The anisotropic Calderón problem on 3-dimensional conformally Stäckel manifolds

Niky Kamran

McGill University

Joint work with Thierry Daudé (Cergy-Pontoise) and François Nicoleau (Nantes)

Mathematical Physics Seminar, Regensburg, August 26, 2022

Setting and Goal

Conformally Stäckel manifolds: n-dimensional Riemannian manifolds (M, g) on which Laplace-Beltrami equation $-\Delta_g \psi = 0$ is solvable by R-separation of variables.

They provide explicit examples of metrics admitting a set of $n-1$ commuting conformal symmetry operators K of order 2 for Δ_g:

$$[\Delta_g, K] = r\Delta_g \, .$$

Our goal is to show that for $n = 3$ and $M = [0,1] \times \mathbb{T}^2$, any conformally Stäckel metric g on M is uniquely determined by the Dirichlet-to-Neumann map, meaning that the anisotropic Calderón problem has a unique solution in this setting.

Ref: T. Daudé, N.K. and F. Nicoleau, 2021, *J. Spectr. Theory*, **11**, *No. 4, pp. 1359-1398.*

The anisotropic Calderón problem

- (M, g) a smooth compact connected Riemannian manifold with boundary ∂M
- $-\Delta_g$ the positive Laplace-Beltrami operator M.
- Dirichlet problem at a fixed frequency $\notin$ Dirichlet spectrum $\sigma(-\Delta_g)$

$$\begin{cases} -\Delta_g u = \lambda u, & \text{on } M, \\ u = \psi, & \text{on } \partial M. \end{cases} \tag{1}$$

- Given $\psi \in H^{1/2}(\partial M)$, there exists a unique solution $u \in H^1(M)$.
- Dirichlet-to-Neumann (DN) map: $\Lambda_g(\lambda) : H^{1/2}(\partial M) \to H^{-1/2}(\partial M)$, where

$$\Lambda_g(\lambda)(\psi) := (\partial_\nu u)_{|\partial M} .$$

where u is the unique solution of (1) and $(\partial_\nu u)_{|\partial M}$ is the normal derivative of u along ∂M.

The anisotropic Calderón problem

Question initially posed by Calderón (1980): *Does the knowledge of the DN map $\Lambda_g(\lambda)$ at a given frequency λ determine uniquely the metric g?*

Due to a number of gauge invariances, the answer to this initial question is negative:

- The DN map $\Lambda_g(\lambda)$ is invariant under pullbacks of the metric by diffeomorphisms of M that restrict to the identity on ∂M, *i.e.* $\forall \phi \in \mathrm{Diff}(M)$ such that $\phi_{|\partial M} = Id$, we have
$$\Lambda_{\phi^* g}(\lambda) = \Lambda_g(\lambda). \tag{2}$$
- In dimension 2 and when $\lambda = 0$, the conformal covariance of $-\Delta_g$ implies that for all $c \in C^\infty(M)$ such that $c > 0$ and $c_{|\partial M} = 1$,
$$\Lambda_{cg}(0) = \Lambda_g(0). \tag{3}$$

The anisotropic Calderón problem

- The question should be: *Let M be a smooth compact connected manifold with boundary ∂M and let $g, \tilde{g}$ be smooth Riemannian metrics on M. Let λ be a fixed frequency that does not belong to $\sigma(-\Delta_g) \cup \sigma(-\Delta_{\tilde{g}})$. If*

$$\Lambda_g(\lambda) = \Lambda_{\tilde{g}}(\lambda),$$

is it then true that

$$g = \tilde{g},$$

up to the above gauge equivalences?

Some known results for $\lambda = 0$

- If (M, g) is a smooth 2-dimensional Riemannian manifold, then g is uniquely determined by $\Lambda_g(0)$ up to the gauge invariances (2) - (3). See [Lee, Uhlmann](1993).
- If (M, g) is real analytic, dim $M \geq 3$, then g is uniquely determined by $\Lambda_g(0)$ up to the gauge invariance (2). See [Lee, Uhlmann] (1993), [Lassas, Uhlmann] (2001).
- If (M, g) is Einstein (and thus analytic in its interior), then g is uniquely determined by $\Lambda_g(0)$ up to the gauge invariance (2). See [Guillarmou, Sa Barreto] (2009).

Known results on the anisotropic Calderón problem

- For the class of *conformally transversally anisotropic (CTA)* Riemannian manifolds, *i.e.* (M, g) such that

$$M \subset\subset \mathbb{R} \times M_0, \quad g = c(e \oplus g_0),$$

where:

- (M_0, g_0) $n - 1$ dimensional smooth compact Riemannian manifold with boundary,
- e Euclidean metric on the real line,
- c smooth positive function in the cylinder $\mathbb{R} \times M_0$.

If the transverse manifold (M_0, g_0) is simple (geodesic flow with no conjugate points, boundary is strictly convex), the conformal factor c can be uniquely determined by $\Lambda_g(0)$. See [Dos Santos Ferreira, Kenig, Sjöstrand, Uhlmann] (2009), [Dos Santos Ferreira, Lassas, Kurylev, Salo] (2013).

Conformally Stäckel Cylinders

Let $\mathcal{M}$ be a smooth compact 3-dimensional manifold with smooth boundary having the topology of a toric cylinder,

$$\mathcal{M} = [0, A] \times \mathbb{T}^2 \,,$$

and let (x^1, x^2, x^3) be global coordinates on $\mathcal{M}$. Note that $\partial\mathcal{M}$ of $\mathcal{M}$ has two connected components

$$\partial = \mathcal{M}_0 \cup \mathcal{M}_1 \,, \quad \mathcal{M}_0 = \{0\} \times \mathbb{T}^2 \,, \quad \mathcal{M}_1 = \{A\} \times \mathbb{T}^2 \,.$$

A conformally Stäckel metric on $\mathcal{M}$ with a smooth Riemannian metric G of the form

$$G = c^4 g = \sum_{i=1}^{3} H_i^2 (dx^i)^2 \,, \tag{4}$$

where g is a Stäckel metric, that is

$$g = \sum_{i=1}^{3} h_i^2 (dx^i)^2 \,, \quad h_i^2 = \frac{\det S}{s^{i1}} \,, \tag{5}$$

with S being a Stäckel matrix, that is a non-singular matrix of the form

$$S = \begin{pmatrix} s_{11}(x^1) & s_{12}(x^1) & s_{13}(x^1) \\ s_{21}(x^2) & s_{22}(x^2) & s_{23}(x^2) \\ s_{31}(x^3) & s_{32}(x^3) & s_{33}(x^3) \end{pmatrix} \,, \tag{6}$$

and

$$s^{ij} = \mathrm{cofact}(s_{ij}) \,.$$

denotes the cofactor of the component s_{ij} of the matrix S.

Furthermore, the conformal factor c^4 is assumed to be a positive solution of the linear elliptic PDE on $\mathcal{M}$ given by:

$$-\Delta_g c - \sum_{i=1}^{3} h_i^2 \big(\phi_i + \frac{1}{4}\gamma_i^2 - \frac{1}{2}\partial_i \gamma_i\big) c = 0 \,, \tag{7}$$

where Δ_g is the Laplace-Beltrami operator associated to g,

$$\gamma_i := -\partial_i \log \frac{h_1 h_2 h_3}{h_i^2} \,,$$

are the contracted Christoffel symbols of g and $\phi_i = \phi_i(x^i)$ are arbitrary smooth functions of the indicated variable.

One shows that all the solutions of the Laplace equation

$$-\Delta_G \psi = 0, \quad \text{on } \mathcal{M}, \tag{8}$$

can be written as

$$\psi = R(x^1, x^2, x^3) \sum_{m=1}^{\infty} u_m(x^1) Y_m(x^2, x^3), \quad Y_m(x^2, x^3) = v_m(x^2) w_m(x^3), \tag{9}$$

where

$$R = \left(\frac{s^{11} s^{21} s^{31}}{c^4 \det S} \right)^{\frac{1}{4}}, \tag{10}$$

and where u_m, v_m, w_m satisfy the coupled separated ODEs :

$$- u_m'' + [\,\mu_m^2 s_{12}(x^1) + \nu_m^2 s_{13}(x^1) - \phi_1(x^1)]\, u_m = 0, \tag{11}$$

$$- v_m'' + [\,\mu_m^2 s_{22}(x^2) + \nu_m^2 s_{23}(x^2) - \phi_2(x^2)]\, v_m = 0, \tag{12}$$

$$- w_m'' + [\,\mu_m^2 s_{32}(x^3) + \nu_m^2 s_{33}(x^3) - \phi_3(x^3)]\, w_m = 0. \tag{13}$$

Here the constants of separation (μ_m^2, ν_m^2) can be understood as the joint spectrum of the commuting elliptic selfadjoint operators (H, L) on $\mathbb{T}^2$ defined by :

$$\begin{pmatrix} H \\ L \end{pmatrix} = \frac{1}{s^{11}} \begin{pmatrix} -s_{33} & s_{23} \\ s_{32} & -s_{22} \end{pmatrix} \begin{pmatrix} A_2 \\ A_3 \end{pmatrix}, \tag{14}$$

where for all $j = 1, 2, 3$, we set :

$$A_j = -\partial_j^2 - \phi_j(x^j). \tag{15}$$

The joint eigenfunctions of (H, L) take the form $Y_m = v_m(x^2) w_m(x^3)$ and satisfy:

$$HY_m = \mu_m^2 Y_m, \quad LY_m = \nu_m^2 Y_m, \quad \forall m \geq 1. \tag{16}$$

Finally, the eigenfunctions Y_m form a Hilbert basis of $L^2(\mathbb{T}^2)$ in the following sense :

$$L^2(\mathbb{T}^2\,; s^{11} dx^2 dx^3) = \bigoplus_{m \geq 1} \langle Y_m \rangle. \tag{17}$$

Denote by $\{c_0, s_0\}$ and $\{c_1, s_1\}$ the fundamental systems of solutions of the ODE

$$-u'' + [\mu^2 s_{12}(x^1) + \nu^2 s_{13}(x^1) - \phi_1(x^1)]u = 0, \qquad (18)$$

satisfying the boundary conditions of sine and cosine type at $x^1 = 0$ and $x^1 = A$ given by

$$c_0(0) = 1,\ c_0'(0) = 0,\ s_0(0) = 0,\ s_0'(0) = 1,$$
$$c_1(A) = 1,\ c_1'(A) = 0,\ s_1(A) = 0,\ s_1'(A) = 1.$$

The functions $c_j,\ s_j,\ j = 0, 1$ are analytic separately in the parameters $\mu, \nu \in \mathbb{C}$ and their Wronskians satisfy

$$W(c_j, s_j) = 1,\ j = 0, 1\,.$$

We associate to (18) the characteristic function

$$\Delta(\mu^2, \nu^2) = W(s_0, s_1)\,, \tag{19}$$

and the Weyl solutions given by

$$\Psi = c_0 + M(\mu^2, \nu^2)s_0, \quad \Phi = c_1 - N(\mu^2, \nu^2)s_1,$$

by demanding that they satisfy Dirichlet boundary condition at $x = A$ and $x = 0$ respectively. The coefficients M, N are the Weyl-Titchmarsh functions and one has

$$\begin{aligned} M(\mu^2, \nu^2) &= -\tfrac{W(c_0, s_1)}{\Delta(\mu^2, \nu^2)} = -\tfrac{D(\mu^2, \nu^2)}{\Delta(\mu^2, \nu^2)}, \\ N(\mu^2, \nu^2) &= \tfrac{W(s_0, c_1)}{\Delta(\mu^2, \nu^2)} = \tfrac{E(\mu^2, \nu^2)}{\Delta(\mu^2, \nu^2)}. \end{aligned} \tag{20}$$

Finally,

$$\begin{aligned} u_m'(0) &= M(\mu_m^2, \nu_m^2)\,\varphi_m^0 + \tfrac{1}{\Delta(\mu_m^2, \nu_m^2)}\,\varphi_m^1, \\ u_m'(A) &= \tfrac{1}{\Delta(\mu_m^2, \nu_m^2)}\,\varphi_m^0 + N(\mu_m^2, \nu_m^2)\,\varphi_m^1. \end{aligned} \tag{21}$$

The DN map can be "almost" diagonalized in the Hilbert basis $(Y_m)_{m\geq 1}$. Indeed, writing

$$H^s(\partial\mathcal{M}) = H^s(\mathcal{M}_0) \oplus H^s(\mathcal{M}_1), \quad H^s(\mathcal{M}_j) \simeq H^s(\mathbb{T}^2),\ j = 0, 1,$$

and using a 2×2-matrix notation for the DN map $\Lambda_G : H^{\frac{1}{2}}(\partial\mathcal{M}) \longrightarrow H^{-\frac{1}{2}}(\partial\mathcal{M})$, *i.e.*

$$\Lambda_G = \begin{pmatrix} \Lambda_{G,\mathcal{M}_0,\mathcal{M}_0} & \Lambda_{G,\mathcal{M}_0,\mathcal{M}_1} \\ \Lambda_{G,\mathcal{M}_1,\mathcal{M}_0} & \Lambda_{G,\mathcal{M}_1,\mathcal{M}_1} \end{pmatrix},$$

where the operators $\Lambda_{G,\mathcal{M}_i,\mathcal{M}_j} : H^{\frac{1}{2}}(\mathbb{T}^2) \longrightarrow H^{-\frac{1}{2}}(\mathbb{T}^2)$ correspond to the DN map when the Dirichlet data are imposed on $\mathcal{M}_i$ and the Neumann data are measured on $\mathcal{M}_j$, the DN map reads:

$$\Lambda_G = \begin{pmatrix} \frac{-1}{H_1(0,x^2,x^3)} & 0 \\ 0 & \frac{1}{H_1(A,x^2,x^3)} \end{pmatrix} \left[\begin{pmatrix} \frac{\Gamma_1(0,x^2,x^3)}{2} & 0 \\ 0 & \frac{\Gamma_1(A,x^2,x^3)}{2} \end{pmatrix} \right.$$

$$\left. + \begin{pmatrix} R(0,x^2,x^3) & 0 \\ 0 & R(A,x^2,x^3) \end{pmatrix} A_G \begin{pmatrix} \frac{1}{R(0,x^2,x^3)} & 0 \\ 0 & \frac{1}{R(A,x^2,x^3)} \end{pmatrix} \right]$$

where

$$\Gamma_i := -\partial_i \log \frac{H_1 H_2 H_3}{H_i^2}, \quad i = 1,2,3,$$

and where the operator A_G is completely diagonalizable in the Hilbert basis $(Y_m)_{m\geq 1}$, its restriction on $\langle Y_m \rangle$ being defined by :

$$(A_G)_{|\langle Y_m \rangle} := \begin{pmatrix} M(\mu_m^2, \nu_m^2) & \frac{1}{\Delta(\mu_m^2,\nu_m^2)} \\ \frac{1}{\Delta(\mu_m^2,\nu_m^2)} & N(\mu_m^2, \nu_m^2) \end{pmatrix}, \tag{22}$$

Conformally Stäckel Manifold

By definition, a conformally Stäckel manifold is a smooth compact connected Riemannian manifold with boundary (M, G) embedded isometrically by inclusion in a conformally Stäckel cylinder $\mathcal{M}$, that is

$$M \subset\subset \mathcal{M} = [0, A] \times \mathbb{T}^2, \qquad (23)$$

and G is a Riemannian metric on M that possesses a smooth extension (still denoted by G) to the whole cylinder $\mathcal{M}$ given by (4) - (6).

Main Result

Theorem 1

Let (M, G) and $(M, \tilde{G})$ be two conformally Stäckel manifolds satisfying (4) - (6) and (23). Assume that

$$\Lambda_G = \Lambda_{\tilde{G}}.$$

Then there exists a diffeomorphism $\varphi : M \longrightarrow M$ with $\varphi_{|\partial M} = Id$ whose pull-back satisfies

$$\tilde{G} = \varphi^* G,$$

Let us make a comment on this result in relation to the results of Dos Santos Ferreira, Kenig, Sjöstrand, Uhlmann (2009), and Dos Santos Ferreira, Lassas, Kurylev, Salo (2013) for CTA manifolds:

Conformally Stäckel manifolds are generically not CTA manifolds. They could be if for example one of the rows the Stäckel matrix S, say s_{1j}, was a row of constant functions. Then ∂_{x^1} would be a Killing vector field for g and thus a conformal Killing vector field for G. This suggests that (M, G) could perhaps lie within the class of CTA manifolds since G can be written as

$$G = \left(c^4 \frac{\det(S)}{s^{11}}\right) \left[(dx^1)^2 + g_0\right], \quad g_0 = \frac{s^{11}}{s^{21}}(dx^2)^2 + \frac{s^{11}}{s^{31}}(dx^3)^2.$$

However, the injectivity of the geodesic X-ray transform on the transversal manifold

$$(M_0, g_0) = (\mathbb{T}^2, g_0),$$

is not guaranteed in general.

Sketch of Proof

The proof of Theorem 1 is divided in four steps.

Step 1. Extension to the whole cylinder $\mathcal{M}$:

The Laplace equation $-\Delta_G \psi = 0$ on M is usually not separable since the boundary ∂M need not be compatible with variable separation, unlike the case on the whole cylinder $(\mathcal{M}, G)$. Hence we cannot use a priori the form (9) for the solutions of the Laplace equation as well as the structure (17) of the DN map.

However we can reduce the Calderón problem on (M, G) to the Calderón problem on the extended cylinder $(\mathcal{M}, G)$ by a result which is similar to the corresponding result on asymptotically hyperbolic manifolds from Isozaki-Kurylev, 2014.

Step 2. Boundary determination:

We use the standard boundary determination results [1] and the particular structure of the metrics G and $\tilde{G}$ given by (4) - (6) to prove in a successive series of steps that first (from the equality of the metrics on the boundary)

$$\begin{pmatrix} s_{22} & s_{23} \\ s_{32} & s_{33} \end{pmatrix} = \begin{pmatrix} \tilde{s}_{22} & \tilde{s}_{23} \\ \tilde{s}_{32} & \tilde{s}_{33} \end{pmatrix}, \tag{24}$$

as functions of x^2, x^3 and

$$\left\{ \begin{array}{c} (c^4 \ \det S)(x^1, x^2, x^3) = (\tilde{c}^4 \ \det \tilde{S})(x^1, x^2, x^3), \\ R(x^1, x^2, x^3) = \tilde{R}(x^1, x^2, x^3), \\ H_1(x^1, x^2, x^3) = \tilde{H}_1(x^1, x^2, x^3), \end{array} \right. \qquad x^1 = 0, A, \ \forall x^2, x^3. \tag{25}$$

[1] Precisely we use the fact $\Lambda_{G,\mathcal{M}} = \Lambda_{\tilde{G},\mathcal{M}}$ imply the equality of $G_{|\partial\mathcal{M}}$ and $\tilde{G}_{|\partial\mathcal{M}}$ as well as the equality between the normal derivatives $(\partial_\nu G)_{|\partial\mathcal{M}}$ and $(\partial_{\tilde{\nu}} \tilde{G})_{|\partial\mathcal{M}}$ on the boundary $\partial\mathcal{M}$.

and second (from the equality of the normal derivatives of the metrics on the boundary)

$$\left\{ \begin{array}{c} (\partial_1 \log c^4 \ \det S)(x^1, x^2, x^3) = (\partial_1 \log \tilde{c}^4 \ \det \tilde{S})(x^1, x^2, x^3), \\ \Gamma_1(x^1, x^2, x^3) = \tilde{\Gamma}_1(x^1, x^2, x^3), \end{array} \right. \qquad x^1 = 0, A, \tag{26}$$

where

$$\Gamma_i := -\partial_i \log \frac{H_1 H_2 H_3}{H_i^2}, \quad i = 1, 2, 3.$$

Then, using the special structure (17) of the DN map, we obtain

$$A_G = A_{\tilde{G}}, \tag{27}$$

where the operator A_G is defined in (22). From this and some additional work, we can show the equality of the eigenfunctions Y_m

$$Y_m = \tilde{Y}_m, \ \forall m, \tag{28}$$

the equality of the joint spectra

$$(\mu_m^2, \nu_m^2) = (\tilde{\mu}_m^2, \tilde{\nu}_m^2), \quad \forall m, \tag{29}$$

and the equality between the ϕ_2 and ϕ_3

$$\phi_2 = \tilde{\phi}_2, \quad \phi_3 = \tilde{\phi}_3. \tag{30}$$

Hence at the end of the second step, we will have recovered most of the unknown functions of one variable depending on one of the angular variables x^2, x^3, and in fact all of them if we keep in mind the possibility of removing some of these unknown functions thanks to an admissible change of variables.

Step 3. The multi-parameter CAM method:

At this stage, it remains to determine the unknown functions depending on the radial variable x^1 and the conformal factor c.

To determine the former, we start from the equality

$$M(\mu_m^2, \nu_m^2) = \tilde{M}(\mu_m^2, \nu_m^2), \quad \forall m, \tag{31}$$

which is a consequence of (27) and (29). Recall that the WT function M only depends on the radial ODE (11) and contains all the information on the functions s_{12}, s_{13}, ϕ_1 through the Borg-Marchenko Theorem.

Our first task is thus to extend the equality (31) which is initially true on the joint spectrum $J = \{(\mu_m^2, \nu_m^2),\ m \geq 1\}$ to the whole plane $\mathbb{C}^2$, that is we complexify the angular momenta following Regge. For this, we use a multi-parameter CAM method which allows us to prove that

$$M(\mu^2, \nu^2) = \tilde{M}(\mu^2, \nu^2), \quad \forall \mu, \nu \in \mathbb{C} \backslash \{poles\} \tag{32}$$

This will be done below.

Once it is done, an application of Borg-Marchenko Theorem leads to

$$\phi_1 = \tilde{\phi}_1, \quad s_{12} = \tilde{s}_{12}, \quad s_{13} = \tilde{s}_{13}\,. \tag{33}$$

Note that the multi-parameter CAM method that permits to infer (32) from (31) is not as simple as in the case of a single angular momentum. A good understanding of the joint spectrum J is needed.

Step 4. A Unique Continuation Argument for the Conformal Factor:

We finish the proof of our main Theorem by remarking first that the metric G can be written as

$$G = \alpha\, g_0, \quad \alpha = c^4 \det S, \quad g_0 = \frac{1}{s^{11}}(dx^1)^2 + \frac{1}{s^{21}}(dx^2)^2 + \frac{1}{s^{31}}(dx^3)^2,$$

Note from the results of Steps 1 to 3 that we have

$$g_0 = \tilde{g}_0, \tag{34}$$

Thus it only remains to prove that $\alpha = \tilde{\alpha}$. The second crucial remark consists in using (7) to show that the conformal factor α satisfies the elliptic PDE

$$-\Delta_{g_0}\alpha - Q_{g_0,\phi_i}\alpha = 0, \tag{35}$$

where

$$Q_{g_0,\phi_i} = \sum_{i=1}^{3} g_0^{ii}\left[\frac{\partial_{ii}^2 \log\det g_0}{4} + \frac{\partial_i \log\det g_0}{8} + \frac{(\partial_i \log\det g_0)^2}{16} + \phi_i\right]. \tag{36}$$

Thanks to (30), (33) and (34), we thus observe one additional (and last) remarkable fact: the conformal factors α and $\tilde{\alpha}$ satisfy the *same* second order elliptic PDE (35). Finally, we use (25), (26) and a classical unique continuation principle to prove $\alpha = \tilde{\alpha}$. As a consequence, we find that

$$G = \tilde{G},$$

up to diffeomorphisms that preserve the boundary.

We spend the remaining time on the details of Step 3, the multi-parameter CAM method.

Multi-parameter CAM Method

Our first task is to show that the equality (31) on the discrete subset J can be extended to the whole plane $\mathbb{C}^2$, *i.e.*

$$M(\mu^2, \nu^2) = \tilde{M}(\mu^2, \nu^2), \quad \forall (\mu, \nu) \in \mathbb{C}^2 \backslash \{poles\}. \tag{37}$$

Note first that (**??**) can be rewritten using (20) as

$$D(\mu_m^2, \nu_m^2)\tilde{\Delta}(\mu_m^2, \nu_m^2) - \tilde{D}(\mu_m^2, \nu_m^2)\Delta(\mu_m^2, \nu_m^2) = 0, \quad \forall m \geq 1. \tag{38}$$

Define now the function

$$F(\mu, \nu) := D(\mu^2, \nu^2)\tilde{\Delta}(\mu^2, \nu^2) - \tilde{D}(\mu^2, \nu^2)\Delta(\mu^2, \nu^2). \tag{39}$$

Then F is clearly analytic [2] on $\mathbb{C}^2$ and vanishes on the "square-root" of the joint spectrum J thanks to (38). Hence, in order to prove (37), it will be enough to prove that F vanishes identically.

To go further, we will use the following result of Berndtsson which provides a sufficient condition for a discrete set to be a *uniqueness set* of a bounded analytic function of several variables:

[2] The functions $c_j,\ s_j,\ j=0,1$ and thus the functions Δ, D and F are analytic in the variables μ and ν independently thanks to standard theorems on ODE depending analytically on parameters. Hence the function F is analytic on $\mathbb{C}^2$ due to the Hartogs Theorem.

Theorem 2 (Berndtsson, 1978)

Let K be an open cone in $\mathbb{R}^n$ with vertex at the origin and $T(K) = \{z \in \mathbb{C}^n \ / \ \Re(z) \in K\}$. Suppose f is bounded and analytic on $T(K)$. Let E be a discrete subset of K such that for some constant $h > 0$, $e_1, e_2 \in E$ implies that $|e_1 - e_2| \geq h$. Let $n(r) = \# E \cap B(0, r)$. Assume that f vanishes on E. Then f is identically 0 if

$$\overline{\lim_{r \to \infty}} \frac{n(r)}{r^n} > 0.$$

In order to apply Theorem 2, we need to define an analytic function that is bounded on a conical set of the form $T(K)$ and that satisfies the above properties.

The natural candidate - the function F - is not bounded and we need to rescale it in a convenient way. Hence we first need some universal estimates for F:

Proposition 3

There exist positive constants $\bar{A}, \bar{B}, C > 0$ *such that for all* $(\mu, \nu) \in \mathbb{C}^2$

$$|D(\mu^2, \nu^2)|,\ |\Delta(\mu^2, \nu^2)| \ \leq \ C\, e^{\frac{\bar{A}}{2}|\Re(\mu)| + \frac{\bar{B}}{2}|\Re(\nu)|}.$$

As a consequence,

$$|F(\mu, \nu)| \ \leq \ C\, e^{\bar{A}|\Re(\mu)| + \bar{B}|\Re(\nu)|}.$$

We then have independent estimates μ and ν:

Lemma 4

1. *For each* $\nu \in \mathbb{C}$ *fixed, there exists positive constants* $\bar{A}$, $C(\nu) > 0$ *such that*

$$|D(\mu^2, \nu^2)|, \ |\Delta(\mu^2, \nu^2)|, \ |F(\mu, \nu)| \ \leq \ C(\nu)\, e^{\bar{A}|\Re(\mu)|}.$$

2. *For each* $\mu \in \mathbb{C}$ *fixed, there exists positive constants* $\bar{B}$, $C(\nu) > 0$ *such that*

$$|D(\mu^2, \nu^2)|, \ |\Delta(\mu^2, \nu^2)|, \ |F(\mu, \nu)| \ \leq \ C(\mu)\, e^{\bar{B}|\Re(\nu)|}.$$

We also have a uniform estimate when $(\mu, \nu) = (iy, iy') \in (i\mathbb{R})^2$.

Lemma 5

There exists a constant $C > 0$ such that for all $(y, y') \in \mathbb{R}^2$

$$|D(-y^2, -y'^2)|, \, |\Delta(-y^2, -y'^2)|, \, |F(iy, iy')| \;\leq\; C.$$

We can now finish the proof of Proposition 3 by applying the Phragmen-Lindelöf principle twice.

Proof of Proposition 3

First we fix $\nu \in i\mathbb{R}$. According to Lemmas 4 and 5, the analytic function $\mu \longrightarrow F(\mu, \nu)$ satisfies

$$\begin{cases} |F(\mu,\nu)| \leq C(\nu) e^{\bar{A}|\Re(\mu)|}, & \forall \mu \in \mathbb{C}, \\ |F(\mu,\nu)| \leq C, & \forall \mu \in i\mathbb{R}. \end{cases}$$

Hence the Phragmen-Lindelöf principle (see for instance [**?**], Lecture 6., Theorem 3) yields

$$|F(\mu,\nu)| \leq C e^{\bar{A}|\Re(\mu)|}, \quad \forall (\mu,\nu) \in (\mathbb{C}, i\mathbb{R}). \tag{40}$$

Second we fix $\mu \in \mathbb{C}$. Then, according to Lemma 4 and (40), the analytic function $\nu \longrightarrow F(\mu, \nu)$ satisfies

$$\begin{cases} |F(\mu,\nu)| \leq C(\mu) e^{\bar{B}|\Re(\nu)|}, & \forall \nu \in \mathbb{C}, \\ |F(\mu,\nu)| \leq C e^{\bar{A}|\Re(\mu)|}, & \forall \nu \in i\mathbb{R}. \end{cases}$$

Applying once again the Phragmen-Lindelöf principle, we obtain

$$|F(\mu,\nu)| \leq C e^{\bar{A}|\Re(\mu)| + \bar{B}|\Re(\nu)|}, \quad \forall (\mu,\nu) \in \mathbb{C}^2,$$

which proves the Proposition.

We can now apply Theorem 2. First define the analytic function

$$f(\mu,\nu) := F(\mu,\nu)e^{-\bar{A}\mu-\bar{B}\nu},$$

where $\bar{A}$ and $\bar{B}$ are the positive constants appearing in Proposition 3. Then it follows fro Proposition 3 that f is bounded and analytic on the set

$$T((\mathbb{R}^+)^2) = \{(\mu,\nu) \in \mathbb{C}^2 \ \mid \ \Re(\mu,\nu) \in (\mathbb{R}^+)^2\}.$$

Second define the cone

$$\mathcal{C}_\epsilon = \{(\mu,\theta\mu) \in (\mathbb{R}^+)^2 \ / \ \mu \in \mathbb{R}^+, \ \sqrt{c_1+\epsilon} \le \theta \le \sqrt{c_2-\epsilon}\}, \quad 0 < \epsilon << 1, \tag{41}$$

where

$$c_1 = \max\left(-\frac{s_{32}}{s_{33}}\right), \quad c_2 = \min\left(-\frac{s_{22}}{s_{23}}\right). \tag{42}$$

Define also the discrete set

$$E_M = \{(\mu_m, \nu_m) \in (\mathbb{R}^+)^2 \ / \ m \geq M\}, \tag{43}$$

where M is chosen large enough to ensure that for all $m \geq M$, the joint spectrum (μ_m^2, ν_m^2) of the angular operators (H, L) belongs to $(\mathbb{R}^+)^2$. In that case, (μ_m, ν_m) simply denotes the positive square root of (μ_m^2, ν_m^2). We now have

Lemma 6

1. *There exists* $h > 0$ *such that*

$$|e_1 - e_2| \geq h, \quad \forall (e_1, e_2) \in (E_m \cap \mathcal{C}_\epsilon)^2.$$

2. *Set* $N(r) = \#(E_m \cap \mathcal{C}_e) \cap B(0, r)$. *Then*

$$\overline{\lim_{r \to \infty}} \frac{N(r)}{r^2} > 0.$$

Hence applying Theorem 2 to the bounded and analytic function $f(\mu, \nu)$ on $T(\mathcal{C}_\epsilon)$, we see that f vanishes identically on $T(\mathcal{C}_\epsilon)$ and thus on $\mathbb{C}^2$ by analytic continuation. Using the definition of f, we infer that the function $F(\mu, \nu)$ vanishes identically on $\mathbb{C}^2$ and by definition (39) of F, this means that

$$M(\mu^2, \nu^2) = \tilde{M}(\mu^2, \nu^2), \quad \forall(\mu, \nu) \in \mathbb{C}^2 \backslash \{poles\} \tag{44}$$

Perspectives

- There exists a theory of *non-orthogonal* Stäckel manifolds (in the sense that the metrics are non-diagonal) for which the HJ and Helmholtz equations admit a complete set of classical separated solutions. These contain (and generalize enormously) the well-known family of Kerr black holes in General Relativity and their Riemannian counterparts. It would be interesting to address the question of uniqueness for the anisotropic Calderón problem in this non-orthogonal setting.
- The methods employed in this paper should work in more general situations in which the Laplace equation could be separated with respect to one variable only. Such models have been studied recently by us in and named conformally Painlevé manifolds. This class of manifolds contains Riemannian manifolds of dimension n for which the geodesic flow is not completely integrable, but rather possesses $1 \leq r < n-1$ hidden symmetries. In such manifolds, the HJ and Helmholtz equations can be separated in groups of variables, leading to r coupled PDEs.

Thank you for your attention!