Møller operators and Hadamard states for Proca fields in paracausally related spacetimes

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Framework and scope

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Answers:

1. Technology of (geometric) Møller operators.
2. **Geometric tool**: Paracausal deformations of globally hyperbolic metrics.

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Based on recent papers with V.Moretti and S.Murro:
Paracausal deformations of Lorentzian metrics and Møller isomorphisms in algebraic quantum field theory (2021).
The quantization of Proca fields on globally hyperbolic spacetimes: Hadamard states and Møller operators (2022).

Preliminaries: globally hyperbolic spacetimes

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- **Globally hyperbolic** $\iff$ a Cauchy hypersurface exists.

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- an isometry $h = -\beta^2 dt^2 + h_t$, h_t family of Riemannian metrics on the slices.

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g_χ **will be important later!**

The paracausal relation

Definition (Paracausal relation)

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g is **paracausally related** to g ($g \simeq g'$) if there is a finite sequence $g = g, g', \ldots, g_N = g' \in \mathcal{GH}_{\mathrm{M}}$ such that, for $k = 0, \ldots, N-1$,
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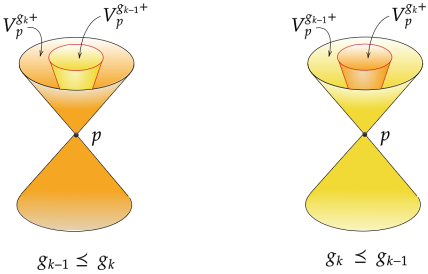

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- We can have more: g_u can even be of *bounded geometry*.

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Easiest examples of n.h.o: Klein-Gordon operator $K = \Box_g + m^2$ on the trivial bundle or the vectorial (unphysical) Klein-Gordon operator defined on one-forms.

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The kernel is characterized by the **causal propagator**

$$G := G^{+}|_{\Gamma_c(E)} - G^{-}|_{\Gamma_c(E)} : \Gamma_c(E) \to \Gamma(E)\,.$$

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Problem 1: It is **not** normally hyperbolic, but it **is** equivalent to a constrained Klein-Gordon operator.

$$\begin{cases} \mathsf{N}A = 0 \\ \delta_g A = 0 \end{cases}$$

So it is Green hyperbolic: $\mathsf{G}^\pm_{\mathsf{P}} := \left(\mathrm{Id} + \frac{d\delta_g}{m^2}\right) \mathsf{G}^\pm_{\mathsf{N}} = \mathsf{G}^\pm_{\mathsf{N}} \left(\mathrm{Id} + \frac{d\delta_g}{m^2}\right)$

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- We build the interpolating spacetime with $g_\chi = (1-\chi)g + \chi g'$ and an operator N_χ.
- g_χ is globally hyperbolic and so N_χ is Green hyperbolic.

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What's the relation between the solution spaces of N and N′, normally hyperbolic respectively w.r.t g and g'? What about the solution spaces of the Proca operators P and P′?

Gluing spacetimes!

- Let $\chi \in C^\infty(\mathrm{M}, [0, 1])$ with $\chi = 0$ before t_0 and $\chi = 1$ after t_1 $(t_1 > t_0)$.
- We build the interpolating spacetime with $g_\chi = (1 - \chi)g + \chi g'$ and an operator N_χ.
- g_χ is globally hyperbolic and so N_χ is Green hyperbolic.
- The couple $(\mathrm{g}_\chi, \mathrm{N}_\chi)$ equals (g, N) in the past of t_0 and $(\mathrm{g}', \mathrm{N}')$ in the future of t_1.

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The latter is solved by the existence of the linear isometry $\kappa_{g'g} : \Gamma(\mathsf{V}_g) \to \Gamma(\mathsf{V}_{g'})$

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The first by the introduction of smooth functions $\rho, \rho' : \mathsf{M} \to (0, +\infty)$ depending on ratios of volume forms.

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Notice that the same construction for Klein-Gordon fields would just have Klein-Gordon Green operators inside.

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- **General Møller operator**: $\mathsf{R} = \mathsf{R}_0 \cdots \mathsf{R}_{N-1}$.

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is said to be the **adjoint of** T **with respect to** g, g' (with the said order) if it satisfies

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Key properties of the adjoint Møller operator

$$\mathsf{R}\mathsf{G}_{\mathsf{P}}\mathsf{R}^{\dagger_{gg'}} = \mathsf{G}_{\mathsf{P}'}$$

$$\mathsf{R}^{\dagger_{gg'}}\mathsf{P}'\kappa_{g'g}|_{\Gamma_c(\mathsf{V}_g)} = \mathsf{P}|_{\Gamma_c(\mathsf{V}_g)}\,.$$

Quantization: the CCR Proca algebra

On-shell Proca *CCR* $*$**-algebra**: the $*$-algebra

$$\mathcal{A}_g = \mathfrak{A}_g / \mathfrak{I}_g$$

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4. $\mathfrak{a}(\mathsf{P}\mathfrak{f})\ \forall \mathfrak{f} \in \Gamma_c(\mathsf{V}_g)$.

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If the state is also continuous we associate $n-$point distributional kernels $\to \widetilde{\omega}_n$
A state is called **quasi-free** if all its $n-$point distributions can be recovered by $\widetilde{\omega}_2$.

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which is an isomorphism since it preserves all ideals, in particular R and $\mathsf{R}^{\dagger_{gg'}}$ it intertwine the causal propagators.

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- The isomorphism preserves **Hadamard states** defined by the wavefrontset of their two point bidistributions

$$WF(\omega_2) = \{(x, k_x; y, -k_y) \in T^*\mathsf{M}^2 \setminus \{0\} | (x, k_x) \sim_{\parallel} (y, k_y), k_x \rhd 0\}.$$

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The "bounded geometry" hypotesis has been used to explicitly construct the Klein-Gordon Hadamard states on ultrastatic spacetimes.
By using these result we constructed an Hadamard state for the Proca field on the ultrastatic spacetime with Cauchy surfaces of bounded geometry.

Proca Hadamard states

Fewster-Pfenning's definition of Proca Hadamard state

A quasi-free state $\omega : \mathcal{A}_g \to \mathbb{C}$ is called **Hadamard** if its two-point function has the form

$$\omega(\hat{\mathfrak{a}}(\mathfrak{f})\hat{\mathfrak{a}}(\mathfrak{h})) = W_g(\mathfrak{f}, Q\mathfrak{h}) \tag{0.1}$$

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- $Q = \mathrm{Id} + m^{-2}(d\delta_g)$.
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- G_{N} is Klein-Gordon causal propagator
- W_g satisfies the microlocal spectrum condition.

Proca Hadamard states

Fewster-Pfenning's definition of Proca Hadamard state

A quasi-free state $\omega : \mathcal{A}_g \to \mathbb{C}$ is called **Hadamard** if its two-point function has the form

$$\omega(\hat{\mathfrak{a}}(\mathfrak{f})\hat{\mathfrak{a}}(\mathfrak{h})) = W_g(\mathfrak{f}, Q\mathfrak{h}) \tag{0.1}$$

$\forall \mathfrak{f}, \mathfrak{h} \in \Gamma_c(\mathsf{V}_g)$, where

- $Q = \mathrm{Id} + m^{-2}(d\delta_g)$.
- $W_g \in \Gamma'_c(\mathsf{V}_g \otimes \mathsf{V}_g)$ is a Klein-Gordon bisolution such that

$$W_g(\mathfrak{f}, \mathfrak{g}) - W_g(\mathfrak{g}, \mathfrak{f}) = i\mathsf{G}_\mathsf{N}(\mathfrak{f}, \mathfrak{g}) \mod C^\infty, \tag{0.2}$$

- G_N is Klein-Gordon causal propagator
- W_g satisfies the microlocal spectrum condition.

The authors proved the existence of states of this form just for compact Cauchy surfaces.

Proca Hadamard states

Actually this result came out:

Theorem

Consider quasifree Hadamard state $\omega : \mathcal{A}_g \to \mathbb{C}$ *for the* $*$*-algebra of observables on* (M, g) *of the real Proca field. Let* $\omega_2 \in \Gamma'_c(\mathsf{V}_g \otimes \mathsf{V}_g)$ *be the two-point function of* ω*. The following facts are true.*

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(a) *If* ω *is Hadamard according to the standard microlocal definition, then it is also Hadamard according to Fewster and Pfenning.*

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(a) *If ω is Hadamard according to the standard microlocal definition, then it is also Hadamard according to Fewster and Pfenning.*

(b) *If in (M, g) admits Proca quasifree Hadamard states in the sense of Fewster and Pfenning then, then if ω is Hadamard in the standard sense, it is Hadamard in Fewster-Pfenning sense.*

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This means that for Cauchy compact spacetimes the two definitions have been proved to be equivalent.

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1. Free classical and quantum field theories for massive spin-1 theories on curved backgrounds are structurally comparable if the background metrics are paracausally related.
2. In the standard sense Proca Hadamard states **exist** for general g.h. spacetime and the classical deformation argument is implemented by concrete Møller operators.
3. The two definitions of Hadamard states for these fields (almost) agree.

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5. Employ these techniques to incorporate the (non normally hyperbolic) abelian gauge Maxwell fields.

Thanks for the attention!