Sharp KMS-inequalities in all dimensions

joint work with P. Lewintan & P. Neff

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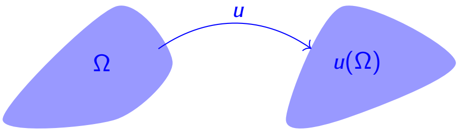
Seminar Mathematical Physics, Regensburg, Nov 29, 2022

Joint work with ...

Patrizio Neff & Peter Lewintan (Duisburg-Essen)

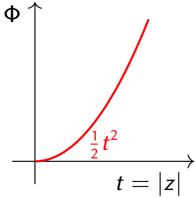
Elasticity and Korn inequalities



- Symmetric gradient: $\varepsilon(u) := \frac{1}{2}(Du + Du^{\top})$
- Trace-free symmetric gradient: $\varepsilon^D(u) := \varepsilon(u) - \frac{1}{n}\operatorname{div}(u)\mathbf{1}_n$

$$\text{minimise} \quad \mathscr{F}[u] := \int_\Omega \Phi(|\varepsilon^D(u)|)\,\mathrm{d}x + \frac{1}{2}\int_\Omega |\operatorname{div}(u)|^2\,\mathrm{d}x - \int_\Omega F\cdot u\,\mathrm{d}x$$

subject to suitable side constraints (forces, tensions)



Korn inequalities: $1 < p < \infty$

$$\|Du\|_{\mathrm{L}^p(\mathbb{R}^n)} \lesssim \|\varepsilon(u)\|_{\mathrm{L}^p(\mathbb{R}^n)}$$
$$\|Du\|_{\mathrm{L}^p(\mathbb{R}^n)} \lesssim \|\varepsilon^D(u)\|_{\mathrm{L}^p(\mathbb{R}^n)}$$

Korn & KMS: Introduction and context

Let $\mathcal{A}\colon \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ be linear. In general, for $1 < p < \infty$

$$\|P\|_{\mathrm{L}^p(\mathbb{R}^n)} \lesssim \|\mathcal{A}[P]\|_{\mathrm{L}^p(\mathbb{R}^n)} \quad \text{for all } P \in \mathrm{C}_c^\infty(\mathbb{R}^n;\mathbb{R}^{n\times n})$$ **is impossible**.

If we impose more structure...

... and e.g. require P to be curl-free, so $P = \nabla u$ for $u\colon \mathbb{R}^n \to \mathbb{R}^n$, then

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and this can be true indeed!

Example (Classical Korn inequalities)

- ✓ $\mathcal{A}[P] = \mathrm{sym}(P) \rightsquigarrow \|\nabla u\|_{\mathrm{L}^p(\mathbb{R}^n)} \lesssim \|\varepsilon(u)\|_{\mathrm{L}^p(\mathbb{R}^n)}$
- ✓ $\mathcal{A}[P] = \mathrm{sym}(P) - \frac{1}{n}\mathrm{tr}(P)\mathbf{1}_n \rightsquigarrow \|\nabla u\|_{\mathrm{L}^p(\mathbb{R}^n)} \lesssim \|\varepsilon^D(u)\|_{\mathrm{L}^p(\mathbb{R}^n)}$

$\longrightarrow$ **Goal: Introduce curl-correctors in** $\|P\|_{\mathrm{L}^p(\mathbb{R}^n)} \lesssim \|\mathcal{A}[P]\|_{\mathrm{L}^p(\mathbb{R}^n)}$!

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Additive correction by curl terms

Korn-Maxwell-Sobolev inequalities

Given

- the space dimension $n \in \mathbb{N}$,
- an integrability $1 \leq p \leq \infty$ and
- an algebraic part map $\mathcal{A}\colon \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$,

classify all constellations $(n, p, \mathcal{A})$ such that

$$\|P\|_{\mathrm{L}^q(\mathbb{R}^n)} \leq c\Big(\|\mathcal{A}[P]\|_{\mathrm{L}^q(\mathbb{R}^n)} + \|\mathrm{Curl}(P)\|_{\mathrm{L}^p(\mathbb{R}^n)}\Big) \qquad \text{for all } P \in \mathrm{C}_c^\infty(\mathbb{R}^n;\mathbb{R}^{n\times n}).$$

- Scaling determines q in terms of n and p, e.g. $1 \leq p < n \Rightarrow q = \frac{np}{n-p}$
- $P = \nabla u$ for $u \in \mathrm{C}_c^\infty(\mathbb{R}^n)$ gives neoclassical Korn-type inequalities

$$\|\nabla u\|_{\mathrm{L}^q(\mathbb{R}^n)} \leq c\|\mathbb{A}u\|_{\mathrm{L}^q(\mathbb{R}^n)} \qquad \text{for all } u \in \mathrm{C}_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$$

with $\mathbb{A}u := \mathcal{A}[\nabla u]$.

$\rightsquigarrow$ already yields strong conditions on $\mathcal{A}$!

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Some history – up to now only special cases

- ✓ Garroni, Leoni & Ponsiglione '10: $(n, p, \mathcal{A}) = (2, 1, \mathrm{sym})$
- ✓ Conti & Garroni '21: $(n, p, \mathcal{A}) = (3, 1, \mathrm{sym})$
- ✓ FXG & Spector '21: $(n, p, \mathcal{A}) = (3, p, \mathcal{A})$

- ✓ Lewintan, Müller & Neff '21: (dev)sym-(dev)sym-curl, $p \neq 1$
- ✓ (Lewintan &)Neff '09– now: a wealth of related inequalities

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Aims now:

Systematize these findings and explain the underlying mechanisms!

Terminology and examples

Let V, W be real, finite dimensional vector spaces, $\mathbb{A}_j \in \mathscr{L}(V; W)$ and

$$\mathbb{A}u := \sum_{j=1}^{n} \mathbb{A}_j \partial_j \quad \text{with Fourier symbol} \quad \mathbb{A}[\xi] := \sum_{j=1}^{n} \xi_j \mathbb{A}_j.$$

Definition (Notions of ellipticity)

We say that $\mathbb{A}$ is

- ($\mathbb{R}$-)**elliptic** $\Longleftrightarrow$ $\mathbb{A}[\xi]\colon V \to W$ injective for all $\xi \in \mathbb{R}^n \setminus \{0\}$,
- $\mathbb{C}$-**elliptic** $\Longleftrightarrow$ $\mathbb{A}[\xi]\colon V + \mathrm{i}V \to W + \mathrm{i}W$ injective for all $\xi \in \mathbb{C}^n \setminus \{0\}$.

Example

Suppose that $u\colon \mathbb{R}^n \to \mathbb{R}^n$.

- Symmetric gradient: $\varepsilon(u) := \frac{1}{2}(Du + Du^\top)$ is $\mathbb{C}$-elliptic for all $n \geq 1$.
- Trace-free symmetric gradient: $\varepsilon^D(u) := \varepsilon(u) - \frac{1}{n}\mathrm{div}(u)\mathbf{1}_n$
  - $\rightsquigarrow$ $\mathbb{R}$-elliptic $\Leftrightarrow n \geq 2$
  - $\rightsquigarrow$ $\mathbb{C}$-elliptic $\Leftrightarrow n \geq 3$

Necessity of ellipticity

Suppose the validity of the KMS-inequality

$$\|P\|_{\mathrm{L}^{p^*}(\mathbb{R}^n)} \lesssim \|\mathcal{A}[P]\|_{\mathrm{L}^{p^*}(\mathbb{R}^n)} + \|\mathrm{Curl}(P)\|_{\mathrm{L}^{p}(\mathbb{R}^n)}, \qquad P \in \mathrm{C}_c^\infty(\mathbb{R}^n; \mathbb{R}^{n\times n})$$

$\Rightarrow$ For $P = Du$, $u \in \mathrm{C}_c^\infty(\mathbb{R}^n; \mathbb{R}^n)$, obtain

$$\|Du\|_{\mathrm{L}^{p^*}(\mathbb{R}^n)} \lesssim \|\mathcal{A}[Du]\|_{\mathrm{L}^{p^*}(\mathbb{R}^n)}, \qquad u \in \mathrm{C}_c^\infty(\mathbb{R}^n; \mathbb{R}^{n\times n})$$

$\Rightarrow$ $\mathbb{A}u := \mathcal{A}[Du]$ **must be elliptic**: If not, there exist

$$\xi \in \mathbb{R}^n \setminus \{0\} \text{ and } v \in \mathbb{R}^n \setminus \{0\}: \ \mathbb{A}[\xi]v = 0.$$

Towards a contradiction, take the **plane wave**

$$u(x) = \eta(x\cdot\xi)v \Longrightarrow \mathbb{A}u(x) = (\eta')(x\cdot\xi)\mathbb{A}[\xi]v = 0.$$

- Ellipticity **often** also suffices, but not always.

Sharp conditions on $\mathcal{A}$?

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The case $(p, n) = (1, 2)$

Example (Ellipticity does not always suffice)

Let $\mathcal{A}[P] = \operatorname{devsym}(P) = \operatorname{sym}(P) - \frac{1}{n}\operatorname{tr}(P)\mathbf{1}_n$.

$\longrightarrow$ put for $f \in C_c^\infty(\mathbb{R}^2)$

$$P_f = \begin{pmatrix} \partial_2 f & -\partial_1 f \\ \partial_1 f & \partial_2 f \end{pmatrix}$$

$\longrightarrow$ then

$$\operatorname{devsym}[P_f] = \mathbf{0} \quad \text{and} \quad \operatorname{Curl}(P_f) = \begin{pmatrix} -\Delta f \\ 0 \end{pmatrix}$$

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$$\|P\|_{L^2(\mathbb{R}^2)} \lesssim \|\mathcal{A}[P]\|_{L^2(\mathbb{R}^2)} + \|\operatorname{Curl}(P)\|_{L^1(\mathbb{R}^2)}$$

implies the contradictory

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Sharp conditions – Main theorem

Theorem (FXG, Lewintan & Neff, '22)

For a linear map $\mathcal{A}\colon \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ *the KMS-inequality*

$$\|P\|_{L^{p^*}(\mathbb{R}^n)} \lesssim \|\mathcal{A}[P]\|_{L^{p^*}(\mathbb{R}^n)} + \|\operatorname{Curl}(P)\|_{L^{p}(\mathbb{R}^n)}, \qquad P \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^{n\times n})$$

***is equivalent** to* $\mathbb{A}u := \mathcal{A}[Du]$ *satisfying the following:*

	$n = 1$	$n = 2$	$n \geq 3$
$p = 1$		**$\mathbb{C}$-ellipticity**	
$1 < p < n$		**$\mathbb{R}$-ellipticity**	

Two useful tools

Calderón-Zygmund-Korn-type inequalities

The following are equivalent for a first order differential operator $\mathbb{A}$:

1. $\mathbb{A}$ is elliptic
2. $\|Du\|_{\mathrm{L}^q} \lesssim_q \|\mathbb{A}u\|_{\mathrm{L}^q}$ for all $1 < q < \infty$ and all $u \in \mathrm{C}_c^\infty(\mathbb{R}^n; V)$.

Idea of proof: Write

$$\partial_j u \approx \mathscr{F}^{-1}[\xi_j(\mathbb{A}^*[\xi]\mathbb{A}[\xi])^{-1}\mathbb{A}^*[\xi]\widehat{\mathbb{A}u}]$$

and use Mihlin-Hörmander.

Fractional integration theorem (FIT)

Let $1 < p < n$. Then the Riesz potential

$$I_1 f(x) := \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-1}}\,\mathrm{d}y, \qquad x \in \mathbb{R}^n,$$

is a bounded linear operator $I_1 : \mathrm{L}^p(\mathbb{R}^n) \to \mathrm{L}^{p^*}(\mathbb{R}^n)$.

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The case $(p, n) \neq (1, 2)$

Idea: Reduction of KMS to Korn-type inequalities!

P

Helmholtz decomposition

P_{curl} | P_{div}

Korn-part | **FIT-part**

$\approx \nabla(|\cdot|^{2-n} * (\mathrm{div}(P))$ | $\approx \mathrm{curl}(|\cdot|^{2-n} * \mathrm{curl}(P))$

Korn-part

$$\begin{aligned}\|P_{\mathrm{curl}}\|_{L^{p^*}} &= \|Du\|_{L^{p^*}} \lesssim \|\mathcal{A}[Du]\|_{L^{p^*}}\\ &= c\|\mathcal{A}[P - P_{\mathrm{div}}]\|_{L^{p^*}} \lesssim \|\mathcal{A}[P]\|_{L^{p^*}} + \|P_{\mathrm{div}}\|_{L^{p^*}}\,.\end{aligned}$$

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Fractional integration part

$$\begin{aligned}\|P_{\mathrm{div}}\|_{L^{p^*}} &\lesssim \|I_1(\mathrm{curl}(P))\|_{L^{p^*}} \\ &\lesssim \|\mathrm{curl}(P)\|_{L^p}\end{aligned}$$

...**but fails if $p = 1$!**

The case $(p, n) \neq (1, 2)$

Idea: Reduction of KMS to Korn-type inequalities!

P

Helmholtz decomposition

P_{curl} — **Korn-part** $\approx \nabla(|\cdot|^{2-n} * (\mathrm{div}(P))$

P_{div} — **FIT-part** $\approx \mathrm{curl}(|\cdot|^{2-n} * \mathrm{curl}(P))$

Fractional integration part

$$
\begin{aligned}
\|P_{\mathrm{div}}\|_{\mathrm{L}^{p^*}} &\lesssim \|I_1(\mathrm{curl}(P))\|_{\mathrm{L}^{p^*}} \\
&\lesssim \|\mathrm{curl}(P)\|_{\mathrm{L}^p}
\end{aligned}
$$

...**but fails if $p = 1$!**

The case $(p, n) \neq (1, 2)$

Idea: Reduction of KMS to Korn-type inequalities!

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Helmholtz decomposition

P_{curl} | P_{div}

Korn-part $\approx \nabla(|\cdot|^{2-n} * (\mathrm{div}(P))$

FIT-part $\approx \mathrm{curl}(|\cdot|^{2-n} * \mathrm{curl}(P))$

Bourgain-Brezis-type estimate

$$u \in C^{\infty}_{c,\mathrm{div}}(\mathbb{R}^n;\mathbb{R}^n) \Longrightarrow \|u\|_{L^{1^*}} \lesssim \|\mathrm{curl}(u)\|_{L^1(\mathbb{R}^n)}$$
$$\Longrightarrow \|P_{\mathrm{div}}\|_{L^{\frac{n}{n-1}}} \lesssim \|\mathrm{Curl}(P_{\mathrm{div}})\|_{L^1} = c\|\mathrm{Curl}(P)\|_{L^1}.$$

The case $(p, n) \neq (1, 2)$: Conclusion

$\|P\|_{L^{p^*}}$

Helmholtz decomposition

$\|P_{\text{curl}}\|_{L^{p^*}}$ — **Korn-part**: $\lesssim \|\mathcal{A}[P]\|_{L^{p^*}} + \|P_{\text{div}}\|_{L^{p^*}}$

$\|P_{\text{div}}\|_{L^{p^*}}$ — **FIT-part**: $\lesssim \|\text{Curl}(P)\|_{L^{p}}$

KMS-type inequalities

$$\|P\|_{L^{p^*}} \lesssim \|\mathcal{A}[P]\|_{L^{p^*}} + \|\text{Curl}(P)\|_{L^{p}} \qquad \text{for all } P \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^{n\times n}).$$

... but what if $(p, n) = (1, 2)$?

The case $(p, n) \neq (1, 2)$: Conclusion

$$\|P\|_{L^{p^*}}$$

Helmholtz decomposition

$\|P_{\mathrm{curl}}\|_{L^{p^*}}$ — **Korn-part**: $\lesssim \|\mathcal{A}[P]\|_{L^{p^*}} + \|P_{\mathrm{div}}\|_{L^{p^*}}$

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... but what if $(p, n) = (1, 2)$?

Existence of good almost complementary parts

Theorem (Existence of good almost complementary parts)

Let $\mathcal{A}\colon \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ be such that $\mathbb{A}u = \mathcal{A}[\nabla u]$ is $\mathbb{C}$-elliptic. Then there exist

- *a linear map $\mathcal{L}\colon \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$,*
- *a linear map $\gamma\colon \mathbb{R}^{2\times 2} \to \mathbb{R}$, and*
- *$\mathbf{Q} \in \mathrm{GL}(2)$ such that*

$$X - \mathcal{L}\mathcal{A}[X] = \gamma(X)\mathbf{Q} \qquad \text{for all } X \in \mathbb{R}^{2\times 2}.$$

Idea: $\mathcal{A}[\mathbf{e} \otimes \mathbf{f}] = \mathbb{A}[\mathbf{f}]\mathbf{e} =: \mathbf{e} \otimes_{\mathbb{A}} \mathbf{f}$. Need:

$$\exists \gamma_{ij}\ (\text{not all } = 0)\exists \mathbf{Q} \in \mathrm{GL}(2)\colon\ \forall i,j\colon\ \mathbf{e}_i \otimes \mathbf{e}_j - \mathcal{L}(\mathbf{e}_i \otimes_{\mathbb{A}} \mathbf{e}_j) = \gamma_{ij}\mathbf{Q}$$

$\longrightarrow$ uses interplay between $\mathbb{C}$-ellipticity and cancellation.

Finishing the proof

Write

$$P = \mathcal{L}[\mathcal{A}[P]] + \gamma(P)\mathbf{Q} \quad \text{with } \mathbf{Q} \in \mathrm{GL}(2)$$

and note that

$$\mathrm{Curl}(\gamma(P)\mathbf{Q}) = \mathbf{Q}\begin{pmatrix} -\partial_2\gamma \\ \partial_1\gamma \end{pmatrix}$$

Then

$$\mathbf{F} := \mathbf{Q}^{-1}\mathrm{Curl}(P) = \mathbf{Q}^{-1}\mathrm{Curl}(\mathcal{L}[\mathcal{A}[P]]) + \cancel{\mathbf{Q}^{-1}\mathbf{Q}}\begin{pmatrix} -\partial_2\gamma \\ \partial_1\gamma \end{pmatrix}$$

and so with a second order differential operator $\mathbb{B}$

$$\mathrm{div}(\mathbf{F}) = \mathrm{div}\Big(\mathbf{Q}^{-1}\mathrm{Curl}(\mathcal{L}[\mathcal{A}[P]])\Big) =: \mathbb{B}[\mathcal{A}[P]].$$

Bourgain-Brezis in two dimensions

$$\|f\|_{\dot{W}^{-1,2}} \lesssim \|\mathrm{div}(f)\|_{\dot{W}^{-2,2}} + \|f\|_{L^1} \qquad \text{for all } f \in C_c^\infty(\mathbb{R}^2;\mathbb{R}^2).$$

Finishing the proof

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Bourgain-Brezis in two dimensions

$$\|f\|_{\dot{\mathrm{W}}^{-1,2}} \lesssim \|\mathrm{div}(f)\|_{\dot{\mathrm{W}}^{-2,2}} + \|f\|_{\mathrm{L}^1} \qquad \text{for all } f \in \mathrm{C}_c^\infty(\mathbb{R}^2;\mathbb{R}^2).$$

Finishing the proof

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Bourgain-Brezis in two dimensions

$$\|\mathbf{F}\|_{\dot{W}^{-1,2}} \lesssim \|\mathrm{div}(\mathbf{F})\|_{\dot{W}^{-2,2}} + \|\mathbf{F}\|_{L^1} \qquad \text{for all } P \in C_c^\infty(\mathbb{R}^2; \mathbb{R}^{2\times 2}).$$

Finishing the proof

Write

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Bourgain-Brezis in two dimensions

$$\|\mathbf{F}\|_{\dot{\mathrm{W}}^{-1,2}} \lesssim \|\mathbb{B}[\mathcal{A}[P]]\|_{\dot{\mathrm{W}}^{-2,2}} + \|\mathbf{F}\|_{\mathrm{L}^1} \qquad \text{for all } P \in \mathrm{C}_c^\infty(\mathbb{R}^2;\mathbb{R}^{2\times 2}).$$

Finishing the proof

Write

$$P=\mathcal{L}[\mathcal{A}[P]]+\gamma(P)\mathbf{Q}\quad\text{with } \mathbf{Q}\in\mathrm{GL}(2)$$

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Bourgain-Brezis in two dimensions

$$\|\mathbf{F}\|_{\dot{\mathrm{W}}^{-1,2}}\lesssim\|\mathcal{A}[P]\|_{\mathrm{L}^2}+\|\mathbf{F}\|_{\mathrm{L}^1}\qquad\text{for all } P\in\mathrm{C}_c^\infty(\mathbb{R}^2;\mathbb{R}^{2\times 2}).$$

Finishing the proof

Write

$$P=\mathcal{L}[\mathcal{A}[P]]+\gamma(P)\mathbf{Q}\quad\text{with } \mathbf{Q}\in\mathrm{GL}(2)$$

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$$\mathrm{div}(\mathbf{F})=\mathrm{div}\Big(\mathbf{Q}^{-1}\mathrm{Curl}(\mathcal{L}[\mathcal{A}[P]])\Big)=:\mathbb{B}[\mathcal{A}[P]].$$

Bourgain-Brezis in two dimensions

$$\|\mathrm{Curl}(P)\|_{\dot{\mathrm{W}}^{-1,2}}\lesssim\|\mathcal{A}[P]\|_{\mathrm{L}^2}+\|\mathrm{Curl}(P)\|_{\mathrm{L}^1}\qquad\text{for all } P\in\mathrm{C}_c^\infty(\mathbb{R}^2;\mathbb{R}^{2\times2}).$$

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$$\mathrm{div}(\mathbf{F}) = \mathrm{div}\Big(\mathbf{Q}^{-1}\mathrm{Curl}(\mathcal{L}[\mathcal{A}[P]])\Big) =: \mathbb{B}[\mathcal{A}[P]].$$

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$$\|\mathrm{Curl}(P)\|_{\dot{\mathrm{W}}^{-1,2}} \lesssim \|\mathcal{A}[P]\|_{\mathrm{L}^2} + \|\mathrm{Curl}(P)\|_{\mathrm{L}^1} \qquad \text{for all } P \in \mathrm{C}_c^\infty(\mathbb{R}^2;\mathbb{R}^{2\times 2}).$$

$(p,n)=(1,2)$: Conclusion

- Now put $f:=\operatorname{Curl}(P)$ and solve $-\Delta u=f$. Then

$$\begin{aligned}\|Du\|_{L^2(\mathbb{R}^2)} &\lesssim \|(-\Delta u)\|_{\dot{W}^{-1,2}(\mathbb{R}^2)} = c\|f\|_{\dot{W}^{-1,2}(\mathbb{R}^2)}\\ &= c\|\operatorname{Curl}(P)\|_{\dot{W}^{-1,2}(\mathbb{R}^2)}\\ &\lesssim \|\mathcal{A}[P]\|_{L^2}+\|\operatorname{Curl}(P)\|_{L^1}\end{aligned}$$

- Introduce

$$T=\begin{pmatrix}\partial_2 u_1 & -\partial_1 u_1\\ \partial_2 u_2 & -\partial_1 u_2\end{pmatrix}$$

$$\text{then: } \|T\|_{L^2(\mathbb{R}^2)} \lesssim \|Du\|_{L^2(\mathbb{R}^2)} \lesssim \|\mathcal{A}[P]\|_{L^2}+\|\operatorname{Curl}(P)\|_{L^1}$$

- $\operatorname{Curl}(T-P)=-\Delta u-f=0$, hence $T-P=Dv$, so

$$\begin{aligned}\|P\|_{L^2(\mathbb{R}^2)} &\leq \|T-P\|_{L^2(\mathbb{R}^2)}+\|T\|_{L^2(\mathbb{R}^2)}\\ &\lesssim \|Dv\|_{L^2(\mathbb{R}^2)}+\|\mathcal{A}[P]\|_{L^2(\mathbb{R}^2)}+\|\operatorname{Curl}(P)\|_{L^1(\mathbb{R}^2)}\end{aligned}$$

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- $\mathrm{Curl}(T-P) = -\Delta u - f = 0$, hence $T-P = Dv$, so

$$\begin{aligned}\|P\|_{\mathrm{L}^2(\mathbb{R}^2)} &\leq \|T-P\|_{\mathrm{L}^2(\mathbb{R}^2)} + \|T\|_{\mathrm{L}^2(\mathbb{R}^2)} \\ &\lesssim \|\mathcal{A}[Dv]\|_{\mathrm{L}^2(\mathbb{R}^2)} + \|\mathcal{A}[P]\|_{\mathrm{L}^2(\mathbb{R}^2)} + \|\mathrm{Curl}(P)\|_{\mathrm{L}^1(\mathbb{R}^2)}\end{aligned}$$

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$(p, n) = (1, 2)$: Conclusion

- Now put $f := \mathrm{Curl}(P)$ and solve $-\Delta u = f$. Then

$$
\begin{aligned}
\|Du\|_{L^2(\mathbb{R}^2)} &\lesssim \|(-\Delta u)\|_{\dot{W}^{-1,2}(\mathbb{R}^2)} = c\|f\|_{\dot{W}^{-1,2}(\mathbb{R}^2)} \\
&= c\|\mathrm{Curl}(P)\|_{\dot{W}^{-1,2}(\mathbb{R}^2)} \\
&\lesssim \|\mathcal{A}[P]\|_{L^2} + \|\mathrm{Curl}(P)\|_{L^1}
\end{aligned}
$$

- Introduce

$$T = \begin{pmatrix} \partial_2 u_1 & -\partial_1 u_1 \\ \partial_2 u_2 & -\partial_1 u_2 \end{pmatrix}$$

then: $\|T\|_{L^2(\mathbb{R}^2)} \lesssim \|Du\|_{L^2(\mathbb{R}^2)} \lesssim \|\mathcal{A}[P]\|_{L^2} + \|\mathrm{Curl}(P)\|_{L^1}$

- $\mathrm{Curl}(T - P) = -\Delta u - f = 0$, hence $T - P = Dv$, so

$$
\begin{aligned}
\|P\|_{L^2(\mathbb{R}^2)} &\leq \|T - P\|_{L^2(\mathbb{R}^2)} + \|T\|_{L^2(\mathbb{R}^2)} \\
&\lesssim \|T\|_{L^2(\mathbb{R}^2)} + \|\mathcal{A}[P]\|_{L^2(\mathbb{R}^2)} + \|\mathcal{A}[P]\|_{L^2(\mathbb{R}^2)} + \|\mathrm{Curl}(P)\|_{L^1(\mathbb{R}^2)}
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Main Theorem:

Many thanks for your attention!