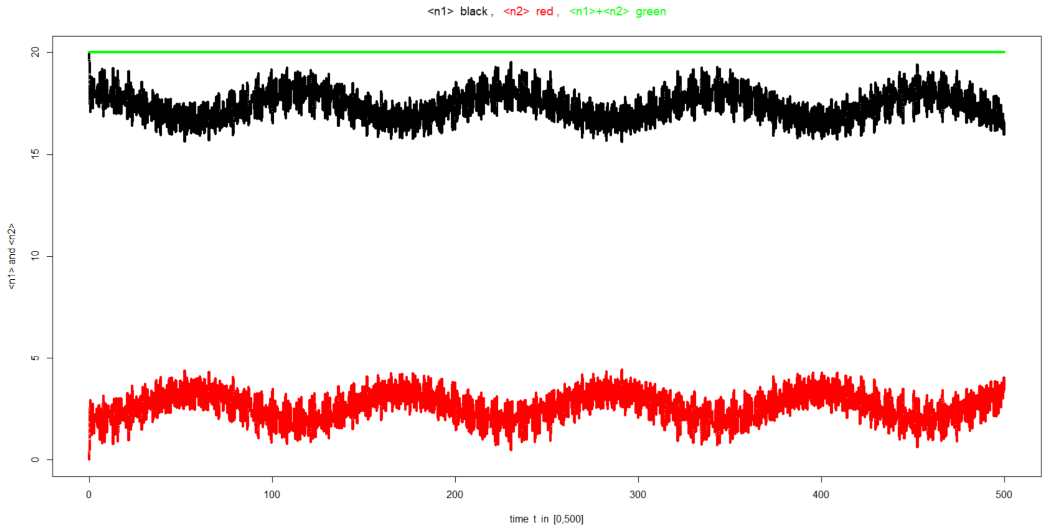
time t in [0,500]

$g = 3$



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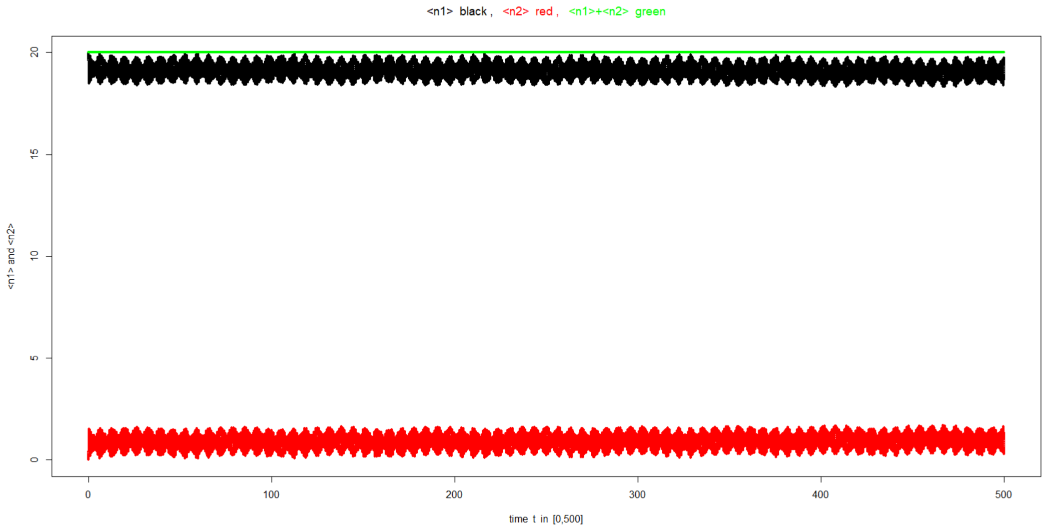
time t in [0,500]

$g = 4$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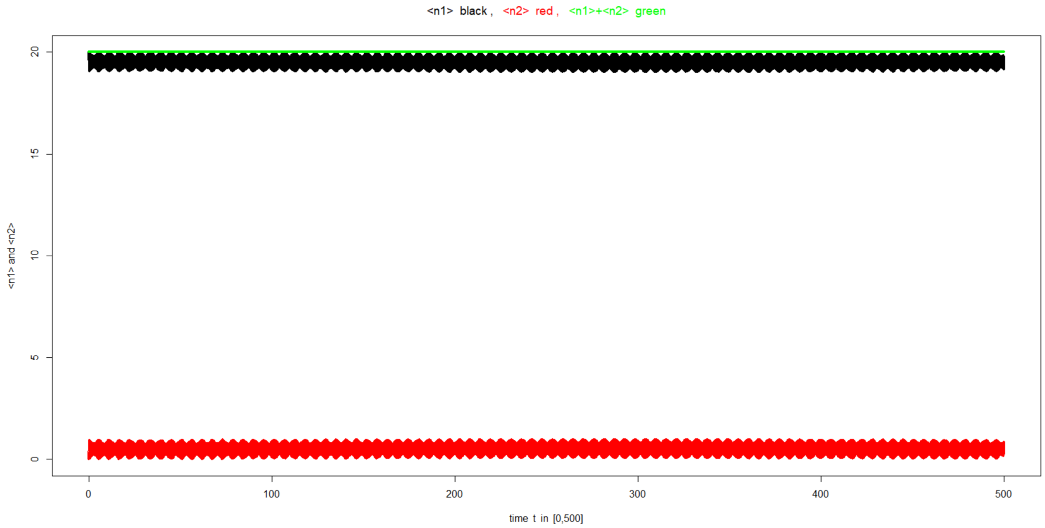
time t in [0,500]

$g = 5$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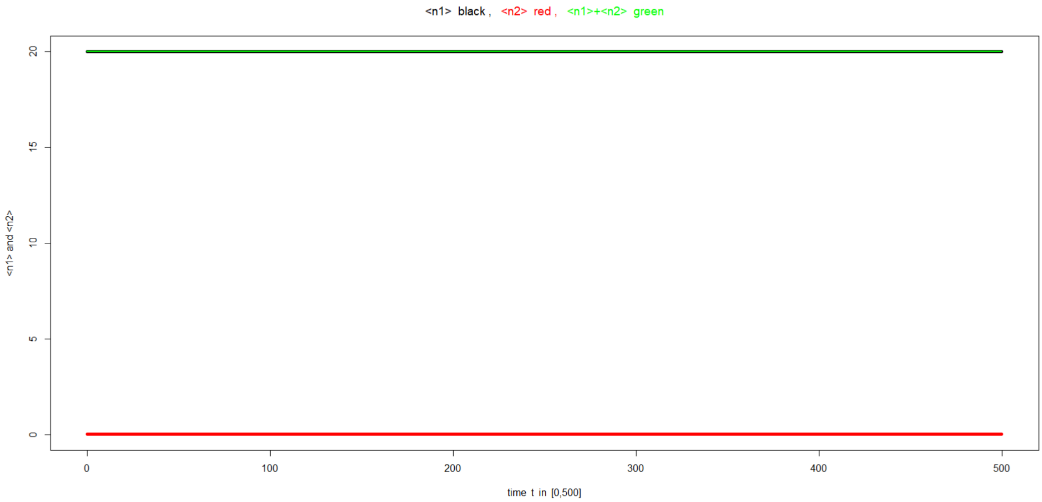
time t in [0,500]

$g = 25$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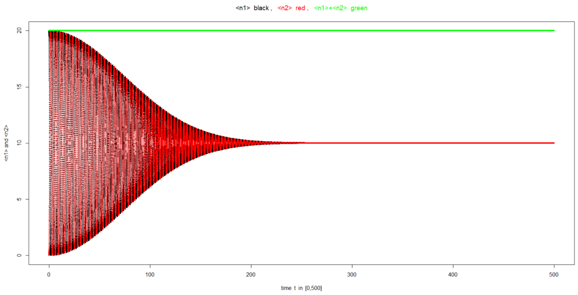
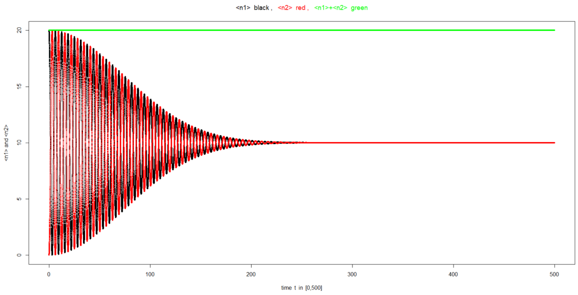
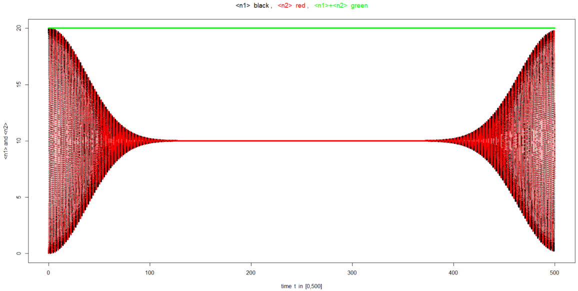


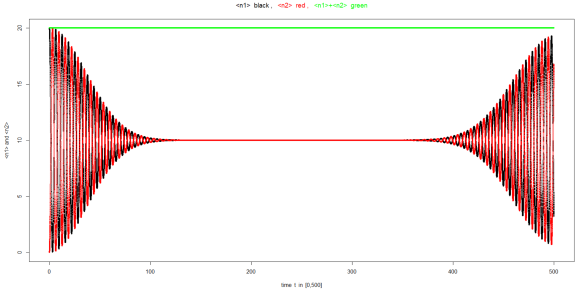
$$\varepsilon = 1, \;\; g = 1/16$$



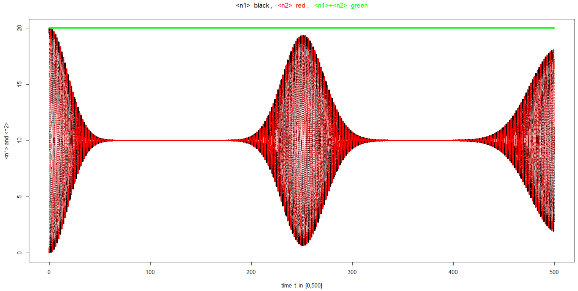
$$\varepsilon = 0.5, \;\; g = 1/16$$



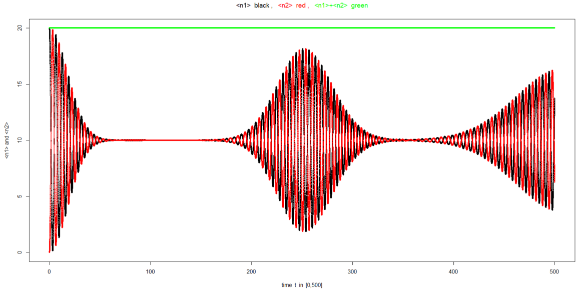
$$\varepsilon = 1, \ \ g = 1/8$$



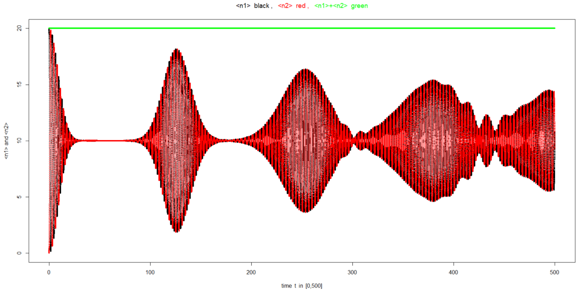
$$\varepsilon = 0.5, \ \ g = 1/8$$



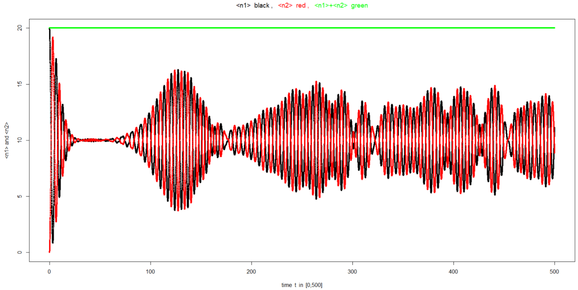
$$\varepsilon = 1, \;\; g = 1/4$$



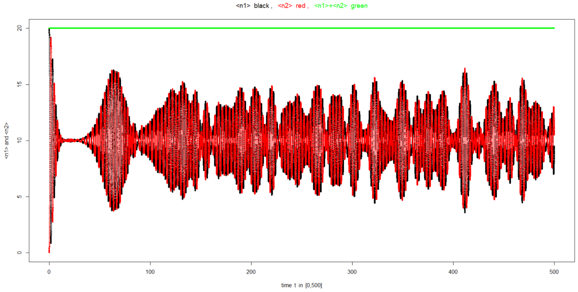
$$\varepsilon = 0.5, \;\; g = 1/4$$



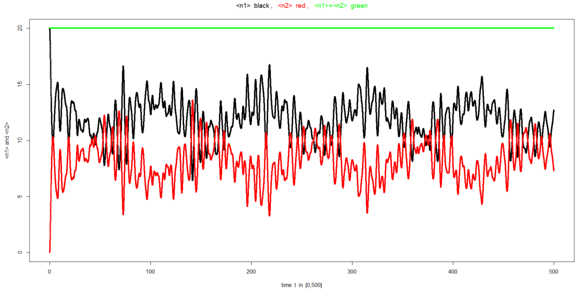
$\varepsilon = 1, \;\; g = 1/2$



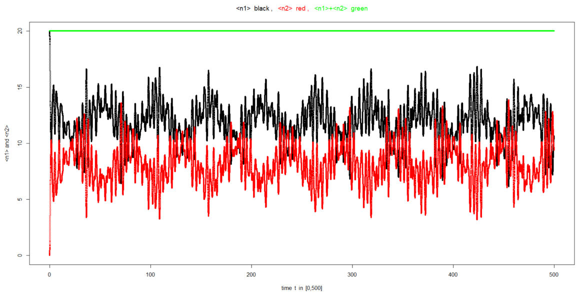
$\varepsilon = 0.5, \;\; g = 1/2$



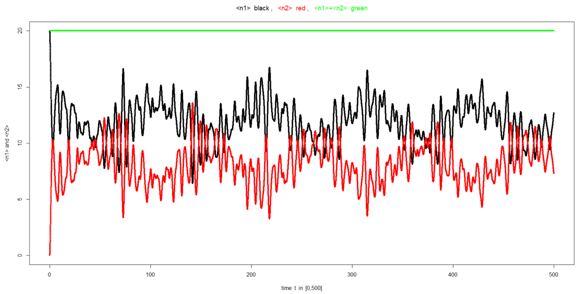
$$\varepsilon = 1, \quad g = 1, \qquad g/\varepsilon = 1$$



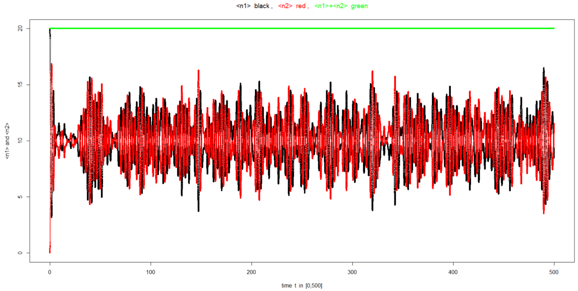
$$\varepsilon = 0.5, \quad g = 1, \qquad \boldsymbol{g/\varepsilon = 2}$$



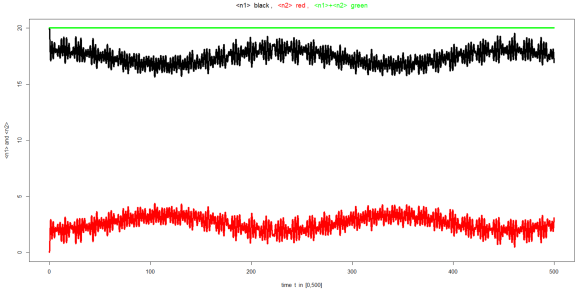
$$\varepsilon = 1, \quad \boldsymbol{g} = \mathbf{2}, \qquad \boldsymbol{g}/\varepsilon = \mathbf{2}$$



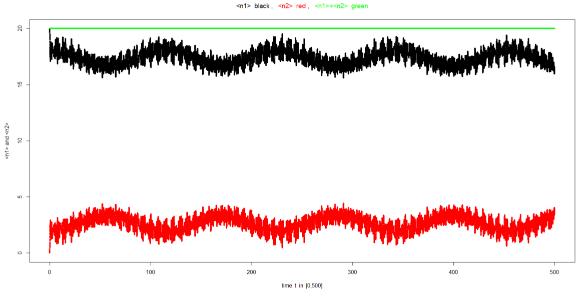
$$\varepsilon = 0.5, \quad g = 1, \qquad \boldsymbol{g}/\varepsilon = \mathbf{2}$$



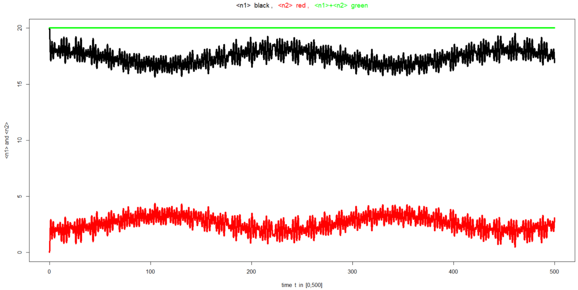
$$\varepsilon = 1, \quad g = 1.5, \qquad g/\varepsilon = 1.5 < 2$$



$$\varepsilon = 0.5, \quad g = 1.5, \qquad \boldsymbol{g/\varepsilon = 3 > 2}$$



$\varepsilon = 1,\ \ \boldsymbol{g} = \mathbf{3.0},\quad \boldsymbol{g/\varepsilon = 3}$



$\varepsilon = 0.5,\ \ g = 1.5,\quad \boldsymbol{g/\varepsilon = 3}$

1. Setup and Motivation

Central Problem of Many Body Quantum Mechanics:

Calculate Density Matrix Elements or Correlation Functions

$$\langle \psi_t \, , \, a_i^+ a_j \, \psi_t \rangle \, , \qquad \langle \psi_t , a_i^+ a_j^+ a_k \, a_\ell \, \psi_t \rangle \, , \; \ldots$$

this talk: new proposal for doing that

1. Setup and Motivation: Bargmann-Segal Representation

Bargmann-Segal representation:

$$a_j \;=\; \tfrac{\partial}{\partial z_j}\,, \qquad a_j^+ \;=\; z_j$$

acting on the Hilbert space of analytic functions of $|\Gamma| = L^d$ complex variables:

$$\mathcal{F} \;:=\; \Big\{\, f = f(\,\{z_j\}\,) : \mathbb{C}^{|\Gamma|} \to \mathbb{C} \;\text{ analytic} \;\Big|\; \|f\|^2_{\mathcal{F}} = \langle f, f\rangle_{\mathcal{F}} \,<\, \infty \,\Big\}$$

with scalar product

$$\langle f, g\rangle_{\mathcal{F}} \;:=\; \int_{\mathbb{C}^{|\Gamma|}=\mathbb{R}^{2|\Gamma|}} f(z)\,\overline{g(z)}\,d\mu(z)$$

$$d\mu(z) \;:=\; \prod_j e^{-|z_j|^2}\,\frac{d\mathrm{Re}z_j\,d\mathrm{Im}z_j}{\pi}$$

1. Setup and Motivation: Bargmann-Segal Representation

Hamiltonian:

$$h \;=\; \sum_{i,j} \varepsilon_{ij}\, z_i\, \frac{\partial}{\partial z_j} \;+\; u \sum_j z_j^2\, \frac{\partial^2}{\partial z_j^2} \;=:\; h_0 + h_{int}$$

We can allow for a general hopping matrix which should be real and symmetric:

$$\varepsilon \;:=\; (\varepsilon_{ij})_{i,j\in\Gamma} \;\in\; \mathbb{R}^{|\Gamma|\times|\Gamma|}, \qquad \varepsilon_{i,j} = \varepsilon_{j,i}$$

For a nearest neighbor hopping J and on-diagonal trapping potentials ϵ_j:

$$\varepsilon_{ij} \;=\; \begin{cases} -J & \text{if } \; |i-j| = 1 \\ +\epsilon_j & \text{if } \; i = j \\ \;\;0 & \text{otherwise} \;. \end{cases}$$

1. Setup and Motivation: Coherent States and Number States

Coherent and Number states, $\lambda = \{\lambda_j\} \in \mathbb{C}^{|\Gamma|}$:

$$\psi_{coh}(z) \;=\; \psi_{coh}(\,\{z_j\}\,) \;=\; \prod_j e^{\lambda_j z_j}\, e^{-\frac{|\lambda_j|^2}{2}} \;=:\; e^{\lambda z}\, e^{-\frac{|\lambda|^2}{2}}$$

$$\psi_{num}(z) \;=\; \psi_{num}(\,\{z_j\}\,) \;=\; \frac{1}{\sqrt{N!N^N}} \left(\sum_j \lambda_j z_j\right)^N \;=:\; \frac{(\lambda z)^N}{\sqrt{N!N^N}}$$

In both cases: $t = 0$ expected number of particles at site j:

$$\langle\, \psi_0\,,\, a_j^+ a_j\; \psi_0 \,\rangle_{\mathcal{F}} \;=\; |\lambda_j|^2$$

Thus: Total number of particles N:

$$N \;=\; \sum_j N_j \;=\; \sum_j |\lambda_j|^2 \;=\; |\lambda|^2$$

2. Overall Strategy and Main Results

2. Overall Strategy and Main Results

1) Time Evolution of States as Fresnel Expectation Values, $t = t_k = k dt$

$$\psi_t(z) = (e^{-ith}\psi_0)(z) = \mathsf{E}_\phi[\,\psi_0(U_{\phi,t} z)\,]$$

with unitary evolution matrix

$$U_{kdt}(\phi) = e^{-i\,dt\,\varepsilon} e^{-i\,D_{dt}(\phi_1)} \cdots e^{-i\,dt\,\varepsilon} e^{-i\,D_{dt}(\phi_k)} \in \mathbb{C}^{|\Gamma|\times|\Gamma|}$$

2) Correlation Functions as Fresnel Expectation Values

$$\langle\, \psi_t\,,\, a_i^+ a_j\ \ \psi_t\,\rangle = \mathsf{E}_\phi \bar{\mathsf{E}}_\theta\big[\ F_{i,j}^{\psi_0}\,(U_{\phi,t}\,,\, \bar{U}_{\theta,t})\ \big]$$

3) Stochastic Differential Equation (SDE) for Unitary Evolution Matrix $U_{\phi,t}$

$$dU_t = -i\,U_t\left(\varepsilon\,dt + \sqrt{2u}\,dx_t\right), \qquad dx_{kdt} = \sqrt{dt}\,\phi_k$$

2. Overall Strategy and Main Results

4) Correlation Functions from SDEs, $(v_t, \bar v_t) := (U_{\phi,t}^{T}\lambda, \bar U_{\theta,t}^{T}\bar\lambda)$

$$\langle\, \psi_t\,, a_i^+ a_j\;\; \psi_t\,\rangle_{coh/num} \;=\; \mathsf{E}\bar{\mathsf{E}}\big[\, v_{i,t}\, \bar v_{j,t}\; P_{\psi_0}(v_t \bar v_t)\,\big] \,/\, P_{\psi_0}(v_0 \bar v_0)$$

with

$$\begin{aligned} dv_j &= -\,i\,(\varepsilon v)_j\, dt \;-\; i\,\sqrt{2u}\; v_j\, dx_j \\ d\bar v_j &= +\,i\,(\varepsilon \bar v)_j\, dt \;+\; i\,\sqrt{2u}\; \bar v_j\, dy_j\;, \end{aligned} \qquad P_{coh/num}(x) = e^x \text{ or } x^{N-1}$$

5) Correlation Functions from Girsanov-Transformed SDEs

$$\langle\, \psi_t\,, a_i^+ a_j\;\; \psi_t\,\rangle_{coh/num} \;=\; \mathsf{E}\bar{\mathsf{E}}[\, v_{i,t}\, \bar v_{j,t}\,]$$

with

$$\begin{aligned} dv_j &= -\,i\,(\varepsilon v)_j\, dt \;-\; i\,2u\,[\log P_{\psi_0}]'(v\bar v)\; v_j \bar v_j\; v_j\; dt \;-\; i\sqrt{2u}\, v_j\, dx_j \\ d\bar v_j &= +\,i\,(\varepsilon \bar v)_j\, dt \;+\; i\,2u\,[\log P_{\psi_0}]'(v\bar v)\; v_j \bar v_j\; \bar v_j\; dt \;+\; i\sqrt{2u}\, \bar v_j\, dy_j \end{aligned}$$

2. Overall Strategy and Main Results

6) Large N Limit: $g := uN$ fixed, $(w, \bar{w}) := (v, \bar{v})/\sqrt{N}$, then

$$\langle \psi_t , a_i^+ a_j \; \psi_t \rangle_{coh/num} \;\; = \;\; N \; \times \; \mathsf{E}\bar{\mathsf{E}}[\, w_{i,t} \, \bar{w}_{j,t} \,]$$

with

$$\begin{aligned} dw_j \;\; &= \;\; - \, i \, (\varepsilon w)_j \, dt \; - \; i \, 2g \; [\log P_{\psi_0}]' \; w_j \bar{w}_j \, w_j \; dt \; - \; i \sqrt{2g/N} \, w_j \, dx_j \\ d\bar{w}_j \;\; &= \;\; + \, i \, (\varepsilon \bar{w})_j \, dt \; + \; i \, 2g \; [\log P_{\psi_0}]' \; w_j \bar{w}_j \, \bar{w}_j \; dt \; + \; i \sqrt{2g/N} \, \bar{w}_j \, dy_j \end{aligned}$$

Large N Limit can be read off:

$$\langle \psi_t , a_i^+ a_j \; \psi_t \rangle_{coh/num} \;\; = \;\; N \; \times \; w_i(t) \, \bar{w}_j(t)$$

with $w_j, \bar{w}_j$ given by the ODE system $\;\; (\, [\log P_{coh/num}]' \to 1 \,)$

$$\begin{aligned} \dot{w}_j \;\; &= \;\; - \, i \, (\varepsilon w)_j \; - \; i \, 2g \; w_j \bar{w}_j \, w_j \\ \dot{\bar{w}}_j \;\; &= \;\; + \, i \, (\varepsilon \bar{w})_j \; + \; i \, 2g \; w_j \bar{w}_j \, \bar{w}_j \end{aligned}$$

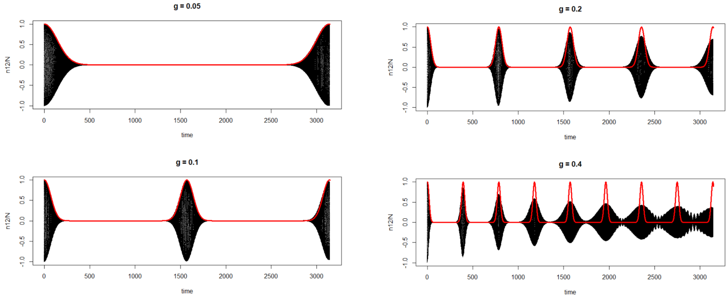
which is the time dependent discrete Gross-Pitaevskii equation.

2. Overall Strategy: Additional Results (appendix has more detail)

- Various Exact PDE Representations (confirming stochastic calculus formalism)

- Collapse and Revivals can be obtained from an approximate ODE System

  Black numbers are from exact diagonalization, the red curve comes from an analytical calculation:



- Equivalence between Mathematical Pendulum and Quartic Double-Well Potential

3. Time Evolution of States as Fresnel Expectation Values

3. Time Evolution of States as Fresnel Expectation Values

Time evolution with Trotter formula, $t = t_k = k\,dt$:

$$e^{-i\,t\,h} = e^{-i\,k\,dt\,(h_0 + h_{int})} \approx \left(e^{-i\,dt\,h_0}\; e^{-i\,dt\,h_{int}} \right)^k$$

Let's consider the action of $e^{-i\,dt\,h_0}$ and $e^{-i\,dt\,h_{int}}$:

3. Time Evolution of States as Fresnel Expectation Values

Action of $e^{-i\,t\,h_0}$:

$$\left(e^{-\,i\,t\,h_0}\,f\right)(z) \;=\; f\left(e^{-\,i\,t\,\varepsilon}z\right)$$

since with $z_t := e^{-i\varepsilon t}z$

$$\begin{aligned}\frac{\partial}{\partial t}\left[f(z_t)\right] \;&=\; \frac{\partial}{\partial t}\left[f(e^{-\,i\,t\,\varepsilon}z)\right] \;=\; \sum_j \frac{\partial z_{t,j}}{\partial t}\,\frac{\partial f}{\partial z_j}(z_t) \;=\; \sum_j (-i\varepsilon z_t)_j\,\frac{\partial f}{\partial z_j}(z_t)\\ &=\; \Big\{-i\sum_{j,k}\varepsilon_{j,k}z_k\,\frac{\partial f}{\partial z_j}\Big\}(z=z_t) \;=\; \Big\{-i\,h_0\,f\Big\}(z_t)\end{aligned}$$

3. Time Evolution of States as Fresnel Expectation Values

Action of $e^{-i\,t\,h_{int}}$:

$$(e^{-\,i\,t\,h_{int}}\,f)(z) \;=\; \int_{\mathbb{R}^{|\Gamma|}} f(e^{-i\,D_t(\phi)}z)\;\prod_j\, e^{\,i\,\frac{\phi_j^2}{2}}\,\frac{d\phi_j}{\sqrt{2\pi i}} \qquad (1)$$

with $D_t(\phi)$ the $|\Gamma|\times|\Gamma|$ diagonal matrix

$$D_t(\phi) \;=\; \mathrm{diag}\big(\,\{\,\sqrt{2ut}\,\phi_j - ut\,\}_{j\in\Gamma}\,\big) \;\in\; \mathbb{R}^{|\Gamma|\times|\Gamma|}$$

3. Time Evolution of States as Fresnel Expectation Values

Proof of (1):

$$h_{int} \prod_j z_j^{n_j} = \{ u \sum_i z_i^2 \frac{\partial^2}{\partial z_i^2} \} \prod_j z_j^{n_j} = \{ u \sum_i n_i(n_i - 1) \} \prod_j z_j^{n_j}$$

$$e^{-it\, h_{int}} \prod_j z_j^{n_j} = e^{-iut \sum_j n_j(n_j-1)} \prod_j z_j^{n_j} = \prod_j \left(e^{+iut\, n_j}\, e^{-i\, ut\, n_j^2}\, z_j^{n_j} \right)$$

Now use Fresnel integral ($\sqrt{i} := e^{i\frac{\pi}{4}}$)

$$\int_{\mathbb{R}} e^{-i\lambda\phi}\, e^{i\,\frac{\phi^2}{2}}\, \frac{d\phi}{\sqrt{2\pi i}} = e^{-\,i\,\frac{\lambda^2}{2}}$$

to obtain

$$e^{-it\, h_{int}} \prod_j z_j^{n_j} = \int_{\mathbb{R}^{|\Gamma|}} \prod_j \left(e^{+iut\, -\, i\sqrt{2ut}\,\phi_j} z_j \right)^{n_j} \prod_j e^{i\,\frac{\phi_j^2}{2}}\, \frac{d\phi_j}{\sqrt{2\pi i}} \;. \qquad \blacksquare$$

3. Time Evolution of States as Fresnel Expectation Values

Thus, single Trotter step $\left(\phi^2 := \sum_j \phi_j^2\right)$:

$$
\begin{aligned}
(e^{-i\,dt\,h} f)(z) &= \left[e^{-i\,dt\,h_{int}}\left(e^{-i\,dt\,h_0} f\right)\right](z) \\
&= \int_{\mathbb{R}^{|\Gamma|}} f\left(e^{-i\,dt\,\varepsilon} e^{-i\,D_{dt}(\phi)} z\right)\; e^{i\frac{\phi^2}{2}} \frac{d^{|\Gamma|}\phi}{(2\pi i)^{|\Gamma|/2}}
\end{aligned}
$$

3. Time Evolution of States as Fresnel Expectation Values

Iterating, $t = t_k = k dt$:

$$(e^{-i\,kdt\,h} f)(z) = \int_{\mathbb{R}^{k|\Gamma|}} f\left(e^{-i\,dt\,\varepsilon} e^{-i\,D_{dt}(\phi_1)} \dots e^{-i\,dt\,\varepsilon} e^{-i\,D_{dt}(\phi_k)}\, z \right) \prod_{\ell=1}^{k} e^{i\,\frac{\phi_\ell^2}{2}} \frac{d^{|\Gamma|}\phi_\ell}{(2\pi i)^{|\Gamma|/2}}$$

$$=: \int_{\mathbb{R}^{k|\Gamma|}} f\left(U_{kdt}(\phi)\, z \right) dF\left(\{\phi_\ell\}_{\ell=1}^{k} \right)$$

$$=: \mathrm{E}\left[f\left(U_{\phi,kdt}\, z \right) \right]$$

with notations $(\, k, \ell \in \mathbb{N}$ time indices, $j = (j_1, \cdots, j_d) \in \Gamma$ lattice site index $)$

$$\phi_\ell := \left(\{\phi_{j,\ell}\}_{j\in\Gamma} \right) \in \mathbb{R}^{|\Gamma|}$$

$$\phi_\ell^2 := \textstyle\sum_j \phi_{j,\ell}^2 \in \mathbb{R}$$

$$d^{|\Gamma|}\phi_\ell := \textstyle\prod_j d\phi_{j,\ell}$$

3. Time Evolution of States as Fresnel Expectation Values

Theorem 1: Let $h = h_0 + h_{int}$ be the d-dimensional Bose-Hubbard hamiltonian with cubic lattice $\Gamma = \{1, ..., L\}^d$, hopping matrix ε and interaction u. Then, for $t = kdt$, the time evolution of some initial state $\psi_0(z)$ is given by

$$(e^{-i\,kdt\,h}\psi_0)(z) \;=\; \int_{\mathbb{R}^{k|\Gamma|}} \psi_0\big(\,U_{kdt}(\phi)\,z\,\big)\;dF\big(\,\{\phi_\ell\}_{\ell=1}^k\,\big) \;=\; \mathsf{E}\big[\,\psi_0\big(\,U_{\phi,kdt}\,z\,\big)\,\big]$$

with unitary evolution matrix

$$U_{kdt}(\phi) \;=\; e^{-i\,dt\,\varepsilon}e^{-i\,D_{dt}(\phi_1)}\cdots e^{-i\,dt\,\varepsilon}e^{-i\,D_{dt}(\phi_k)} \;\in\; \mathbb{C}^{|\Gamma|\times|\Gamma|}$$

$$D_{dt}(\phi_\ell) \;=\; \mathrm{diag}\big(\,\{\,\sqrt{2u\,dt}\;\phi_{j,\ell} \,-\, u\,dt\,\}_{j\in\Gamma}\,\big) \;\in\; \mathbb{R}^{|\Gamma|\times|\Gamma|}$$

and Fresnel measure given by

$$\mathsf{E}[\,\cdot\,] \;=\; \int_{\mathbb{R}^{k|\Gamma|}} \cdot\;\; dF\big(\,\{\phi_\ell\}_{\ell=1}^k\,\big) \;=\; \int_{\mathbb{R}^{k|\Gamma|}} \cdot\;\; \prod_{\ell=1}^k e^{\,i\,\frac{\phi_\ell^2}{2}}\,\frac{d^{|\Gamma|}\phi_\ell}{(2\pi i)^{|\Gamma|/2}}$$

4. Correlation Functions as Fresnel Expectation Values

4. Correlation Functions as Fresnel Expectation Values

Correlation Functions in Bargmann-Segal Representation:

$$\begin{aligned}\langle \psi_t , a_i^+ a_j \ \psi_t \rangle &= \langle \psi_t , [a_j \ a_i^+ - \delta_{i,j}] \psi_t \rangle \\ &= \int_{\mathbb{C}^{|\Gamma|}} z_j \, \bar{z}_i \, |\psi_t(z)|^2 \, d\mu(z) \ - \ \delta_{i,j}\end{aligned}$$

From Theorem 1,

$$\begin{aligned}\psi_{kdt}(z) &= \int_{\mathbb{R}^{k|\Gamma|}} \psi_0(\, U_{\phi,kdt} z \,) \, dF(\phi) \\ \overline{\psi_{kdt}(z)} &= \int_{\mathbb{R}^{k|\Gamma|}} \overline{\psi_0(\, U_{\theta,kdt} z \,)} \, d\bar{F}(\theta)\end{aligned}$$

with

$$dF(\phi) \ = \ \prod_{\ell=1}^{k} e^{+i\frac{\phi_\ell^2}{2}} \frac{d^{|\Gamma|}\phi_\ell}{(2\pi i)^{|\Gamma|/2}} \ , \qquad d\bar{F}(\theta) \ = \ \prod_{\ell=1}^{k} e^{-i\frac{\theta_\ell^2}{2}} \frac{d^{|\Gamma|}\theta_\ell}{[2\pi(-i)]^{|\Gamma|/2}}$$

4. Correlation Functions as Fresnel Expectation Values

Thus, with $t = kdt$,

$$\langle\, \psi_t \,, a_j a_i^+ \psi_t \,\rangle \;=\; \int_{\mathbb{C}^{|\Gamma|}} z_j\, \bar{z}_i\, |\psi_t(z)|^2\, d\mu(z)$$

$$= \int_{\mathbb{C}^{|\Gamma|}} z_j\, \bar{z}_i \int_{\mathbb{R}^{k|\Gamma|}} \psi_0(U_{\phi,t} z)\, dF(\phi) \int_{\mathbb{R}^{k|\Gamma|}} \overline{\psi_0(U_{\theta,t} z)}\, d\bar{F}(\theta)\;\; d\mu(z)$$

$$= \int_{\mathbb{R}^{k|\Gamma|}} \int_{\mathbb{R}^{k|\Gamma|}} \left\{ \int_{\mathbb{C}^{|\Gamma|}} z_j\, \bar{z}_i\, \psi_0(U_{\phi,t} z)\, \overline{\psi_0(U_{\theta,t} z)}\, d\mu(z) \right\} dF(\phi)\, d\bar{F}(\theta)$$

The red integral is the expectation over the bosonic Fock space and can be calculated:

4. Correlation Functions as Fresnel Expectation Values

The integrand is:

For coherent states:

$$
\begin{aligned}
z_j\,\bar{z}_i\,\psi_0(U_{\phi,t}z)\,\overline{\psi_0(U_{\theta,t}z)} &= z_j\,\bar{z}_i\ \exp\{\,\lambda\cdot U_{\phi,t}z\,\}\ \exp\{\,\bar{\lambda}\cdot\bar{U}_{\theta,t}\bar{z}\,\}\ e^{-|\lambda|^2} \\
&= z_j\,\bar{z}_i\ \exp\{\,U_{\phi,t}^T\lambda\cdot z\,\}\ \exp\{\,\bar{U}_{\theta,t}^T\bar{\lambda}\cdot\bar{z}\,\}\ e^{-|\lambda|^2}
\end{aligned}
$$

For number states:

$$
\begin{aligned}
z_j\,\bar{z}_i\,\psi_0(U_{\phi,t}z)\,\overline{\psi_0(U_{\theta,t}z)} &= z_j\,\bar{z}_i\ \tfrac{1}{N!N^N}\ \left(\,\lambda\cdot U_{\phi,t}z\,\right)^N\left(\,\bar{\lambda}\cdot\bar{U}_{\theta,t}\bar{z}\,\right)^N \\
&= z_j\,\bar{z}_i\ \tfrac{1}{N!N^N}\ \left(\,U_{\phi,t}^T\lambda\cdot z\,\right)^N\left(\,\bar{U}_{\theta,t}^T\bar{\lambda}\cdot\bar{z}\,\right)^N
\end{aligned}
$$

4. Correlation Functions as Fresnel Expectation Values

Using (with $(\lambda, \bar\lambda) \to (U^T_{\phi,t}\lambda \, , \, \bar U^T_{\theta,t}\bar\lambda)$ and recall $d\mu(z) \;=\; \prod_j \, e^{-|z_j|^2} \, \frac{d\mathrm{Re}z_j \, d\mathrm{Im}z_j}{\pi}$)

$$\int_{\mathbb{C}^{|\Gamma|}} e^{\lambda z + \bar\lambda \bar z} \, d\mu(z) \;=\; e^{\lambda\bar\lambda}$$

$$\int_{\mathbb{C}^{|\Gamma|}} z_j \, \bar z_i \; e^{\lambda z + \bar\lambda \bar z} \, d\mu(z) \;=\; \frac{\partial}{\partial\lambda_j}\frac{\partial}{\partial\bar\lambda_i} \, e^{\lambda\bar\lambda} \;=\; (\lambda_i \, \bar\lambda_j + \delta_{i,j}) \, e^{\lambda\bar\lambda}$$

and

$$\tfrac{1}{N!} \int_{\mathbb{C}^{|\Gamma|}} (\lambda z)^N \, (\bar\lambda \bar z)^N \, d\mu(z) \;=\; (\lambda\bar\lambda)^N$$

$$\tfrac{1}{N!} \int_{\mathbb{C}^{|\Gamma|}} z_j \, \bar z_i \; (\lambda z)^N \, (\bar\lambda \bar z)^N \, d\mu(z) \;=\; \frac{\partial}{\partial\lambda_j}\frac{\partial}{\partial\bar\lambda_i} \, \frac{(\lambda\bar\lambda)^{N+1}}{N+1} \;=\; N \, \lambda_i \, \bar\lambda_j \, (\lambda\bar\lambda)^{N-1} \;+\; \delta_{i,j} \, (\, \lambda\bar\lambda \,)^N$$

we find

4. Correlation Functions as Fresnel Expectation Values

Theorem 2: For $t = kdt$ let $\mathrm{E}\bar{\mathrm{E}}[\,\cdot\,] \;=\; \int_{\mathbb{R}^{k|\Gamma|}} \int_{\mathbb{R}^{k|\Gamma|}} \;\cdot\; dF(\phi)\, d\bar{F}(\theta)$. Then:

a) For an arbitrary initial state ψ_0,

$$\psi_t(z) \;=\; (e^{-ith}\psi_0)(z) \;=\; \mathrm{E}[\,\psi_0(U_{\phi,t}z)\,]$$

(reminder, same as Theorem 1)

b) For a coherent state $\psi_0(z) = e^{\lambda z} e^{-|\lambda|^2/2}$,

$$\langle\, \psi_t \,,\, a_i^+ a_j \;\; \psi_t \,\rangle \;=\; \mathrm{E}\bar{\mathrm{E}}\Big[\; [U^T_{\phi,t}\lambda]_i \;\; [\bar{U}^T_{\theta,t}\bar{\lambda}]_j \;\; \exp\{\, U^T_{\phi,t}\lambda \cdot \bar{U}^T_{\theta,t}\bar{\lambda} \,\} \;\Big] \,/\, e^{\lambda\bar{\lambda}}$$

c) For a number state $\psi_0(z) = (\lambda z)^N / \sqrt{N!N^N}$,

$$\langle\, \psi_t \,,\, a_i^+ a_j \;\; \psi_t \,\rangle \;=\; \mathrm{E}\bar{\mathrm{E}}\Big[\; [U^T_{\phi,t}\lambda]_i \;\; [\bar{U}^T_{\theta,t}\bar{\lambda}]_j \;\; \big(\, U^T_{\phi,t}\lambda \cdot \bar{U}^T_{\theta,t}\bar{\lambda} \,\big)^{N-1} \;\Big] \,/\, (\lambda\bar{\lambda})^{N-1}$$

5. SDE Representation

5. SDE Representation: Reminder Brownian Motion and Wiener Measure

Standard Brownian motion in discrete time $t = t_k = kdt$:

$$x_{t_k} \quad = \quad \sqrt{dt}\,\sum_{\ell=1}^{k} \phi_\ell \qquad\qquad dx_{t_k} := x_{t_k} - x_{t_{k-1}} = \sqrt{dt}\,\phi_k$$

Wiener measure in discrete time with fixed time horizon $T = t_n = ndt$:

$$dW \quad = \quad \prod_{\ell=1}^{n} e^{-\frac{\phi_\ell^2}{2}}\,\frac{d\phi_\ell}{\sqrt{2\pi}} \quad = \quad \prod_{\ell=1}^{n} e^{-\frac{(x_{t_\ell}-x_{t_{\ell-1}})^2}{2\,dt}}\,\frac{dx_{t_\ell}}{\sqrt{2\pi\,dt}}$$

Brownian motion calculation rule, basic to stochastic calculus:

$$(dx_t)^2 \quad = \quad dt$$

For more background, see the appendix compact summary stochastic calculus in

https://arxiv.org/pdf/2205.02010.pdf

5. SDE Representation: Fresnel Brownian Motion and Fresnel Measure

Then Fresnel Brownian motion is

$$x_{t_k} \quad = \quad \sqrt{dt}\, \sum_{\ell=1}^{k} \phi_\ell \qquad\qquad dx_{t_k} := x_{t_k} - x_{t_{k-1}} = \sqrt{dt}\, \phi_k$$

with the ϕ_ℓ to be integrated against Fresnel measure,

$$dF \quad = \quad \prod_{\ell=1}^{n} e^{\,i\frac{\phi_\ell^2}{2}} \frac{d\phi_\ell}{\sqrt{2\pi i}} \quad = \quad \prod_{\ell=1}^{n} e^{\,i\,\frac{(x_{t_\ell}-x_{t_{\ell-1}})^2}{2\,dt}} \frac{dx_{t_\ell}}{\sqrt{2\pi i\, dt}}$$

Fresnel Brownian motion calculation rule:

$$(dx_t)^2 \quad = \quad i\,dt$$

For more background, see the appendix compact summary stochastic calculus in

`https://arxiv.org/pdf/2205.02010.pdf`

5. SDE Representation: Unitary Evolution Matrix U_{kdt}

Unitary evolution matrix of Theorem 1,

$$U_{kdt} \;=\; e^{-i\,dt\,\varepsilon}e^{-i\,D_{dt}(\phi_1)}\ldots e^{-i\,dt\,\varepsilon}e^{-i\,D_{dt}(\phi_k)} \;=\; U_{(k-1)dt}\; e^{-i\,dt\,\varepsilon}e^{-i\,D_{dt}(\phi_k)}$$

with diagonal matrix $D_{dt}(\phi_k)$ given by

$$\begin{aligned} D_{dt}(\phi_k) &= \operatorname{diag}\big(\{\sqrt{2u}\,\sqrt{dt}\;\phi_{j,k}\; - u\,dt\,\}_{j\in\Gamma}\big) \\ &= \operatorname{diag}\big(\{\sqrt{2u}\;dx_{j,kdt}\; - u\,dt\,\}_{j\in\Gamma}\big) \\ &=: \sqrt{2u}\;dx_{kdt}\; - \; u\,dt\,Id \end{aligned}$$

and diagonal matrix of Fresnel Brownian motions

$$dx_{kdt} \;:=\; \operatorname{diag}\big(\{\,dx_{j,kdt}\,\}_{j\in\Gamma}\big) \;\in\; \mathbb{R}^{|\Gamma|\times|\Gamma|}$$

which satisfies the matrix equation $(dx_{kdt})^2 \;=\; i\,dt\,Id$.

5. SDE Representation: Unitary Evolution Matrix U_{kdt}

Up to terms $O(dt^{3/2})$,

$$\begin{aligned}
e^{-i\,D_{dt}(\phi_k)} &= 1 - i\,D_{dt}(\phi_k) - \tfrac{1}{2}\,[D_{dt}(\phi_k)]^2 \\
&= 1 - i\left[\sqrt{2u}\,dx_{kdt} - u\,dt\,Id\right] - \tfrac{1}{2}\left[\sqrt{2u}\,dx_{kdt} - u\,dt\,Id\right]^2 \\
&= 1 - i\sqrt{2u}\,dx_{kdt} + i\,u\,dt\,Id - \tfrac{1}{2}\left[\sqrt{2u}\,dx_{kdt}\right]^2 \\
&= 1 - i\sqrt{2u}\,dx_{kdt} + i\,u\,dt\,Id - i\,u\,dt\,Id \\
&= 1 - i\sqrt{2u}\,dx_{kdt}
\end{aligned}$$

Thus,

$$\begin{aligned}
U_{kdt} &= U_{(k-1)dt}\; e^{-i\,dt\,\varepsilon} e^{-i\,D_{dt}(\phi_k)} \\
&= U_{(k-1)dt}\left(1 - i\,dt\,\varepsilon\right)\left(1 - i\sqrt{2u}\,dx_{kdt}\right) \\
&= U_{(k-1)dt}\left(1 - i\,dt\,\varepsilon - i\sqrt{2u}\,dx_{kdt}\right)
\end{aligned}$$

$$\Rightarrow \quad dU_{t_k} := U_{t_k} - U_{t_{k-1}} = -i\,U_{t_{k-1}}\left(dt\,\varepsilon + \sqrt{2u}\,dx_{t_k}\right)$$

5. SDE Representation

Theorem 3: a) The unitary evolution matrix U_t satisfies the SDE

$$dU_t \;=\; -i\,U_t\left(\varepsilon\,dt \;+\; \sqrt{2u}\,dx_t\right)$$

b) The correlation functions have the representations

$$\begin{aligned}\langle\,\psi_t\,,a_i^+a_j\;\psi_t\,\rangle_{coh} &\;=\; \mathrm{E}\bar{\mathrm{E}}\left[\,v_i\,\bar{v}_j\;e^{v\bar{v}}\,\right]\,/\,e^{\lambda\bar{\lambda}}\\ \langle\,\psi_t\,,a_i^+a_j\;\psi_t\,\rangle_{num} &\;=\; \mathrm{E}\bar{\mathrm{E}}\left[\,v_i\,\bar{v}_j\;(v\,\bar{v})^{N-1}\,\right]\,/\,(\lambda\bar{\lambda})^{N-1}\end{aligned}$$

with $v,\bar{v}\in\mathbb{C}^{|\Gamma|}$, $v=(v_{j,x,t})$, $\bar{v}=(\bar{v}_{j,y,t})$, given by the SDE system

$$\begin{aligned}dv_j &\;=\; -\,i\,dt\,(\varepsilon v)_j \;-\; i\,\sqrt{2u}\,v_j\,dx_j\\ d\bar{v}_j &\;=\; +\,i\,dt\,(\varepsilon\bar{v})_j \;+\; i\,\sqrt{2u}\,\bar{v}_j\,dy_j\end{aligned}$$

with initial conditions $(v_{x,0},\bar{v}_{y,0})=(\lambda,\bar{\lambda})$ and $\mathrm{E}\bar{\mathrm{E}} = \mathrm{E}_x\bar{\mathrm{E}}_y$.

5. SDE Representation

Proof 3b) From Theorem 2,

$$\langle \psi_t \,, a_i^+ a_j \;\; \psi_t \rangle_{coh} \;=\; \mathsf{E}\bar{\mathsf{E}}\Big[\; [U_{\phi,t}^T \lambda]_i \;\, [\bar{U}_{\theta,t}^T \bar{\lambda}]_j \;\; \exp\{\, U_{\phi,t}^T \lambda \cdot \bar{U}_{\theta,t}^T \bar{\lambda} \,\} \;\Big] \,/\, e^{\lambda\bar{\lambda}}$$

$$\langle \psi_t \,, a_i^+ a_j \;\; \psi_t \rangle_{num} \;=\; \mathsf{E}\bar{\mathsf{E}}\Big[\; [U_{\phi,t}^T \lambda]_i \;\, [\bar{U}_{\theta,t}^T \bar{\lambda}]_j \;\; \big(\, U_{\phi,t}^T \lambda \cdot \bar{U}_{\theta,t}^T \bar{\lambda} \,\big)^{N-1} \;\Big] \,/\, (\lambda\bar{\lambda})^{N-1}$$

With $\;(x_{kdt} = \sqrt{dt} \sum_{j=1}^{k} \phi_j \;, \;\; y_{kdt} = \sqrt{dt} \sum_{j=1}^{k} \theta_j \;)$

$$v \;=\; v_{x,t} \;\;:=\;\; U_{x,t}^T \lambda \;\in\; \mathbb{C}^{|\Gamma|}$$

$$\bar{v} \;=\; \bar{v}_{y,t} \;\;:=\;\; \bar{U}_{y,t}^T \bar{\lambda} \;\in\; \mathbb{C}^{|\Gamma|}$$

this looks as follows

$$\langle \psi_t \,, a_i^+ a_j \;\; \psi_t \rangle_{coh} \;=\; \mathsf{E}\bar{\mathsf{E}}\big[\; v_i \,\bar{v}_j \; e^{v\bar{v}} \;\big] \,/\, e^{\lambda\bar{\lambda}}$$

$$\langle \psi_t \,, a_i^+ a_j \;\; \psi_t \rangle_{num} \;=\; \mathsf{E}\bar{\mathsf{E}}\big[\; v_i \,\bar{v}_j \; (v\bar{v})^{N-1} \;\big] \,/\, (\lambda\bar{\lambda})^{N-1}$$

5. SDE Representation

Since

$$dU_t \;=\; -i\,U_t\left(\varepsilon\,dt \;+\; \sqrt{2u}\;dx_t\right)$$

and because of $\varepsilon^T = \varepsilon$, $dx_t^T = dx_t$, we obtain

$$\begin{aligned} dU_t^T &= -i\left(\varepsilon\,dt \;+\; \sqrt{2u}\;dx_t\right)U_t^T \\ d\bar{U}_t^T &= +i\left(\varepsilon\,dt \;+\; \sqrt{2u}\;dy_t\right)\bar{U}_t^T \end{aligned}$$

Thus, with $v_t = U_t^T\lambda$, $\bar{v}_t = \bar{U}_t^T\bar{\lambda}$,

$$\begin{aligned} dv &= -\,i\left(dt\,\varepsilon \;+\; \sqrt{2u}\,dx_t\right)v \\ d\bar{v} &= +\,i\left(dt\,\varepsilon \;+\; \sqrt{2u}\,dy_t\right)\bar{v} \end{aligned}$$

with initial conditions $v_0 = U_0^T\lambda = \lambda$, $\bar{v}_0 = \bar{U}_0^T\bar{\lambda} = \bar{\lambda}$. ∎

6. Girsanov Transformed SDE Representation

6.1 Unitary Time Evolution as a Martingale

6.1 Unitary Time Evolution as a Martingale

As in Theorem 3, one finds just for the norms $\|\psi_t\|^2$:

$$\begin{aligned} \langle \psi_t, \psi_t \rangle_{coh} &= \mathsf{E}\bar{\mathsf{E}}[\, e^{v\bar{v}} \,] \,/\, e^{\lambda\bar{\lambda}} \\ \langle \psi_t, \psi_t \rangle_{num} &= \mathsf{E}\bar{\mathsf{E}}[\, (v\bar{v})^{N-1}] \,/\, (\lambda\bar{\lambda})^{N-1} \end{aligned}$$

with $v\bar{v} = \sum_j v_j \bar{v}_j$ and $v_j, \bar{v}_j$ given by the SDE system

$$\begin{aligned} dv_j &= \; - \, i \, dt \, (\varepsilon v)_j \; - \; i \sqrt{2u} \, v_j \, dx_j \\ d\bar{v}_j &= \; + \, i \, dt \, (\varepsilon \bar{v})_j \; + \; i \sqrt{2u} \, \bar{v}_j \, dy_j \end{aligned}$$

These quantities have to be independent of time, we have to have

$$\begin{aligned} \mathsf{E}\bar{\mathsf{E}}[\, e^{v\bar{v}} \,] &= e^{v_0 \bar{v}_0} = e^{\lambda\bar{\lambda}} \\ \mathsf{E}\bar{\mathsf{E}}[\, (v\bar{v})^{N-1}] &= (v_0 \bar{v}_0)^{N-1} = (\lambda\bar{\lambda})^{N-1} \end{aligned}$$

How can this be understood in an SDE context?

6.1 Unitary Time Evolution as a Martingale

From
$$\begin{aligned} dv &= -\,i\,(\,\varepsilon\,dt + \sqrt{2u}\,dx_t\,)\,v \\ d\bar v &= +\,i\,(\,\varepsilon\,dt + \sqrt{2u}\,dy_t\,)\,\bar v \end{aligned}$$

we get
$$dv^T = -\,i\,v^T(\,\varepsilon\,dt + \sqrt{2u}\,dx_t\,)\ ,$$

$$\begin{aligned} d(v\bar v) \overset{notation}{=} d(v^T\bar v) &= dv^T\bar v + v^T d\bar v + dv^T d\bar v \\ &= -\,i\,v^T(\,\varepsilon\,dt + \sqrt{2u}\,dx_t\,)\,\bar v + i\,v^T(\,\varepsilon\,dt + \sqrt{2u}\,dy_t\,)\,\bar v + 0 \\ &= -\,i\,v^T\varepsilon\,\bar v\,dt + i\,v^T\varepsilon\,\bar v\,dt - i\sqrt{2u}\,v^T(dx_t - dy_t)\,\bar v \\ &= -\,i\sqrt{2u}\,\textstyle\sum_j v_j\bar v_j\,(dx_{j,t} - dy_{j,t}) \end{aligned}$$

The quantity $v\bar v$ is a martingale, its $d(v\bar v)$ has no drift part. Since $\mathsf{E}[dx_{j,t}] = \bar{\mathsf{E}}[dy_{j,t}] = 0$,

$$\Rightarrow \qquad \mathsf{E}\bar{\mathsf{E}}[\,(v\bar v)_{t_k}\,] = (v\bar v)_0 + \textstyle\sum_{\ell=1}^{k} \mathsf{E}\bar{\mathsf{E}}[\,d(v\bar v)_{t_\ell}\,] = (v\bar v)_0$$

6.1 Unitary Time Evolution as a Martingale

Ok, but we need more, we need $\mathsf{E}\bar{\mathsf{E}}[\,f(v_t\bar{v}_t)\,] = f(v_0\bar{v}_0)$ for arbitrary f. With Ito-Lemma:

$$df(v_t\bar{v}_t) \;=\; f'(v\bar{v})\,d(v_t\bar{v}_t) \;+\; \tfrac{1}{2}\,f''(v\bar{v})\,[d(v_t\bar{v}_t)]^2$$

Now,

$$\begin{aligned}
[d(v_t\bar{v}_t)]^2 \;&=\; \Big\{ -\,i\sqrt{2u}\,\textstyle\sum_j v_j\bar{v}_j\,(dx_{j,t} - dy_{j,t}) \Big\}^2 \\
&=\; -\,2u\sum_{i,j} v_i\bar{v}_i\; v_j\bar{v}_j\,(dx_{i,t} - dy_{i,t})(dx_{j,t} - dy_{j,t}) \\
&=\; 0
\end{aligned}$$

since $dx_{i,t}dx_{j,t} \;=\; dy_{i,t}dy_{j,t} \;=\; dx_{i,t}dy_{j,t} \;=\; 0$ for $i \neq j$ and for $i = j$:

$$\begin{aligned}
(dx_{j,t} - dy_{j,t})^2 \;&=\; (dx_{j,t})^2 \;-\; 2\,dx_{j,t}dy_{j,t} \;+\; (dy_{j,t})^2 \\
&=\; +\;i\,dt \;-\; 2\cdot 0 \;-\; i\,dt \;=\; 0
\end{aligned}$$

6.1 Unitary Time Evolution as a Martingale

Thus,

$$\begin{aligned} df(v_t\bar{v}_t) &= f'(v\bar{v})\, d(v_t\bar{v}_t) + \tfrac{1}{2} f''(v\bar{v}) \underbrace{[d(v_t\bar{v}_t)]^2}_{=0} \\ &= -i\sqrt{2u}\, f'(v_t\bar{v}_t) \textstyle\sum_j v_j\bar{v}_j\, (dx_{j,t} - dy_{j,t}) \end{aligned} \tag{2}$$

has no drift part which gives $\mathsf{E}\bar{\mathsf{E}}[\, df(v_t\bar{v}_t)\,] = 0$ and therefore

$$\mathsf{E}\bar{\mathsf{E}}[\, f(v_t\bar{v}_t)\,] = f(v_0\bar{v}_0) \;.$$

Equation (2) allows us in the next section 6.2 to absorb the the quantities $e^{v\bar{v}}$ or $(v\bar{v})^{N-1}$ into the Fresnel integration measure and to arrive at the compact expressions

$$\langle\, \psi_t\,,\, a_i^+ a_j\; \psi_t\,\rangle_{coh/num} = \mathsf{E}\bar{\mathsf{E}}[\; v_i\;\bar{v}_j\;]$$

with new, and initial state coh/num dependent, SDE's for the $v_i, \bar{v}_j$. Result is Theorem 4 .

6.2 Girsanov Transformed SDE System

6.2 Girsanov Transformed SDE System

Untransformed representation:

$$\begin{aligned} \langle\, \psi_t\,,\, a_i^{+} a_j\ \psi_t\,\rangle_{coh} &= \mathrm{E}\bar{\mathrm{E}}\big[\ v_i\ \bar{v}_j\ e^{v\bar{v}}\ \big]\ /\ e^{\lambda\bar{\lambda}} \\ \langle\, \psi_t\,,\, a_i^{+} a_j\ \psi_t\,\rangle_{num} &= \mathrm{E}\bar{\mathrm{E}}\big[\ v_i\ \bar{v}_j\ (v\,\bar{v})^{N-1}\ \big]\ /\ (\lambda\bar{\lambda})^{N-1} \end{aligned}$$

We write

$$\langle\, \psi_t\,,\, a_i^{+} a_j\ \psi_t\,\rangle = \mathrm{E}\bar{\mathrm{E}}\big[\ v_{i,t}\ \bar{v}_{j,t}\ P(v_t\bar{v}_t)\ \big]\ /\ P(v_0\bar{v}_0)$$

Now, with $p := \log P$,

$$\begin{aligned} P(v_{t_k}\bar{v}_{t_k})\ /\ P(v_0\bar{v}_0) &= \exp\big\{\ p(v_{t_k}\bar{v}_{t_k}) - p(v_0\bar{v}_0)\,\big\} = \exp\big\{\ \textstyle\sum_{\ell=1}^{k} dp(v_{t_\ell}\bar{v}_{t_\ell})\ \big\} \\ &\overset{(2)}{=} \exp\big\{ -i\sqrt{2u}\ \textstyle\sum_{\ell=1}^{k}\ \sum_j\, [p'(v\,\bar{v})(v_j\bar{v}_j)]_{t_{\ell-1}}\, (dx_{j,t_\ell} - dy_{j,t_\ell})\ \big\} \\ &= \exp\big\{ -i\sqrt{2u\,dt}\ \textstyle\sum_{\ell=1}^{k}\ \sum_j\, [p'(v\,\bar{v})(v_j\bar{v}_j)]_{t_{\ell-1}}\, (\phi_{j,\ell} - \theta_{j,\ell})\ \big\} \end{aligned}$$

6.2 Girsanov Transformed SDE System

Since

$$\mathrm{E}\bar{\mathrm{E}}[\,\cdot\,] \;=\; \int_{\mathbb{R}^{k|\Gamma|}} \int_{\mathbb{R}^{k|\Gamma|}} \;\cdot\; dF(\phi)d\bar{F}(\theta)$$

with

$$dF(\phi) \;=\; \prod_{\ell=1}^{k} e^{+i\frac{\phi_\ell^2}{2}} \frac{d^{|\Gamma|}\phi_\ell}{(2\pi i)^{|\Gamma|/2}} \;, \qquad d\bar{F}(\theta) \;=\; \prod_{\ell=1}^{k} e^{-i\frac{\theta_\ell^2}{2}} \frac{d^{|\Gamma|}\theta_\ell}{[2\pi(-i)]^{|\Gamma|/2}}$$

the quantity

$$P(v_{t_k}\bar{v}_{t_k}) \,/\, P(v_0\bar{v}_0) \;=\; \exp\Big\{ -i\sqrt{2u\,dt}\, \textstyle\sum_{\ell=1}^{k} \sum_j [p'(v\bar{v})(v_j\bar{v}_j)]_{t_{\ell-1}} (\phi_{j,\ell} - \theta_{j,\ell}) \Big\}$$

can be absorbed into the Fresnel measure by completing the square. In stochastic calculus, this is called a Girsanov transformation:

$$\begin{aligned}
\tilde{\phi}_{j,\ell} &:= \phi_{j,\ell} - \sqrt{2u\,dt}\,[\,p'(v\bar{v})\,v_j\bar{v}_j\,]_{(\ell-1)dt} \\
\tilde{\theta}_{j,\ell} &:= \theta_{j,\ell} - \sqrt{2u\,dt}\,[\,p'(v\bar{v})\,v_j\bar{v}_j\,]_{(\ell-1)dt}
\end{aligned}$$

6.2 Girsanov Transformed SDE System

or equivalently

$$
\begin{aligned}
d\tilde{x}_{j,t_\ell} &:= dx_{j,t_\ell} - \sqrt{2u}\, dt\, [\, p'(v\bar{v})\, v_j \bar{v}_j\,]_{(\ell-1)dt} \\
d\tilde{y}_{j,t_\ell} &:= dy_{j,t_\ell} - \sqrt{2u}\, dt\, [\, p'(v\bar{v})\, v_j \bar{v}_j\,]_{(\ell-1)dt}
\end{aligned}
$$

We get a new SDE system:

$$
\begin{aligned}
dv_j &= -\, i\, dt\, (\varepsilon v)_j - i\sqrt{2u}\, v_j\, dx_j \\
&= -\, i\, dt\, (\varepsilon v)_j - i\sqrt{2u}\, v_j \left[d\tilde{x}_j + \sqrt{2u}\, dt\, p'\, v_j \bar{v}_j \right] \\
&= -\, i\, dt\, (\varepsilon v)_j - i\, 2u\, dt\, p'\, v_j \bar{v}_j\, v_j - i\sqrt{2u}\, v_j\, d\tilde{x}_j
\end{aligned}
$$

$$
\begin{aligned}
d\bar{v}_j &= +\, i\, dt\, (\varepsilon \bar{v})_j + i\sqrt{2u}\, \bar{v}_j\, dy_j \\
&= +\, i\, dt\, (\varepsilon \bar{v})_j + i\sqrt{2u}\, \bar{v}_j \left[d\tilde{y}_j + \sqrt{2u}\, dt\, p'\, v_j \bar{v}_j \right] \\
&= +\, i\, dt\, (\varepsilon \bar{v})_j + i\, 2u\, dt\, p'\, v_j \bar{v}_j\, \bar{v}_j + i\sqrt{2u}\, \bar{v}_j\, d\tilde{y}_j
\end{aligned}
$$

6.2 Girsanov Transformed SDE System

Theorem 4: The correlation functions of Theorem 3 have the following equivalent representation:

$$\langle \psi_t \,, a_i^+ a_j \; \psi_t \rangle_{coh/num} \quad = \quad \mathsf{E}\bar{\mathsf{E}}[\, v_i \, \bar{v}_j \,]$$

with the $v_j, \bar{v}_j$ given by the transformed SDE system $\;(v_j = v_{j,x,y,t}, \;\; \bar{v}_j = \bar{v}_{j,x,y,t})$

$$\begin{aligned} dv_j &= \; - \, i \, (\varepsilon v)_j \, dt \; - \; i \, 2u \, p' v_j \bar{v}_j \, v_j \, dt \; - \; i\sqrt{2u} \, v_j \, dx_j \\ d\bar{v}_j &= \; + \, i \, (\varepsilon \bar{v})_j \, dt \; + \; i \, 2u \, p' v_j \bar{v}_j \, \bar{v}_j \, dt \; + \; i\sqrt{2u} \, \bar{v}_j \, dy_j \end{aligned}$$

and

$$p'(v\bar{v}) \;=\; [\log P]'(v\bar{v}) \;=\; \begin{cases} 1 & \text{for a coherent state} \\ (N-1)/(v\bar{v}) & \text{for a number state} \end{cases} .$$

with $P(x) = e^x$ or x^{N-1} for coh/num.

7. Large N Limit: Gross-Pitaevskii Equation

7.1 GP Equation for the d Dim Bose-Hubbard Model

7.1 GP Equation for the d Dim Bose-Hubbard Model

Recall $|\lambda|^2 = N$. Define normalized quantities

$$
\begin{aligned}
w_j &:= v_j \,/\, |\lambda| = v_j \,/\, \sqrt{N} \\
\bar{w}_j &:= \bar{v}_j \,/\, |\lambda| = \bar{v}_j \,/\, \sqrt{N}
\end{aligned}
$$

Then, with $p' = p'(v\bar{v}) = p'(Nw\bar{w})$,

$$
\begin{aligned}
dw_j &= -\, i\, dt\, (\varepsilon w)_j \,-\, i\, 2uN\, dt\, p'\, w_j \bar{w}_j\, w_j \,-\, i\sqrt{2u}\, w_j\, dx_j \\
d\bar{w}_j &= +\, i\, dt\, (\varepsilon \bar{w})_j \,+\, i\, 2uN\, dt\, p'\, w_j \bar{w}_j\, \bar{w}_j \,+\, i\sqrt{2u}\, \bar{w}_j\, dy_j
\end{aligned}
$$

or, with $g := uN$

$$
\begin{aligned}
dw_j &= -\, i\, dt\, (\varepsilon w)_j \,-\, i\, 2g\, dt\, p'\, w_j \bar{w}_j\, w_j \,-\, i\sqrt{2g/N}\, w_j\, dx_j \\
d\bar{w}_j &= +\, i\, dt\, (\varepsilon \bar{w})_j \,+\, i\, 2g\, dt\, p'\, w_j \bar{w}_j\, \bar{w}_j \,+\, i\sqrt{2g/N}\, \bar{w}_j\, dy_j
\end{aligned} \tag{3}
$$

The large N limit can be read off:

7.1 GP Equation for the d Dim Bose-Hubbard Model

Theorem 5: In the limit $N \to \infty$ with $g = uN$ fixed, Theorem 4 reduces to

$$\langle \psi_t \,, a_i^+ a_j \; \psi_t \rangle_{coh/num} \;\; = \;\; N \;\times\; w_i(t)\, \bar{w}_j(t)$$

with $w, \bar{w}$ given by the ODE system, in both cases coh/num,

$$\begin{aligned} \dot{w}_j \;\; &= \;\; -\, i\,(\varepsilon w)_j \; - \; i\, 2g\; w_j \bar{w}_j\, w_j \\ \dot{\bar{w}}_j \;\; &= \;\; +\, i\,(\varepsilon \bar{w})_j \; + \; i\, 2g\; w_j \bar{w}_j\, \bar{w}_j \end{aligned} \tag{4}$$

with initial conditions

$$w_j(0) \;\; = \;\; \lambda_j \,/\, |\lambda| \,, \qquad \bar{w}_j(0) \;\; = \;\; \bar{\lambda}_j \,/\, |\lambda|$$

The ODE system (4) is the time dependent discrete Gross-Pitaevskii equation.

The first equation of (4) is enough since $\bar{w}_j$ is the true complex conjugate of w_j now.

7.1 GP Equation for the d Dim Bose-Hubbard Model

Proof: For a coherent state $p' = 1$ and the statement follows immediately from the SDE system (3) since the diffusive part vanishes. For a number state,

$$p' \;=\; p'(Nw\bar{w}) \;=\; (N-1)/(Nw\bar{w}) \;\stackrel{N\to\infty}{\longrightarrow}\; 1/(w\bar{w})$$

such that we get the following ODE system:

$$\begin{aligned}
\dot{w}_j &= -\,i\,(\varepsilon w)_j \;-\; i\,2g\,\tfrac{1}{w\bar{w}}\,w_j\bar{w}_j\,w_j \\
\dot{\bar{w}}_j &= +\,i\,(\varepsilon \bar{w})_j \;+\; i\,2g\,\tfrac{1}{w\bar{w}}\,w_j\bar{w}_j\,\bar{w}_j
\end{aligned}$$

However,

$$\begin{aligned}
\tfrac{d}{dt}(w\bar{w}) &= \textstyle\sum_j \Big\{ -\,i\,(\varepsilon w)_j\bar{w}_j \;-\; i\,2g\,\tfrac{1}{w\bar{w}}\,(w_j\bar{w}_j)^2 \;+\; i\,w_j(\varepsilon\bar{w})_j \;+\; i\,2g\,\tfrac{1}{w\bar{w}}\,(w_j\bar{w}_j)^2 \Big\} \\
&= -\,i\,(\varepsilon w)\cdot\bar{w} \;+\; i\,w\cdot(\varepsilon\bar{w}) \;\stackrel{\varepsilon=\varepsilon^T}{=}\; 0
\end{aligned}$$

which results in $(w\bar{w})_t = (w\bar{w})_0 = 1\,.$ $\blacksquare$

7.2 GP Equation for the Two Site Bose-Hubbard Model and Numerical Test

7.2 GP Equation for the Two Site Bose-Hubbard Model

For just two lattice sites 1 and 2, we get from Theorem 5

$$\langle n_{j,t} \rangle \quad := \quad \langle \psi_t \,, a_j^+ a_j \; \psi_t \rangle_{coh/num} \quad \overset{N \to \infty}{=} \quad N \, |w_{j,t}|^2 \quad =: \quad N \; \varrho_{j,t}$$

with w_1, w_2 given by the GP system

$$\begin{aligned} \dot{w}_1 \quad &= \quad - \, i \, \varepsilon \, w_2 \; - \; i \, 2g \, |w_1|^2 \, w_1 \\ \dot{w}_2 \quad &= \quad - \, i \, \varepsilon \, w_1 \; - \; i \, 2g \, |w_2|^2 \, w_2 \quad . \end{aligned}$$

Theorem 6 (well known)**:** Introduce the normalized particle imbalance and its integral,

$$\begin{aligned} \varrho_{12} \quad &:= \quad |w_1|^2 \; - \; |w_2|^2 \\ \varphi_t \quad &:= \quad 2g \int_0^t \varrho_{12,s} \, ds \end{aligned}$$

Then φ_t is a solution of

$$\ddot{\varphi}_t \; + \; 4\varepsilon^2 \sin \varphi_t \quad = \quad 0$$

and the density of particles at lattice site 1 is obtained as

$$\varrho_{1,t} \quad = \quad |w_{1,t}|^2 \quad = \quad \tfrac{1}{2} \Big(1 \; + \; \tfrac{\dot{\varphi}_t}{2g} \Big)$$

7.2 GP Equation for the Two Site Bose-Hubbard Model

Initial conditions: Put all particles on lattice site 1 at $t = 0$. Then $\varphi_0 = 0$, always, and

$$\dot{\varphi}_0 \;=\; 2g\,\varrho_{12,0} \;=\; 2g\,(\varrho_{1,0} - \varrho_{2,0}) \;=\; 2g\,(1-0) \;=\; 2g$$

The total energy is

$$E \;=\; \frac{\dot{\varphi}_t^2}{2} - 4\varepsilon^2 \cos\varphi_t \;=\; \frac{\dot{\varphi}_0^2}{2} - 4\varepsilon^2 \cos\varphi_0 \;=\; 2g^2 - 4\varepsilon^2$$

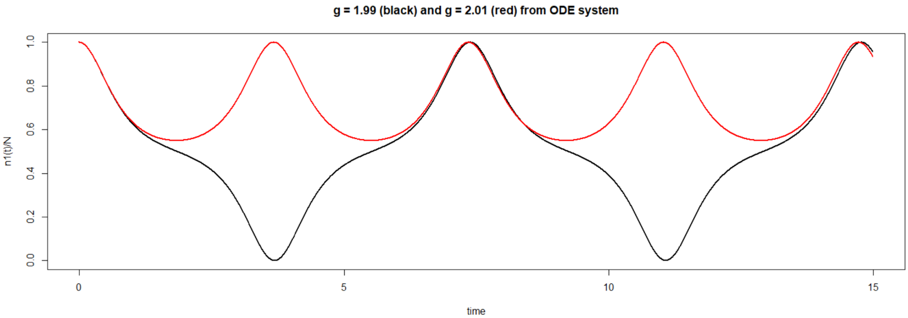
The potential energy at $\varphi = \pi$ is $E_{pot} = +4\varepsilon^2$. We have **rollovers** if the total energy is bigger than that:

$$\begin{aligned} 2g^2 - 4\varepsilon^2 \;&>\; +4\varepsilon^2 \\ \Leftrightarrow \qquad \boldsymbol{g^2} \;&>\; \boldsymbol{(2\varepsilon)^2} \end{aligned}$$

In that case, the velocity $\dot{\varphi}_t = 2g\,(\varrho_{1,t} - \varrho_{2,t})$ which is the particle imbalance between the two lattice sites, does not change its sign and this corresponds to the non-oscillatory or self-trapping regime:

7.2 GP Equation for the Two Site Bose-Hubbard Model

The mathematical pendulum ODE for $\varepsilon = 1$ and $g \in \{1.99, 2.01\}$:



the quantity $\varrho_{1,t} = \frac{1}{2}\big(1 + \frac{\dot\varphi_t}{2g}\big)$ with φ_t from ODE, $\varphi_0 = 0$ and $\dot\varphi_0 = 2g$

7.2 GP Equation for the Two Site Bose-Hubbard Model: Numerical Test

Numerical check of the large N limit: $\varepsilon = 1$ and

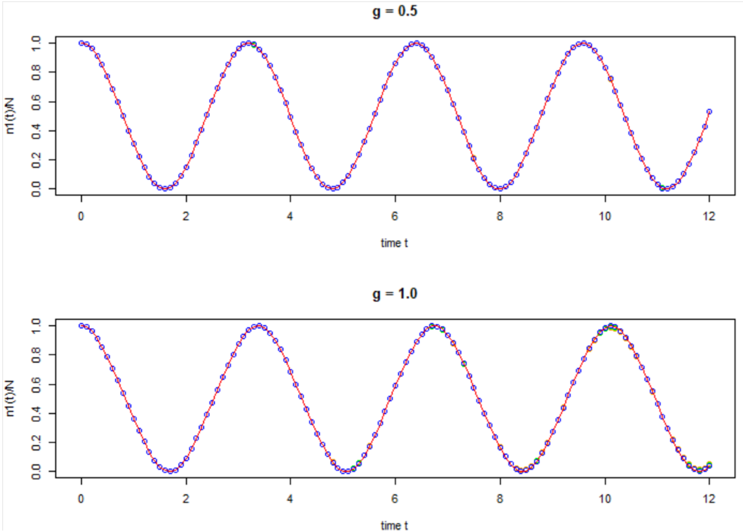
$$N \in \{2500, 5000, 10000, 20000\} = \{\text{orange}, \text{green}, \text{light blue}, \text{dark blue}\}$$
$$g \in \{0.5, 1.0, 1.8, 2.2, 3.0, 6.0\}$$

Quantity:

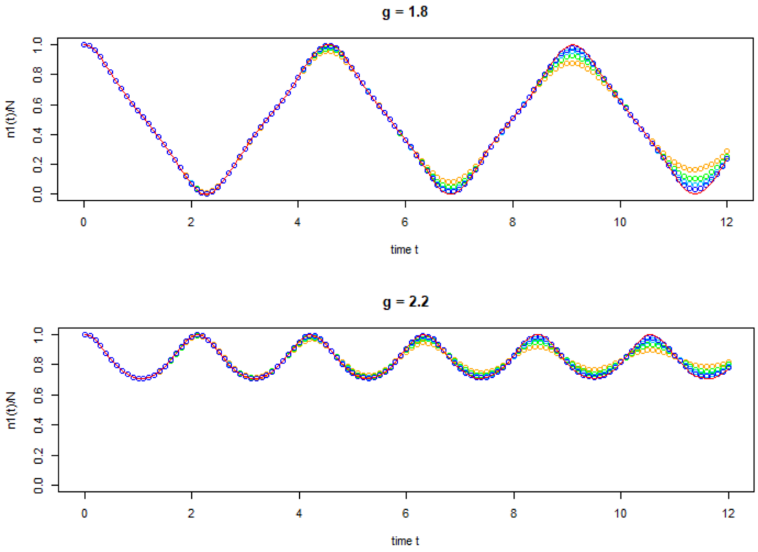
$$\varrho_{1,t} = \begin{cases} \langle \psi_t, a_1^+ a_1\, \psi_t \rangle \,/\, N & \text{from exact diagonalization} \\ \frac{1}{2}\left(1 + \frac{\dot{\varphi}_t}{2g}\right) & \text{from ODE mathematical pendulum} \end{cases}$$

The red line below is the ODE solution and the dots come from exact diagonalization:

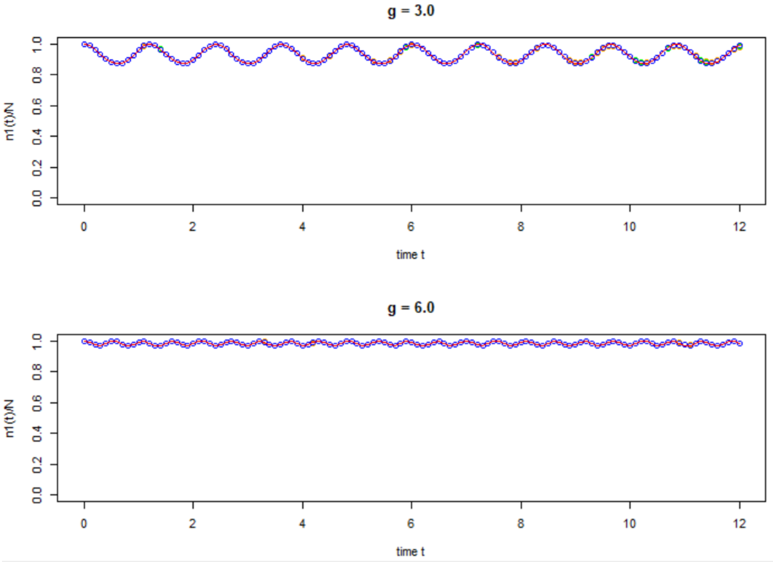
7.2 GP Equation for the Two Site Bose-Hubbard Model: Numerical Test



7.2 GP Equation for the Two Site Bose-Hubbard Model: Numerical Test



7.2 GP Equation for the Two Site Bose-Hubbard Model: Numerical Test



8. Summary

8. Summary

Main Results:

- Simple and Elegant Method to Calculate GP-like Mean Field Equations
- Generic Formalism: Arbitrary Initial State, Arbitrary Hopping Matrix, Arbitrary Dimension
- Various Exact PDE Representations
- collapse and revivals could be reproduced, proper treatment of diffusive part still missing

Outlook:

- Use this for New Numerical or Analytical Calculation Schemes
- Fermi Hubbard Model
- Thermodynamic Quantities

full paper at

`https://arxiv.org/abs/2205.02010`

8. Summary

Theorems:

Theorem 1: Time Evolution of States as Fresnel Expectation Values, Matrix U_{kdt}

Theorem 2: Correlation Functions as Fresnel Expectation Values

Theorem 3: SDE for U_{kdt} and Correlation Functions as Expectations of Fresnel Diffusions

Theorem 4: Girsanov Transformed SDE System: Initial State moves into Drift Part

Theorem 5: Large N-Limit of Theorem 4: Gross-Pitaevskii Equation

Theorem 6: GP Equation for Two Site Bose-Hubbard Model as Mathematical Pendulum

Appendix:

Theorem 7: PDE Representation Correlation Functions, d-dim Bose-Hubbard Model

Theorem 8: PDE Representation Correlation Functions, Two Site Bose-Hubbard Model

Theorem 9: Equivalence Mathematical Pendulum and Quartic Double-Well Potential

Standard Formula Wiener and Fresnel Expectations

Appendix

- PDE Representation d Dim Bose-Hubbard Model
- PDE Representation Two Site Bose-Hubbard Model
- Equivalence Mathematical Pendulum and Quartic Double-Well Potential
- Standard Formula Wiener and Fresnel Expectations

Appendix: PDE Representation d Dim Bose-Hubbard Model

Theorem 7: Define the differential operators

$$\mathcal{L}_0 \;:=\; \sum_{i,j} \varepsilon_{ij} \left\{ v_i \frac{\partial}{\partial v_j} - \bar{v}_i \frac{\partial}{\partial \bar{v}_j} \right\}, \quad \mathcal{L}_{int} \;:=\; u \sum_j \left\{ v_j^2 \frac{\partial^2}{\partial v_j^2} - \bar{v}_j^2 \frac{\partial^2}{\partial \bar{v}_j^2} \right\}$$

and for

$$P(x) \;:=\; \begin{cases} e^x & \text{for coherent state} \\ x^{N-1} & \text{for number state} \end{cases}$$

let

$$\mathcal{L}_P \;:=\; 2u \, \frac{P'(v\bar{v})}{P(v\bar{v})} \sum_j v_j \bar{v}_j \left\{ v_j \frac{\partial}{\partial v_j} - \bar{v}_j \frac{\partial}{\partial \bar{v}_j} \right\}$$

Then, for the d-dimensional Bose-Hubbard model with hopping matrix ε_{ij} and interaction u,

$$\begin{aligned} \langle \psi_t \,, a_i^+ a_j \; \psi_t \rangle_{coh/num} &= e^{-it(\mathcal{L}_0 + \mathcal{L}_{int})} \left\{ v_i \, \bar{v}_j \; P(v\bar{v}) \right\} / \, P(\lambda\bar{\lambda}) \; \big|_{v=\lambda, \bar{v}=\bar{\lambda}} \\ &= e^{-it(\mathcal{L}_0 + \mathcal{L}_{int} + \mathcal{L}_P)} \left\{ v_i \, \bar{v}_j \right\} \big|_{v=\lambda, \bar{v}=\bar{\lambda}} \;. \end{aligned}$$

Appendix: PDE Representation Two Site Bose-Hubbard Model

Theorem 8: Define the differential operators (with $p' = [\log P]'$, $P(x) = e^x$ or x^{N-1} coh/num)

$$\begin{aligned}
\mathcal{L}_\varepsilon &:= -\varepsilon\Big\{ (q-\bar{q})\left(\tfrac{\partial}{\partial n_1} - \tfrac{\partial}{\partial n_2}\right) + (n_1 - n_2)\left(\tfrac{\partial}{\partial q} - \tfrac{\partial}{\partial \bar{q}}\right) \Big\} \\
\mathcal{L}_u &:= +2u\Big\{ \left(n_1 \tfrac{\partial}{\partial n_1} - n_2 \tfrac{\partial}{\partial n_2}\right)\left(q\tfrac{\partial}{\partial q} - \bar{q}\tfrac{\partial}{\partial \bar{q}}\right) + p'(n)(n_1-n_2)\left(q\tfrac{\partial}{\partial q} - \bar{q}\tfrac{\partial}{\partial \bar{q}}\right) \Big\}
\end{aligned}$$

acting on functions of 4 variables $F = F(n_1, n_2, q, \bar{q})$. Then for the two site Bose-Hubbard model

$$\langle \psi_t, a_1^+ a_1 \, \psi_t \rangle_{coh/num} = e^{-it(\mathcal{L}_\varepsilon + \mathcal{L}_u)} \, n_1 \Big|_{(n_1,n_2,q,\bar{q}) = (\,|\lambda_1|^2, |\lambda_2|^2, \lambda_1\bar{\lambda}_2, \bar{\lambda}_1\lambda_2\,)}$$

with actions

$$(e^{-it\mathcal{L}_\varepsilon} F)(n_1, n_2, q, \bar{q}) = F\big(R_t \, (n_1, n_2, q, \bar{q})^T \big)$$

$$R_t = \begin{pmatrix} \cos^2 \varepsilon t & \sin^2 \varepsilon t & +i \sin \varepsilon t \cos \varepsilon t & -i \sin \varepsilon t \cos \varepsilon t \\ \sin^2 \varepsilon t & \cos^2 \varepsilon t & -i \sin \varepsilon t \cos \varepsilon t & +i \sin \varepsilon t \cos \varepsilon t \\ +i \sin \varepsilon t \cos \varepsilon t & -i \sin \varepsilon t \cos \varepsilon t & \cos^2 \varepsilon t & \sin^2 \varepsilon t \\ -i \sin \varepsilon t \cos \varepsilon t & +i \sin \varepsilon t \cos \varepsilon t & \sin^2 \varepsilon t & \cos^2 \varepsilon t \end{pmatrix}$$

Appendix: PDE Representation Two Site Bose Hubbard Model

and

$$e^{-it\mathcal{L}_u}\left\{\, G(n_1, n_2)\, q^b\, \bar{q}^{\,\bar{b}} \,\right\} \;=\;$$

$$G\left(\, e^{-i2ut\,(b-\bar{b})} n_1 \,,\; e^{+i2ut\,(b-\bar{b})} n_2 \,\right) \times \frac{P(\, e^{-i2ut\,(b-\bar{b})} n_1 \,+\, e^{+i2ut\,(b-\bar{b})} n_2 \,)}{P(n_1 + n_2)} \times q^b\, \bar{q}^{\,\bar{b}}$$

where $G = G(n_1, n_2)$ is an arbitrary analytic function and $b, \bar{b}$ are arbitrary natural numbers.

Example, used for Collapse and Revivals:

$$\begin{aligned} e^{-it\mathcal{L}_u}\{\, n_1 \,\} &= n_1 \\ e^{-it\mathcal{L}_u}\{\, q \,\} &= \frac{P(\, e^{-i2ut} n_1 \,+\, e^{+i2ut} n_2 \,)}{P(n_1 + n_2)} \times q \\ e^{-it\mathcal{L}_u}\{\, n_1 q \,\} &= e^{-i2ut}\, n_1 \times \frac{P(\, e^{-i2ut} n_1 \,+\, e^{+i2ut} n_2 \,)}{P(n_1 + n_2)} \times q \end{aligned}$$

such that

$$\frac{e^{-it\mathcal{L}_u}[n_1 q]}{e^{-it\mathcal{L}_u}[n_1]\, e^{-it\mathcal{L}_u}[q]} \;=\; e^{-i2ut}\,, \qquad \frac{e^{-it\mathcal{L}_\varepsilon}[n_1 q]}{e^{-it\mathcal{L}_\varepsilon}[n_1]\, e^{-it\mathcal{L}_\varepsilon}[q]} \;=\; 1\;.$$

Appendix: Equivalence Mathematical Pendulum and Quartic Double-Well Potential

Theorem 9: The mathematical pendulum

$$\ddot{\varphi}_t \; + \; 4\varepsilon^2 \sin\varphi_t \quad = \quad 0$$

with $\varphi_0 = 0$ and $\dot{\varphi}_0 = 2g$ is equivalent to

$$\ddot{x}_t \; + \; (4\varepsilon^2 - 2g^2)\, x_t \; + \; 2g^2\, x_t^3 \quad = \quad 0$$

with $x_0 = 1$ and $\dot{x}_0 = 0$ through the transformation

$$\varphi_t \quad = \quad 2g \int_0^t x_s \, ds \qquad \Leftrightarrow \qquad x_t \quad = \quad \tfrac{1}{2g}\, \dot{\varphi}_t \quad .$$

Appendix: Standard Formula Wiener and Fresnel Expectations

Theorem: Consider m times $0 < t_1 < t_2 < \cdots < t_m \leq T$ and let x_{t_j} be a standard or Fresnel Brownian motion observed at time t_j. Let

$$F \quad = \quad F(x_{t_1}, \cdots, x_{t_m}) \; : \; \mathbb{R}^m \;\; \to \;\; \mathbb{C}$$

be an arbitrary function of m variables and let $\mathrm{E}[F]$ denote its Wiener or Fresnel expectation value. Then, with $t_0 := 0$ and $x_0 := 0$,

$$\mathrm{E}[\,F\,] \quad = \quad \int_{\mathbb{R}^m} F(x_{t_1}, \cdots, x_{t_m}) \, \textstyle\prod_{j=1}^m p_{t_j - t_{j-1}}(x_{t_{j-1}}, x_{t_j}) \, dx_{t_j}$$

with Gaussian or Fresnel kernels given by

$$p_t(x,y) \quad := \quad \begin{cases} \frac{1}{\sqrt{2\pi t}} \; e^{-\frac{(x-y)^2}{2t}} & \text{for Wiener expectations} \\ \frac{1}{\sqrt{2\pi i t}} \; e^{\,i\,\frac{(x-y)^2}{2t}} & \text{for Fresnel expectations}\,. \end{cases}$$

Basic property: $\qquad \int_{\mathbb{R}} p_t(x,y)\, p_s(y,z)\, dy \quad = \quad p_{t+s}(x,z)$