Low Regularity Inextendibility of Spacetimes

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Motivation

1) **Relativity and Determinism:**

Strong Cosmic Censorship Conjecture (Penrose)

For *generic* asymptotically flat initial data for the vacuum Einstein Equations $\mathrm{Ric}(g) = 0$ the maximal globally hyperbolic development is inextendible as a *suitably regular* Lorentzian manifold.

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- Now believed (spoiler!): *suitably regular* means C^0 spacetime (and $\partial g \in L^2_{loc}$).
- Physically, the conjecture encodes determinism in General Relativity.

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2) **Singularity Classification**

Below C^2, classical causality theory breaks down...

$C^{0,1}$ spacetimes:

- Geodesic equation is not a classical ODE: solutions exist (e.g. Filippov solutions) but are not unique.
- Causal maximizers have a fixed causal character almost everywhere.

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C^0 spacetimes:

- Geodesic equation is not well defined.
- Causal maximizers do not have a fixed causal character.
- C^0 Avez-Seifert Theorem: global hyperbolicity guarantees existence of (global) causal maximizers. Moreover, local causal maximizers always exist.

Definition 1: Fix $k \geq 0$ and let $0 \leq l \leq k$. Let (M,g) be a C^k spacetime (i.e. a connected time-oriented Lorentzian manifold) with dimension d:

- A $\boldsymbol{C^l}$**-extension** of (M, g) is a proper isometric embedding ι

$$\iota \ : \ (M, g) \hookrightarrow (M_{\mathrm{ext}}, g_{\mathrm{ext}})$$

  where $(M_{\mathrm{ext}}, g_{\mathrm{ext}})$ is C^l spacetime of dimension d. If such an embedding exists, then (M, g) is said to be $\boldsymbol{C^l}$**-extendible**. The topological boundary of M within M_{ext} is $\partial\iota(M)$.

Definition 2: Let $\iota : (M, g) \to (M_{ext}, g_{ext})$ be a C^k-extension $(k \geq 0)$.

- **Future boundary** of M:
  $\partial^+\iota(M) := \{p \in \partial\iota(M) \ : \ \exists$ f.d.t.l. $\gamma : [0,1] \to M_{ext}$ with $\gamma(1) = p, \ \gamma([0,1) \subset \iota(M)\}$
- **Past boundary** of M:
  $\partial^-\iota(M) := \{p \in \partial\iota(M) \ : \ \exists$ f.d.t.l. $\gamma : [0,1] \to M_{ext}$ with $\gamma(0) = p, \ \gamma((0,1] \subset \iota(M)\}$

Introduction: A first crucial result

Proposition 3 (Sbierski (2018))

Let $\iota \colon (M, g) \to (M_{\mathrm{ext}}, g_{\mathrm{ext}})$ be a C^0extension. Then:

$$\partial^+ \iota(M) \cup \partial^- \iota(M) \neq \emptyset$$

Low regularity inextendibility criteria

Question: Can we get an inextendibility criteria from timelike geodesic completeness?

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Setting: Let (M, g) be a C^2 timelike geodesically complete spacetime and $\iota : M \to M_{\mathrm{ext}}$ a C^0 extension.

1. Fix $p \in \partial^+ \iota(M)$ and a globally hyperbolic neighbourhood V. Then, there exists a causal maximizer (Sämann, 2016) $\gamma_{\max} : [0, 1] \to M_{\mathrm{ext}}$ from $q := \gamma_{\max}(0) \in I^-(p, V) \cap \iota(M)$, to p:

$$L(\gamma) \leq L(\gamma_{\max}) < \infty$$

$\forall$ causal γ with the same endpoints as $\gamma_{\max}$.

Low regularity inextendibility criteria

Question: Can we get an inextendibility criteria from timelike geodesic completeness?

Setting: Let (M, g) be a C^2 timelike geodesically complete spacetime and $\iota : M \to M_{\mathrm{ext}}$ a C^0 extension.

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$$L(\gamma) \leq L(\gamma_{\max}) < \infty$$

   $\forall$ causal γ with the same endpoints as $\gamma_{\max}$.

2. Simple case: Assume $\gamma_{\max}([0,1)) \subset \iota(M)$. Then, $\iota^{-1} \circ \gamma_{\max}|_{[0,1)}$ is a future inext. t.l. geodesic, so $L(\gamma_{\max}) = \infty$ (Contradiction!). So (M, g) is C^0 inextendible.

Low regularity inextendibility criteria

Problems:

- In general, $\gamma_{\max}([0,1)) \not\subset \iota(M)$.
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Solutions:

- Galloway, Ling, Sbierski: Assume (M, g) is also globally hyperbolic. Then the portion of $\gamma_{\max}$ in $\iota(M)$ is a timelike geodesic.

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Solutions:

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- Graf, Ling: Assume $(M_{\text{ext}}, g_{\text{ext}})$ is a $C^{0,1}$ spacetime. Then $\gamma_{\max}$ is timelike a.e.

Low regularity inextendibility criteria

Proposition 4 (Galloway, Ling, Sbierski (2018))

A smooth (at least C^2) spacetime that is timelike geodesically complete and globally hyperbolic is C^0-inextendible.

Proposition 5 (Graf, Ling (2019))

Let (M, g) be a smooth timelike geodesically complete spacetime. Then (M, g) is $C^{0,1}$-inextendible.

Low regularity inextendibility criteria

Theorem 6 (Minguzzi, Suhr (2019))

Let (M, g) be a smooth timelike geodesically complete spacetime. Then it is C^0 inextendible.

Low regularity inextendibility criteria

Examples:

- C^0 inextendibility of Schwarzschild and Minkowski (Sbierski (2018)).
- C^0 inextendibility of anti de Sitter (Galloway, Ling (2018)).
- $C^{0,1}_{\mathrm{loc}}$ inextendibility of FLRW (Sbierski (2022)).

FLRW spacetimes

Definition 7: An open FLRW spacetime is a spacetime (M,g) where $M = (0,\infty) \times \mathbb{R}^d$ with coordinates (t,r,ω)$\in (0,\infty) \times (0,\infty) \times \mathbb{S}^{d-1}$ and:

$$g = \begin{cases} -dt^2 + a^2(t)(dr^2 + r^2 d\Omega^2_{d-1}) & \text{Euclidian} \\ -dt^2 + a^2(t)(dr^2 + sinh^2(r) d\Omega^2_{d-1}) & \text{Hyperbolic} \end{cases} \quad (1)$$

where $a \ : \ (0,\infty) \to (0,\infty)$ satisfies:

1. a has to be smooth
2. $\lim_{t\to 0^+} a(t) = 0$
3. $\exists \ m > 0, b \geq 0$ such that $a(t) \leq mt + b \ \forall t$
4. $a'(t) > 0 \ \forall \ t$

FLRW spacetimes

Theorem 8

Let (M, g) be an open FLRW spacetime. If $(M_{\mathrm{ext}}, g_{\mathrm{ext}})$ is a C^0-extension of (M, g), then $\partial^+ \iota(M) = \emptyset$ and $\partial^- \iota(M)$ is an achronal topological hypersurface.

Strongly spherically symmetric spacetimes

Definition 9: (M, g) is called a **strongly spherically symmetric (StSp) spacetime** if $\forall p \in M$ there exists a neighbourhood U, a change of coordinates ψ_s and coordinates $(T, R) : U \to (0, \infty)$ and $\omega : U \to \mathbb{S}^{d-1}$ such that:

$$g_s := (\psi_s)_* g = -F(T, R)dT^2 + G(T, R)dR^2 + R^2 d\Omega^2_{d-1} \qquad (2)$$

with $F, G : (0, \infty) \times (0, \infty) \to (0, \infty)$ smooth.

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with $F, G : (0, \infty) \times (0, \infty) \to (0, \infty)$ smooth.

Definition 10: Let (M, g) be an FLRW spacetime. We call a change of coordinates $\psi_s : (t, r) \mapsto (T, R)$ such that $g_s := (\psi_s)_* g$ is as in (2) a **natural strongly spherical change of coordinates**. If it exists $\forall p \in M$, we write (M, g, ψ_s).

Theorem 11: Part 1 (Galloway, Ling)

Let (M, g) be a Euclidean FLRW spacetime with $a'(0) := \lim_{t\to 0^+} a'(t) \in (0, \infty]$. Then, up to an initial condition, there exists a unique natural StSp change of coordinates ψ_s with:

$$g_s \coloneqq (\psi_s)_* g = -F(T,R)dT^2 + G(T,R)dR^2 + R^2 d\Omega^2_{d-1} \qquad (3)$$

with F and G are regular a.e.

Strongly spherically symmetric spacetimes

Theorem 11: Part 2 (Galloway, Ling)

Let $\iota : M \to M_{\text{ext}}$ be a C^0-extension M_{ext} and $\gamma : [0,1] \to M_{ext}$ a future directed timelike curve with past endpoint $\gamma(0) \in \partial^- \iota(M)$, and suppose R has a finite positive limit along γ as $t \to 0^+$. Then, the following holds along γ:

- $\lim_{t\to 0^+} G(\iota^{-1} \circ \gamma(t)) = 0$.
- If F has a finite nonzero limit as $t \to 0^+$, then $T \to \pm\infty$ as $t \to 0^+$.

Strongly spherically symmetric spacetimes

Definition 12: Let (M, g, ψ_s) be an FLRW spacetime and $\iota : (M, g) \to (M_{\text{ext}}, g_{\text{ext}})$ be a C^0-extension. It is a **natural strongly spherical C^0-extension** if:

1. $\forall p \in \partial^- \iota(M)$ there exists a StSp change of coordinates ψ_{ext} defined on a neighbourhood U.
2. In $\iota^{-1}(U)$:

$$\psi_s|_{\iota^{-1}(U)} = \psi_{\text{ext}} \circ \iota|_{\iota^{-1}(U)} \tag{4}$$

The previous expression actually implies that:

$$g_{\text{ext},s}|_{\iota(M)\cap U} = (\psi_{\text{ext}})_* \iota_* g = (\psi_s)_* g =: g_s \tag{5}$$

Strongly spherically symmetric spacetimes

Corollary 13

Let (M, g, ψ_s) be a Euclidean FLRW spacetime with $a'(0) \in (0, \infty]$. Then, it has no natural strongly spherically symmetric C^0 extension.

Proof:

- Fix $p \in \partial^- \iota(M)$. There $\exists$ a ψ_{ext} such that $g_{\text{ext},s} := (\psi_{\text{ext}})_* g_{\text{ext}}$ is:

$$g_{\text{ext},s} = -F_{\text{ext}} dT_{\text{ext}}^2 + G_{\text{ext}} dR_{\text{ext}}^2 + R_{\text{ext}}^2 d\Omega_{\text{ext},d-1}^2 \tag{6}$$

  in a neighbourhood U of p.

- Recall that $g_{\text{ext},s}|_{\iota(M)\cap U} = g_s$. This implies that:

$$(T_{\text{ext}}, R_{\text{ext}}, \omega_{\text{ext}})|_{\iota(M)\cap U} = (T, R, \omega) \tag{7}$$

$$(F_{\text{ext}}, G_{\text{ext}})|_{\iota(M)\cap U} = (F, G) \tag{8}$$

- Let $\gamma : [0,1] \to M_{\text{ext}}$ be a f.d. t.l. curve with $\gamma((0,1]) \subset \iota(M)$ and $\gamma(0) = p$. Then, $\lim_{s\to 0} R(\gamma(s)) \neq 0$ as $R_{\text{ext}} \in (0,\infty)$.

Strongly spherically symmetric spacetimes

Proof:

- The metric g_s degenerates as $t \to 0$ because $\lim_{t\to 0^+} G(\gamma(t)) = 0$. Hence, as $g_{\mathrm{ext},s} = g_s$, also $\lim_{t\to 0^+} G_{\mathrm{ext}}(\gamma(t)) = 0$.
  $\to (M_{\mathrm{ext}}, g_{\mathrm{ext}})$ also degenerates!

Strongly cylindrically symmetric spacetimes

Definition 14: Let (M, g) be a 4-dimensional spacetime. It is a **strongly cylindrically symmetric (StSc) spacetime** if $\forall p \in M$ $\exists$ a neighbourhood U, a change of coordinates ψ_c and coordinates $T, \rho : U \to (0, \infty)$, $z : U \to \mathbb{R}$ and $\varphi : U \to (0, 2\pi)$ with $g_c := (\psi_c)_* g$

$$g_c = -A(T, z, \rho)dT^2 + B(T, z, \rho)dz^2 + C(T, z, \rho)d\rho^2 + \rho^2 d\varphi^2 \tag{9}$$

$A, B, C : U \to (0, \infty)$ smooth functions.

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$A, B, C : U \to (0, \infty)$ smooth functions.

Definition 15: Let (M, g) be a StSp spacetime. We call a $\psi_c : (R, \theta) \mapsto (z, \rho)$ such that $g_c = (\psi_c)_*(\psi_s)_* g$ is as in (9) a **natural strongly cylindrical change of coordinates**. If it exists $\forall p \in M$, we write (M, g, ψ_c).

Theorem 16: Let (M, g) be a StSp spacetime. Subject to:

- a suitable initial condition
- the choice of θ and φ

there exists a unique natural StSc transformation ψ_c. Moreover, the metric coefficients A, B, C are regular a.e.

Strongly cylindrically symmetric spacetimes

Proof:

- Ansatz: $\exists\, \psi_c : (R, \theta) \mapsto (z, \rho)$ smooth and invertible. Then:

$$\begin{cases} z = z(R,\theta) \longrightarrow dz^2 = z_R^2 dR^2 + z_\theta^2 d\theta^2 + 2 z_R z_\theta dR d\theta \\ \rho = \rho(R,\theta) \longrightarrow d\rho^2 = \rho_R^2 dR^2 + \rho_\theta^2 d\theta^2 + 2\rho_R \rho_\theta dR d\theta \end{cases}$$

- It holds that $(\psi_c)_*(\psi_s)_* g = (\psi_s)_* g$:

$$-FdT^2 + GdR^2 + R^2(d\theta^2 + \sin^2\theta d\varphi^2) = -AdT^2 + Bdz^2 + Cd\rho^2 + \rho^2 d\varphi^2$$

So, $A = F$ and $\rho = R\sin\theta$.

Strongly cylindrically symmetric spacetimes

- Moreover:

$$Bz_R^2 = G - C\rho_R^2 \tag{10a}$$

$$Bz_\theta^2 = R^2 - C\rho_\theta^2 \tag{10b}$$

$$Bz_R z_\theta = -C\rho_R\rho_\theta \tag{10c}$$

- Squaring (10c), plugging (10a) and (10b) in it:

$$(G - C\rho_R^2)(R^2 - C\rho_\theta^2) = C^2\rho_R^2\rho_\theta^2 \tag{11}$$

where will assume $(G\cos^2\theta + \sin^2\theta) \neq 0$, $z_\theta \neq 0$.

- As $\rho_\theta = R\cos\theta$ and $\rho_R = \sin\theta$, we get:

$$C = \frac{G}{G\cos^2\theta + \sin^2\theta} \tag{12}$$

Strongly cylindrically symmetric spacetimes

Proof:

- Replacing C in (10b):

$$\frac{z_R}{z_\theta} = -\frac{G\cos\theta}{R\sin\theta} \tag{13}$$

- Linear PDE. By method of characteristics:

$$z(R,\theta) = f\left(\int \frac{G(T,R)}{R}dR + \ln|\cos\theta|\right) \tag{14}$$

- Plugging C and $z_\theta = -f'\tan\theta$ in (10b) with $f' \neq 0$:

$$B = \frac{R^2(1 - C\cos^2\theta)}{z_\theta^2} = \frac{R^2\cos^2\theta}{f'^2(G\cos^2\theta + \sin^2\theta)} \tag{15}$$

Proof:

- Summing up:

$$\begin{cases} z(R,\theta) = f\left(\int \frac{G(T,R)}{R} dR + \ln|\cos\theta|\right) \\ \rho(R,\theta) = R\sin\theta \end{cases} \tag{16}$$

$$\begin{cases} A(T,z,\rho) = F(T,R) \\ B(T,z,\rho) = \frac{R^2\cos^2\theta}{f'^2(G\cos^2\theta+\sin^2\theta)} \\ C(T,z,\rho) = \frac{G}{G\cos^2\theta+\sin^2\theta} \end{cases} \tag{17}$$

- So the metric is:

$$g_c = -FdT^2 + \frac{1}{G\cos^2\theta + \sin^2\theta}\left(\frac{R^2\cos^2\theta}{f'^2}dz^2 + GdR^2\right) + \rho^2 d\varphi^2 \tag{18}$$

Strongly cylindrically symmetric spacetimes

- But is this a well defined change of coordinates? Determinant of the Jacobian:

$$J = z_R \rho_\theta - z_\theta \rho_R = f' \frac{G \cos^2\theta + \sin^2\theta}{\cos\theta} \tag{19}$$

- Change of coordinates well defined except on measure zero sets $G = -\tan^2\theta$, $\theta = \pi/2$
- $\underline{f' \neq 0}$: If $f' = 0$, then also $z_R = z_\theta = 0$. Plugging this in (10a), (10b) and (10c):

$$\begin{cases} G = C \sin^2\theta \\ 1 = C \cos^2\theta \\ 0 = CR \cos\theta \sin\theta \end{cases} \tag{20}$$

- $\underline{z_\theta \neq 0}$: If $z_\theta = 0$ we recover the 2 last equations, which cannot be satisfied simultaneously.

Definition 17: Let (M, g) be a StSp spacetime with natural StSc change of coordinates ψ_c and $\iota : (M, g) \to (M_{\text{ext}}, g_{\text{ext}})$ a C^0-extension. It is a **natural strongly cylindrical extension** if:

1. $\forall p \in \partial\iota(M)$, there exists a StSc change of coordinates $\widetilde{\psi}_{ext}$ defined on a neighbourhood U of p.
2. In $\iota^{-1}(U) \subset M$:

$$\psi_c \circ \psi_s|_{\iota^{-1}(U)} = \widetilde{\psi}_{\text{ext}} \circ \iota|_{\iota^{-1}(U)} \tag{21}$$

So:

$$g_{\text{ext},c}|_{\iota(M)\cap U} = (\widetilde{\psi}_{\text{ext}})_* g_{\text{ext}}|_{\iota(M)\cap U} = (\widetilde{\psi}_{\text{ext}})_* \iota_* g = (\psi_c)_*(\psi_s)_* g \tag{22}$$

i.e. $g_{\text{ext},c} = g_c$.

Strongly cylindrically symmetric spacetimes

Theorem 18

Let (M, g) be a StSp spacetime, $\iota : (M, g) \to (M_{\mathrm{ext}}, g_{\mathrm{ext}})$ a C^0 extension and $\gamma : [0, 1] \to M_{\mathrm{ext}}$ a curve with $\gamma(0, 1) \subset \iota(M)$ and $\gamma(0) = p \in \partial\iota(M)$. If one of the following conditions is satisfied:

1. $\lim_{s\to 0} F(\gamma(s)) = 0$.
2. $\lim_{s\to 0} G(\gamma(s)) = 0$ but $\lim_{s\to 0} \theta(\gamma(s)) \notin \{0, \pi\}$.
3. $+$ more ...

Then, there exists no natural StSc extension of (M, g).

Strongly cylindrically symmetric spacetimes

Proof:

- Let $\gamma : [0,1] \to M_{\mathrm{ext}}$ be a curve with $\gamma(0,1) \subset \iota(M)$ and $\gamma(0) = p \in \partial\iota(M)$ such that (at least) one of the conditions of the Theorem is satisfied. Recall:

$$A(T,z,\rho) = F(T,R)$$
$$B(T,z,\rho) = \frac{R^2 \cos^2\theta}{f'^2(G\cos^2\theta + \sin^2\theta)}$$
$$C(T,z,\rho) = \frac{G}{G\cos^2\theta + \sin^2\theta}$$

- If condition 1 is satisfied A vanishes and if condition 2 is satisfied C vanishes
  $\to$ the metric $g_c = (\psi_c)_* g_s$ degenerates, where ψ_c is the natural StSc transf.

Proof:

- The rest is by contradiction: assume there exists a natural StSc transf. $\widetilde{\psi}_{\mathrm{ext}}$ such that $g_{\mathrm{ext},c} = (\widetilde{\psi}_{\mathrm{ext}})_* g_{\mathrm{ext}}$ and $g_{\mathrm{ext},c}|_{\iota(M)\cap U} = g_c$, where U is a neighbourhood of p, so:

$$(A_{\mathrm{ext}}, B_{\mathrm{ext}}, C_{\mathrm{ext}})|_{\iota(M)\cap U} = (A, B, C) \tag{23}$$

$\rightarrow$ Also $(M_{\mathrm{ext}}, g_{\mathrm{ext}})$ degenerates!

Strongly cylindrically symmetric spacetimes

Corollary 19

Let (M, g, ψ_s) be a (4-dimensional) open Euclidean FLRW spacetime satisfying that $a'(0) \in (0, \infty]$. Then, it has no natural strongly cylindrical C^0-extension compatible with the natural ψ_s.

Strongly cylindrically symmetric spacetimes

Proof:

- We apply the following 2-step change of coordinates to (M, g):

$$\{t, r, \theta, \varphi\} \xrightarrow{\psi_s} \begin{cases} T = T(t, r) \\ R = R(t, r) \\ \theta, \varphi \text{ same} \end{cases} \xrightarrow{\psi_c} \begin{cases} z = z(R, \theta) \\ \rho = \rho(R, \theta) \\ T, \varphi \text{ same} \end{cases} \tag{24}$$

so the metric becomes $g_c = (\psi_c)_* (\psi_s)_* g$

Proof:

- The rest of the proof is by contradiction: Let $\iota : (M, g) \to (M_{\rm ext}, g_{\rm ext})$ be a natural StSc C^0-extension compatible with the natural ψ_s in M. Unwinding the definitions:
  1. Fix $p \in \partial^- \iota(M)$. $\exists$ a neighbourhood U and a $\widetilde{\psi}_{\rm ext}$ s.t. $g_{{\rm ext},c} = (\widetilde{\psi}_{\rm ext})_* g_{\rm ext}$ is:
  $$g_{{\rm ext},c} = -A_{\rm ext} dT^2_{\rm ext} + B_{\rm ext} dz^2_{\rm ext} + C_{\rm ext} d\rho^2_{\rm ext} + \rho^2_{\rm ext} d\varphi^2_{\rm ext} \tag{25}$$
  2. In $\iota^{-1}(U)$ it holds that $\psi_c \circ \psi_s|_{\iota^{-1}(U)} = \widetilde{\psi}_{\rm ext} \circ \iota|_{\iota^{-1}(U)}$. So $g_{{\rm ext},c}|_{\iota(M)\cap U} = g_c$. This implies that:
  $$(A_{\rm ext}, B_{\rm ext}, C_{\rm ext})|_{\iota(M)\cap U} = (A, B, C) \tag{26}$$
  $$(T_{\rm ext}, z_{\rm ext}, \rho_{\rm ext})|_{\iota(M)\cap U} = (T, z, \rho) \tag{27}$$

Strongly cylindrically symmetric spacetimes

Proof:

- Let $\gamma : [0,1] \to M_{\text{ext}}$ be a f.d.t.l curve with $\gamma(0,1] \subset \iota(M)$ and $\gamma(0) = p$. By a previous theorem, $\lim_{s\to 0} G(\gamma(s)) = 0$.
- Moreover, $\lim_{s\to 0} \theta(\gamma(s)) \neq 0$ as $\rho = R \sin\theta$ and $\rho|_{\iota(M)\cap U} = \rho_{\text{ext}}|_{\iota(M)\cap U} \in (0,\infty)$.
  $\to$ By previous theorem, g_c degenerates!
- As $g_{\text{ext},c} = g_c$, also $g_{\text{ext},c}$ is degenerate.

Existence of a unique maximal boundary

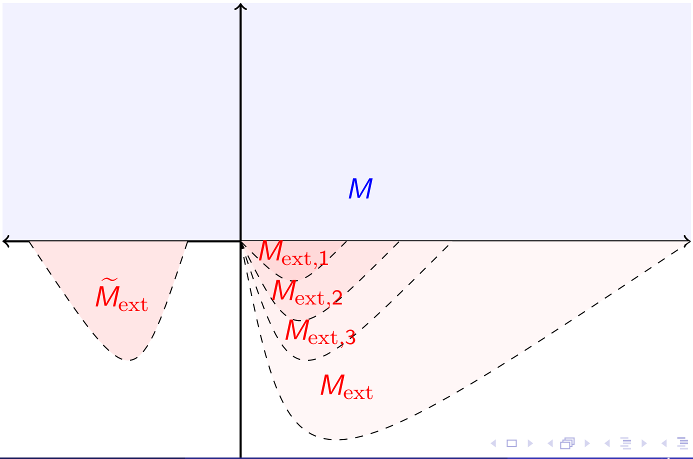
Question: Given a certain (extendible) spacetime (M, g), does there exist a (unique) maximal boundary? Are additional assumptions required?

Strategy:

- Step 1: Define a (well defined) notion of equivalence between extensions and a notion partial ordering on the collection of equivalence classes $\mathcal{I}$.

Strategy

- Step 2: Show that every partially ordered subset has an upper bound. By Zorn's Lemma, there exists a (set theoretic) maximal element: $\iota \in \mathcal{I}$ satisfying that $\iota_{\max} \leq_\partial \iota$, it necessarily holds that $\iota =_\partial \iota_{\max}$.



Existence of a unique maximal boundary

Strategy:

- Step 3 (hypothetical): Show that there exists a maximal element in the desired sense: $\forall \iota \in \mathcal{I}$ it holds that $\iota \leq \iota_{\max}$
  $\rightarrow$ Additional assumptions required? Or consider specific subclasses of spactimes?

Existence of a unique maximal boundary

Definition 20: Let $\iota_1 : M \to M_{\text{ext},1}$ and $\iota_2 : M \to M_{\text{ext},2}$ be 2 extensions. If there exists an embedding $\widehat{\psi}_{12} : \widehat{U}_1 \to \widehat{U}_2$, where $\partial\iota_1(M) \subset \widehat{U}_1$ and $\partial\iota_2(M) \subset \widehat{U}_2$ which is compatible with the extensions, i.e.:

$$\widehat{\psi}_{12} \circ \iota_1\big|_{\iota_1^{-1}(\iota_1(M)\,\cap\,\widehat{U}_1)} = \iota_2\big|_{\iota_1^{-1}(\iota_1(M)\,\cap\,\widehat{U}_1)} \tag{28}$$

and the restriction $\widehat{\psi}_{12} : \overline{\iota_1(M)} \cap \widehat{U}_1 \to \overline{\iota_2(M)} \cap \widehat{U}_2$ is surjective, we define the relation:

$$\iota_1 =_\partial \iota_2$$

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Lemma 21

The relation $=_\partial$ is an equivalence relation.

If $\iota_1 =_\partial \iota_2$, then $\overline{\iota_1(M)} \cap \widehat{U}_1$ is homeomorphic to $\overline{\iota_2(M)} \cap \widehat{U}_2$. In particular, $\partial\iota_1(M)$ is homeomorphic to $\partial\iota_2(M)$.

Existence of a unique maximal boundary

Theorem 22 (Sbierski 2022)

Let (M, g) be a globally hyperbolic spacetime with $g \in C^1$ and let ι_1, ι_2 be two $C^{0,1}_{\mathrm{loc}}$ extensions. Moreover, let $\gamma : [-1, 0) \to M$ be a f.d. causal curve such that

$$p_1 := \lim_{s \to 0} (\iota_1 \circ \gamma)(s) \in \partial^+ \iota_1(M) \quad \text{and} \quad p_2 := \lim_{s \to 0} (\iota_2 \circ \gamma)(s) \in \partial^+ \iota_2(M)$$

Then, there exist neighbourhoods $\widehat{U}_1$ of p_1 and $\widehat{U}_2$ of p_2 such that $\overline{\iota_1(M)} \cap \widehat{U}_1$ is diffeomorphic to $\overline{\iota_2(M)} \cap \widehat{U}_2$.

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Then, there exist neighbourhoods $\widehat{U}_1$ of p_1 and $\widehat{U}_2$ of p_2 such that $\overline{\iota_1(M)} \cap \widehat{U}_1$ is diffeomorphic to $\overline{\iota_2(M)} \cap \widehat{U}_2$.

If the assumptions of the previous theorem hold, we can define $\iota_{1,loc}$ and $\iota_{2,\mathrm{loc}}$ as restrictions of ι_1 and ι_2 and prove that:

$$\iota_{1,\mathrm{loc}} =_\partial \iota_{2,\mathrm{loc}}$$

Existence of a unique maximal boundary

Definition 23: Let $\iota_1 : M \to M_{\mathrm{ext},1}$ and $\iota_2 : M \to M_{\mathrm{ext},2}$ be extensions of (M, g) and U_1, U_2 be open neighbourhoods satisfying that $\partial\iota_1(M) \subset U_1$ and $\partial\iota_2(M) \subset U_2$. Moreover, let $\psi_{12} : U_1 \to U_2$ be an embedding satisfying that $\psi_{12}(\partial\iota_1(M)) \subset \partial\iota_2(M)$ and which is compatible with the extensions:

$$\psi_{12} \circ \iota_1|_{\iota_1^{-1}(\iota_1(M) \cap U_1)} = \iota_2|_{\iota_1^{-1}(\iota_1(M) \cap U_1)} \tag{29}$$

If such an embedding ψ_{12} exists, then we define the relation $\leq_\partial$ as follows:

$$\iota_1 \leq_\partial \iota_2$$

Existence of a unique maximal boundary

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If such an embedding ψ_{12} exists, then we define the relation $\leq_\partial$ as follows:

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Lemma 24

Let $[\iota]$ denote the equivalence class of the extension ι (under the eq. relation $=_\partial$). The collection of equivalence classes of all extensions ι together with the previously defined relation $\leq_\partial$ is a partially ordered set, which will be denoted $\mathcal{I}$.

Existence of a unique maximal boundary

Problems:

- Sbierski's (2022) result does not hold for C^0 extensions... Not even for $C^{0,\alpha}$ with $\alpha \in (0,1)$!

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- Topological issues might appear when constructing the upper bound... if it is too "large" it might not even be Hausdorff! See Choquet-Bruhat, Geroch (1969) and Chrusciel (2010).

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- Sbierski's (2022) result does not hold for C^0 extensions... Not even for $C^{0,\alpha}$ with $\alpha \in (0,1)$!
- Topological issues might appear when constructing the upper bound... if it is too "large" it might not even be Hausdorff! See Choquet-Bruhat, Geroch (1969) and Chrusciel (2010).
- Counterexamples such as the Taub-NUT or Misner spacetime show that additional assumptions on (M, g) might be required (probably on the behaviour of timelike curves reaching $\partial\iota(M)$).

Extra: Comparison with (classical) Lorentzian geometry

Let (M, g) be an arbitrary spacetime. Does it have an extension or not?

- Yes, we explicitly find/construct one.
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Example:

- $M = (0, \infty) \times \mathbb{R}$ and $g = e^{2\sqrt{t}}(-dt^2 + dx^2)$. Then $R = -\frac{1}{2e^{2\sqrt{t}}} \frac{1}{t^{3/2}}$
  $\longrightarrow$ No C^2-extension possible.

Extra

Theorem

Theorem: Let (M, g) be a StSp. spacetime, $\iota : (M, g) \to (M_{\text{ext}}, g_{\text{ext}})$ a C^0 extension and $\gamma : [0, 1] \to M_{\text{ext}}$ a curve with $\gamma(0, 1) \subset \iota(M)$ and $\gamma(0) = p \in \partial M$. If one of the following conditions is satisfied:

1. $\lim_{s\to 0} F(\gamma(s)) = 0$.
2. $\lim_{s\to 0} R(\gamma(s)) = 0$ and/or $\lim_{s\to 0} \theta(\gamma(s)) = \pi/2$ but $\lim_{s\to 0} G(\gamma(s)) \neq \lim_{s\to 0}(-\tan^2(\theta))$.
3. $\lim_{s\to 0} G(\gamma(s)) = 0$ but $\lim_{s\to 0} \theta(\gamma(s)) \notin \{0, \pi\}$.
4. $\lim_{s\to 0} G(\gamma(s)) = \lim_{s\to 0}(-\tan^2(\theta))$ but $\lim_{s\to 0} \theta(\gamma(s)) \notin \{0, \pi\}$.
5. $\lim_{s\to 0} G(\gamma(s)) = \lim_{s\to 0}(-\tan^2(\theta))$ but $\lim_{s\to 0} R(\gamma(s)) \neq 0$ and $\lim_{s\to 0} \theta(\gamma(s)) \neq \pi/2$.

Then, there exists no natural StSc extension of (M, g).

Extra: Properties of extendible spacetimes

Theorem 4 (Galloway, Ling, Sbierski (2018))

Let (M, g) be a smooth (at least C^2) globally hyperbolic spacetime. Suppose $\iota : (M, g) \to (M_{\mathrm{ext}}, g_{\mathrm{ext}})$ is a C^0-extension and $\partial^+ \iota(M) \neq \emptyset$. Then, there exists a future directed timelike geodesic $\gamma : [0, 1] \to M_{ext}$ with $\gamma([0, 1)) \subset \iota(M)$ and $\gamma(1) \in \partial^+ \iota(M)$

Theorem (Galloway, Ling 2018)

Let ι: $(M, g) \to (M_{ext}, g_{ext})$ be a C^0-extension. If $\partial^+ M = \emptyset$, then $\partial^- M$ is an achronal topological hypersurface.