On limits of smooth spacetimes

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What is a spacetime?

Definition

A **spacetime** (M, g) is a connected **time-oriented Lorentzian manifold**, that is,

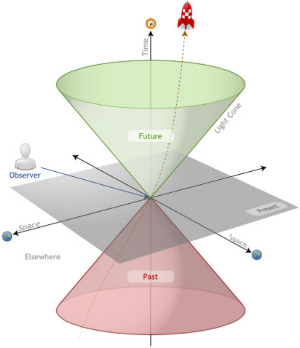
- $g\colon M \to T^*M \otimes T^*M$ is non-degenerate metric of signature $(1, n)$
- $\exists$ continuous choice of a “future cone” for (T_pM, g_p) at every p

Example: Minkowski space $\mathbb{R}^{1,3}$ with

$$v \cdot w = -v_0 w_0 + v_1 w_1 + v_2 w_2 + v_3 w_3$$

Warning (Riemann vs. Lorentz)

Not every manifold admits a Lorentzian metric (but all non-compact ones do).



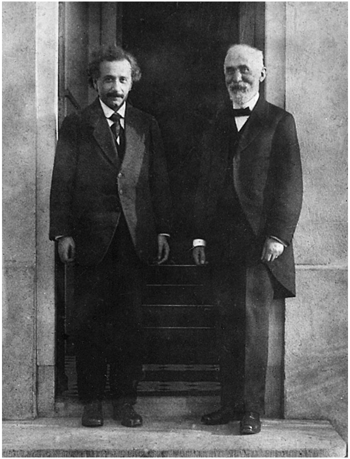
Relevance for physics

General Relativity

models **graviation via geometry**

4-dimensional spacetimes (M, g) satisfying the Einstein equations (1915)

- $Ric[g] - \frac{1}{2}R[g]g = \frac{8\pi G}{c^4}T$
  + equations for T (matter)
- $Ric[g] = 0$ (vacuum)



Einstein and Lorentz (Leiden, 1921)

Important explicit solutions

- **Minkowski spacetime** (flat vacuum)

$$g = -dt^2 + dx_1^2 + dx_2^2 + dx_3^2$$

- **Schwarzschild–Droste spacetime** (static, spherically symmetric black hole)

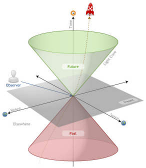
$$g = -\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 g_{\mathbb{S}^2}$$

Kerr (rotating), Reissner–Weyl–Nordström (charged) . . .

- **FLRW spacetimes** (homogeneous, isotropic universe)

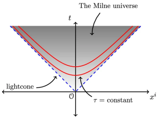
$$g = -dt^2 + a(t)^2 \sigma,$$

where (Σ, σ) is Elliptic/Euclidean/Hyperbolic space



Light cone

Center of M87
(EHT 2019)



Milne model
(Ling 2020)

Concrete occurences of nonsmooth metrics

Matter models: e.g. stars are fluid balls with vacuum exterior

- generic matter-vacuum boundary expected to be C^1 (Makino 2016)
- shocks occur $\rightsquigarrow$ jump in pressure and density
- matching spacetimes also leads to jumps (Mars–Senovilla 1993)
- but some regularization possible (Reintjes–Temple 2020)

(In)extendibility questions:

- Strong cosmic censorship conjecture: maximal globally hyperbolic developments of "generic" initial data of the Einstein equations cannot be extended as "suitably regular" (often C^0 with $\Gamma^k_{ij} \in L^2_{\mathrm{loc}}$) Lorentzian manifold
- (M, g) globally hyperbolic and timelike geodesically complete $\Longrightarrow$ C^0-inextendible (Galloway–Ling–Sbierski 2018)
- related weaker results (discussed later)

Several open problems in GR are related to **stability** and/or **regularity** of solutions to the Einstein eq. $\subseteq$ spacetimes $\subseteq$ Lorentzian mfds., however, we still don't have a full grip on this nonsmooth world itself.

Pathologies of nonsmooth metrics (below $C^{1,1}$)

Geodesics: below regularity $g \in C^1$ is the geodesic equation not necessarily locally solvable, below $g \in C^{1,1}$ not uniquely

$\rightsquigarrow$ "replace" geodesic equation with Lorentzian distance (e.g. in Lorentzian length spaces of Kunzinger–Sämann 2018)

Push-up principle: $p \ll q \leq r \overset{?}{\Longrightarrow} p \ll r$ (wrong for C^0 metrics with "causal bubbles", Chruściel–Grant 2012)

Open futures: $I^{\pm}(p)$ may not be open (Grant et al 2020)

$\rightsquigarrow$ push-up and open futures are implicitely assumed in LLS

Curvature: no longer classically defined

$\rightsquigarrow$ notions of timelike sectional/Ricci curvature *bounds* via triangle comparison/optimal transport (also optimal transport formulation of Einstein equations by Mondino–Suhr 2022, McCann 2020)

Classical results that extend to nonsmooth settings

Singularity theorems

- classically for smooth or C^2 metrics (Penrose, Hawking $\sim$1970)
- now for C^1 metrics with distributional curvature bounds (Graf 2020, Kunzinger et al 2022)
- Hawking singularity theorem also for LLS with synthetic curvature bounds (Alexander et al 2019, Cavalletti–Mondino 2020)

(In)extendibility results

- result for Lorentz–Finsler spaces (Minguzzi–Suhr 2019)
- also not extendible as regular LLS (Grant et al 2019)

Time functions and splitting

- classical characterizations of existence of time functions such as K-causality and global hyperbolicity extend to LLS (B.–García-Heveling 2021)
- but Cauchy “sets” of globally hyperbolic LLS need no longer be homeomorphic

Open questions

1. How close are (and should be) smooth and nonsmooth spacetimes related? How to measure closeness?
2. How reasonable and generic are the assumptions of push-up and open futures for LLS? (geometric and GR perspective needed)
3. Can nonsmooth spacetimes be used to pass from general relativity to some quantum gravitational theories?

How do Lorentzian manifolds **converge**?

$$(M_j, g_j) \longrightarrow (M_\infty, g_\infty)$$

Important questions:

- What kind of **geometric properties** do we want to capture?
- What is a suitable **convergence** and **limit space**?
- Why is this relevant in **general relativity**?

Basic literature

C. Sormani and C. Vega.
Null distance on a spacetime.
Classical Quantum Gravity 33 (2016), no. 8, 085001, 29 p.

C. Sormani.
Spacetime intrinsic flat convergence.
Oberwolfach Report for the Workshop ID 1832: Mathematical General Relativity (2018), 1–3.

B. Allen and A. Burtscher.
Properties of the null distance and spacetime convergence.
Int. Math. Res. Not. IMRN 2022, no. 10, 7729–7808.
https://doi.org/10.1093/imrn/rnaa311

M. Kunzinger and R. Steinbauer.
Null distance and convergence of Lorentzian length spaces.
Ann. Henri Poincaré (2022), 24 p.

Outline

Riemannian distance function

Assume

(M, g) ... Riemannian manifold

$\mathcal{A}$ ... class of piecewise smooth paths (and reparametrizations)

$L_g(\gamma) := \int_a^b \sqrt{g_{\gamma(t)}(\dot\gamma(t), \dot\gamma(t))}dt$ ... length of curve $\gamma\colon [a, b] \to M$ in $\mathcal{A}$

Define Riemannian distance

$$d_g(p, q) := \inf\{L_g(\gamma) \mid \gamma \in \mathcal{A} \text{ between } p \text{ and } q \text{ in } M\}$$

Theorem

The distance function d_g is an intrinsic metric on M that induces the manifold topology.

Some consequences of this metric space structure

- completeness simplifies due to the Hopf–Rinow Theorem (1931)
- can study "closeness" of (compact) Riemannian manifolds (that are not necessarily diffeomorphic) in terms of the **Gromov–Hausdorff distance** d_{GH} (1981) as metric spaces
- add further an integral current structure T, e.g., for oriented compact submanifolds M

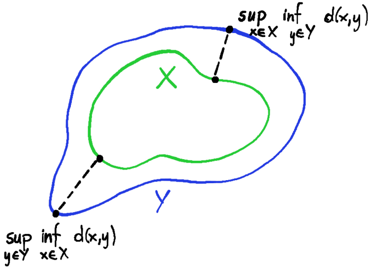
$$T(\omega) := \int_M \omega,$$

and study **Sormani–Wenger intrinsic flat distance** $d_{\mathcal{F}}$ (2011) between (compact) oriented Riemannian manifolds as integral current spaces

Definition of d_{GH}

Hausdorff distance between subsets of a metric space (Z, d):

$$d_{\mathrm{H}}(X, Y) := \max \left\{ \sup_{x \in X} \inf_{y \in Y} d(x, y), \; \sup_{y \in Y} \inf_{x \in X} d(x, y) \right\}$$



Gromov–Hausdorff distance between metric spaces

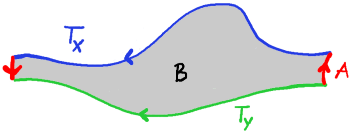
$$d_{\mathrm{GH}}(X, Y) := \inf\{ d_{\mathrm{H}}(\varphi(X), \psi(Y)) \mid \varphi\colon X \to Z, \psi\colon Y \to Z \text{ isometric embeddings into common metric space } (Z, d)\}$$

$d_{\mathrm{GH}}(X, Y) \approx$ how close are (X, d_X) and (Y, d_Y) from being **isometric**

Definition of $d_{\mathcal{F}} = d_{\mathrm{SWIF}}$

Flat distance between currents of a metric space (Z, d)
(Federer–Fleming 1960; Ambrosio–Kirchheim 2000)

$$d_{\mathrm{F}}(T_X, T_Y) := \inf \left\{ \mathbf{M}^n(A) + \mathbf{M}^{n+1}(B) \,\middle|\, A + \partial B = T_X - T_Y \right\}$$



Intrinsic flat distance between integral current spaces
(Sormani–Wenger 2011)

$$d_{\mathcal{F}}(X, Y) := \inf\{ d_{\mathrm{F}}(\varphi_\# T_X, \psi_\# T_Y) \mid \varphi\colon X \to Z, \psi\colon Y \to Z \text{ isometric embed. into common complete metric space } (Z, d)\}$$

$d_{\mathcal{F}}(X, Y) \approx$ how close are (X, d_X, T_X) and (Y, d_Y, T_Y) from being **current-preserving isometric**

Applications of convergence in Riemannian geometry

- GH-convergence works well with respect to **lower sectional and Ricci curvature bounds**
- SWIF-convergence is a weaker notion
- GH- and SWIF-limits agree in certain cases, e.g., $\mathrm{Ric} \geq 0$, $\mathrm{vol} \geq C > 0$ (Sormani–Wenger 2010)
- SWIF-convergence compatible with **scalar curvature bounds**
  - stability of the Positive Mass Theorem, e.g., for M_j rotationally symmetric, asymptotically flat (Lee–Sormani 2014):

$$m_{\mathrm{ADM}}(M_j) \to 0 \Longrightarrow M_j \xrightarrow{\mathcal{F}} M_\infty \text{ Euclidean space}$$

  - related conjecture for almost rigidity of Riemannian Penrose ineq.

Natural analogue: Lorentzian distance function

Assume

(M, g) ... spacetime

$\mathcal{A}_\vee$ ... class of piecewise smooth future-directed causal paths

$L_g(\gamma) := \int_a^b \sqrt{-g_{\gamma(t)}(\dot\gamma(t), \dot\gamma(t))}dt$ ... length of $\gamma\colon [a, b] \to M$ in $\mathcal{A}_\vee$

Definition (Lorentzian distance)

$$d_g(p, q) := \begin{cases} \sup\{L_g(\gamma) \mid \gamma \in \mathcal{A}_\vee \text{ between } p \text{ and } q\} & q \in J^+(p), \\ 0 & q \notin J^+(p). \end{cases}$$

- encodes information about causality
- e.g., globally hyperbolic spacetime $\Longrightarrow d_g$ finite and continuous
- not a metric (not symmetric, reverse triangle inequality)
- still of some use (Noldus 2004, Kunzinger–Sämann 2018 etc.)

Null distance of Sormani–Vega (2016)

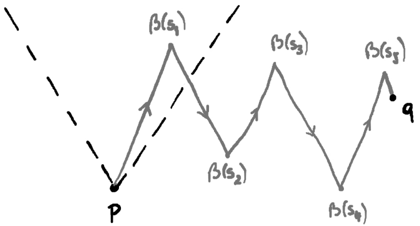
Similar approach, but **assume** a bit more:

(M, g) ... spacetime

τ ... time function $M \to \mathbb{R}$

$\hat{\mathcal{A}}$ ... class of piecewise smooth causal paths (future- <u>and</u> past-directed allowed on pieces)

$\hat{L}_\tau(\beta) := \sum_{i=1}^{k} |\tau(\beta(s_i)) - \tau(\beta(s_{i-1}))|$ ... **null length** of $\beta \in \hat{\mathcal{A}}$



Definition (Null distance)

$$\hat{d}_\tau(p, q) := \inf\{\hat{L}_\tau(\beta) \mid \beta \in \hat{\mathcal{A}} \text{ between } p \text{ and } q\}$$

Properties of null distance: Length metric structure

Always: τ (generalized) time function $\Longrightarrow \hat{d}_\tau$ pseudo-metric

Theorem (Sormani–Vega 2016)

*For sufficiently nice τ (e.g., locally anti-Lipschitz) the **null distance***

$$\hat{d}_\tau(p,q) = \inf\{\hat{L}_\tau(\beta) \,|\, \beta \in \mathcal{A} \text{ from } p \text{ to } q\}$$

is a length metric on M.

Some basic properties.

- $\hat{d}_\tau$ induces the manifold **topology**
- $\hat{d}_\tau$ is **conformally invariant**
- **scaling** for $\lambda > 0$: $\hat{d}_{\tau_2} = \lambda \hat{d}_{\tau_1} \iff \tau_2 = \lambda \tau_1 + C$
- $q \in J^+(p) \Longrightarrow \hat{d}_\tau(p,q) = \tau(q) - \tau(p)$
- $\hat{d}_\tau$ is **bounded on causal diamonds** $J^+(p) \cap J^-(q)$

Properties of null distance: τ (in)dependence

Bad choices of τ hurt $\hat{d}_\tau$:

- e.g., for $\tau = t^3$ on Minkowski space $\hat{d}_\tau$ is not definite at $\{t = 0\}$

Good choices and certain robustness:

- τ locally anti-Lipschitz: $p, q \in U$, $p \leq q \Longrightarrow \tau(q) - \tau(p) \geq d_U(p, q)$
- spacetimes that admit a regular cosmological time function (Andersson–Galloway–Howard 1998)

$$\tau_g(q) := \sup_{p \leq q} d_g(p, q), \quad \text{with } d_g \text{ Lorentzian distance}$$

- $\tau(t, x) = \phi(t)$ with $\phi' > 0$ for warped products $-dt^2 + f(t)^2\sigma$ with σ complete Riemannian metric (equivalent on compact sets)
- τ_1, τ_2 temporal functions $\Longrightarrow \hat{d}_{\tau_1}$ and $\hat{d}_{\tau_2}$ equiv. on compact sets (B.–García-Heveling)

Properties of null distance: Relation to causality

Always: $q \in J^+(p) \Longrightarrow \hat{d}_\tau(p,q) = \tau(q) - \tau(p)$

Definition (causality encoding)

We say that $\hat{d}_\tau$ **encodes causality** if

$$q \in J^+(p) \Longleftrightarrow \hat{d}_\tau(p,q) = \tau(q) - \tau(p)$$

Open problem: When does this hold in general?

- for **warped spacetimes** with $g = -dt^2 + f(t)^2\sigma$ and time functions $\tau(t,x) = \phi(t)$ with $\phi' > 0$ the null distances $\hat{d}_\tau$ encode causality (Sormani–Vega 2016)
- **incompleteness** is an obstruction for $\hat{d}_\tau$ to encode causality (Allen–B. 2022)
- holds for **globally hyperbolic spacetimes** with Cauchy temporal functions τ (B.–García-Heveling)
- $\exists$ causally simple spacetimes with $J^+ = K^+ \subsetneq \hat{d}_\tau$-relation, so $\hat{d}_\tau$ is really weaker (B.–García-Heveling)

Properties of null distance: completeness

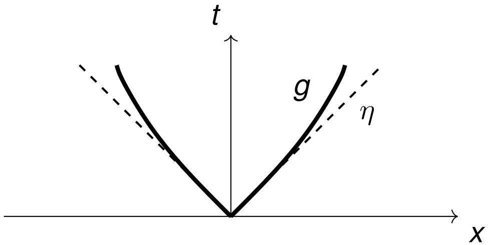
Theorem (Allen–B. 2022)

Let (M, g) be a spacetime with time function τ. If τ is ***anti-Lipschitz on M with respect to a complete distance function*** *d that induces the manifold topology (e.g., induced by a complete Riemannian metric), then $(M, \hat{d}_\tau)$ is a complete metric space.*

Warning!

$(M, \hat{d}_\tau)$ complete $\not\Rightarrow$ null distance is achieved by a piecew. causal path

Example: $g = -dt^2 + (t^2+1)^2 dx^2$ and take $p, q \in \{t = 0\}$



Short pause

- **basic properties of null distance** have been established (Sormani–Vega 2016, Allen–B. 2022)
- good choice of time function needed, but not too restrictive, e.g., temporal functions good for causal theory (B.–García-Heveling)
- warped product spacetimes are well understood

Broad aim

understand **geometric stability of spacetimes**

- obtained a metric space structure (enough for GH) but need integral current space for SWIF convergence (Allen–B. 2022)
- understand spacetime convergence for of **warped product spacetimes** (Allen–B. 2022)
- more general upcoming results (Sakovich–Sormani)

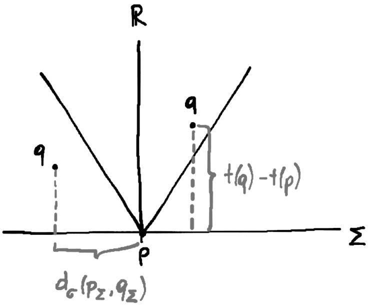
Example 1: Lorentzian products

If (Σ, σ) is a Riemannian manifold and $(\mathbb{R} \times \Sigma, \eta_\sigma)$ where

$$\eta_\sigma = -dt^2 + \sigma,$$

then the null distance between $p = (t(p), p_\Sigma)$ and $q = (t(q), q_\Sigma)$ induced by the canonical time function t is

$$\hat{d}_{t,1}(p,q) = \begin{cases} |t(p) - t(q)| & q \in J^\pm(p), \\ d_\sigma(p_\Sigma, q_\Sigma) & q \notin J^\pm(p). \end{cases}$$



Example 2: Warped spacetimes (e.g. FLRW solutions)

Let I be an interval, (Σ, σ) be a Riemannian manifold, and f be such that

$$0 < f_{\min} \leq f(t) \leq f_{\max}, \qquad t \in I.$$

Then the warped product $M_f = I \times_f \Sigma$ with Lorentzian metric

$$g_f = -dt^2 + f(t)^2 \sigma$$

is such that the null distance $\hat{d}_{t,f}$ of g_f satisfies

$$\begin{aligned} \hat{d}_{t,f}(p,q) &= |t(p) - t(q)|, && q \in J^{\pm}(p), \\ f_{\min} d_\sigma(p_\Sigma, q_\Sigma) \leq \hat{d}_{t,f}(p,q) &\leq f_{\max} d_\sigma(p_\Sigma, q_\Sigma), && q \notin J^{\pm}(p). \end{aligned}$$

$\Longrightarrow \min\{1, f_{\min}\} \hat{d}_{t,1}(p,q) \leq \hat{d}_{t,f}(p,q) \leq \max\{1, f_{\max}\} \hat{d}_{t,1}(p,q)$ for all p, q

Warped products are integral current spaces

Theorem (Allen–B. 2022)

Let I be an interval, (Σ, σ) orientable connected complete Riemannian manifold and $f\colon I \to (0, \infty)$ bounded away from 0 and ∞. Then there is a natural ***local integral current structure on*** *$M = I \times_f \Sigma$* ***with respect to the null distance*** *$\hat{d}_{t,f}$.*

If $I \times_f \Sigma$ is compact, then $(M, \hat{d}_{t,f})$ is integral current space.

Sketch of proof:

- associated Riemannian product $I \times \Sigma$ with standard current

$$T(h, \pi_1, \ldots, \pi_n) = \int_M h \, d\pi_1 \wedge \ldots \wedge d\pi_n$$

- Riemannian product satisfies Pythagorean formula (complete)
  $\Longrightarrow$ identity between products $I \times \Sigma$ is bi-Lipschitz
  $\Longrightarrow$ push-forward current $\mathrm{id}_\# T$ on Lorentzian product $(I \times \Sigma, \hat{d}_{t,1})$
- just as before: identity between $I \times \Sigma$ and $I \times_f \Sigma$ is bi-Lipschitz
  $\Longrightarrow$ push-forward current to $(M, \hat{d}_{t,f})$. $\square$

Globally hyperbolic spacetimes are (local) integral current spaces

For **warped spacetimes** $I \times_f \Sigma$ we know:

$I \times_f \Sigma$ is globally hyperbolic $\Longleftrightarrow$ Σ complete Riemannian manifold

After quite some more work one obtains:

Theorem (Allen–B. 2022)

*Let (M, g) be a **globally hyperbolic spacetime** with smooth time function τ (Bernal–Sanchez 2005). Suppose M admits complete Cauchy hypersurfaces and $(M, \hat{d}_\tau)$ is complete as metric space. Then $(M, \hat{d}_\tau)$ is a local integral current space.*
(Lang–Wenger 2011, Jauregui–Lee 2019)

*If M is **compact**, then $(M, \hat{d}_\tau)$ is an integral current space.*
(Ambrosio–Kirchheim 2000, Sormani–Wenger 2011)

Final stage

For suitable τ a spacetime with null distance $\hat{d}_\tau$

- is a length **metric space**, and
- is a (local) **integral current space.**

$\rightsquigarrow$ We can study/compare (pointed) Gromov–Hausdorff (GH) and Sormani–Wenger intrinsic flat (SWIF) **spacetime convergence**!

Spacetime convergence for warped products

We restrict ourselves to sequences of **warped products** $I \times_{f_j} \Sigma$:

- I closed, (Σ, σ) oriented, connected, compact Riemannian mf
- $f_j \colon I \to (0, \infty)$ continuous, f_j uniformly bounded away from 0
- metric $g_j = -dt^2 + f_j(t)^2 \sigma$

Theorem (Allen–B. 2022)

*Suppose the **warping functions** $f_j \to f_\infty$ **converge uniformly**, then the warped products with corresponding null distances*

$$(I \times_{f_j} \Sigma, \hat{d}_{t,j}) \to (I \times_{f_\infty} \Sigma, \hat{d}_{t,\infty})$$

converge in the uniform, GH, and SWIF sense (to the same limit!).

Sketch of proof

Theorem (Allen–B. 2022)

$f_j \to f_\infty$ *uniform* $\Longrightarrow (I \times_{f_j} \Sigma, \hat{d}_{t,j}) \to (I \times_{f_\infty} \Sigma, \hat{d}_{t,\infty})$ *uniform/GH/SWIF*

1. **Pointwise convergence** $\hat{d}_{t,j}(p,q) \to \hat{d}_{t,\infty}(p,q)$:
   long explicit estimates using ε-close curves β_j between p and q
2. $\exists$ **subsequence** $(\hat{d}_{t,j_k})_k$ **that converges to <u>some</u> length metric space** (M, d_∞) in uniform/GH/SWIF sense:
   show uniform bi-Lipschitz bounds, i.e., that there exists $\lambda > 1$ s.t.
   $$\frac{1}{\lambda} \leq \frac{\hat{d}_{t,j}(p,q)}{\hat{d}_{t,1}(p,q)} \leq \lambda, \qquad \text{for all } j \in \mathbb{N},\ p,q \in M$$
   and then apply a Theorem of Huang–Lee–Sormani (2017)
3. **Uniform/GH/SWIF convergence** of whole sequence to $(M, \hat{d}_\infty)$:
   (1) $\Longrightarrow d_\infty = \hat{d}_{t,\infty}$ is uniform limit of $(\hat{d}_{t,j_k})_k \ldots$ and of $(\hat{d}_{t,j})_j$ by (2)
   HLS Theorem (2017) $\Longrightarrow$ same for GH/SWIF convergence $\square$

Pointwise or L^p convergence $f_j \to f_\infty$ is not enough

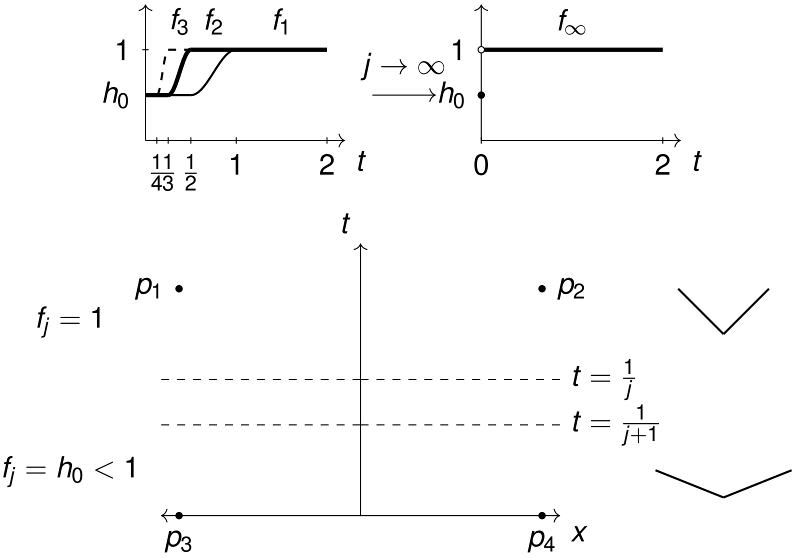
Non-uniform convergence $f_j \to f_\infty$ **can destroy everything**:

- uniform/GF/SWIF convergence but $\hat{d}_{t,j} \to d_\infty \neq \hat{d}_{t,\infty}$ if null cones of g_∞ narrower than of g_j (for large j)
- convergence to a degenerate warped product (g_∞ not Lorentzian) and uniform convergence to pseudo-metric, GH-limit≠SWIF-limit because of "collapse"

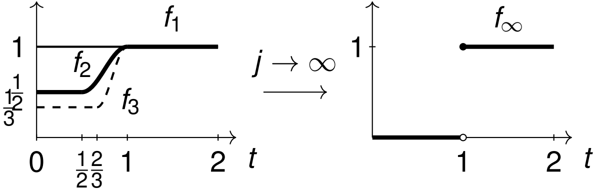
But **uniform convergence** $f_j \to f_\infty$ **is not always necessary**:

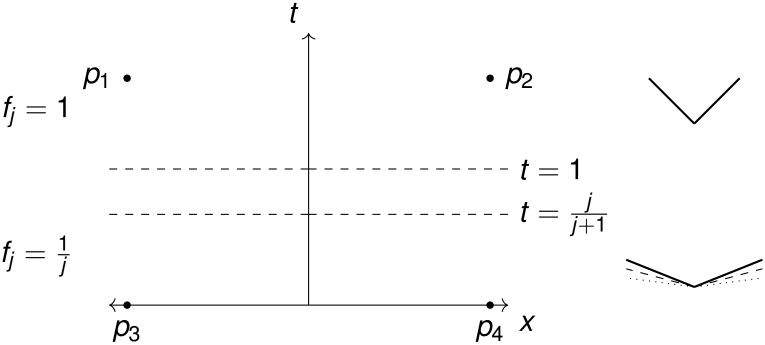
- $f_j \to f_\infty$ pointwise can imply $\hat{d}_{t,j} \to \hat{d}_{t,\infty}$ uniform/GH/SWIF if the null cones of g_∞ wider than of g_j (for large j)

Example 1: convergence but limiting $d_\infty \neq \hat{d}_{t,\infty}$

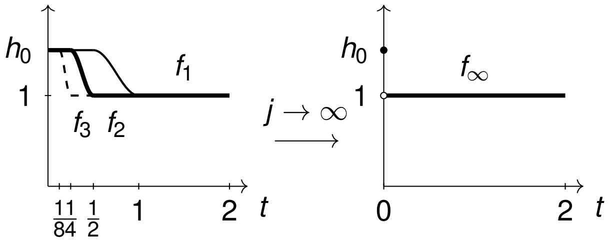


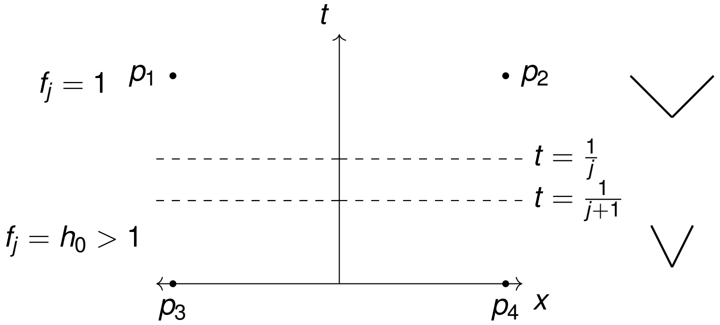
Example 2: SWIF and GH limits exist but disagree





Example 3: convergence to null distance $\hat{d}_{t,\infty}$





Candidate limit spaces

- the **degenerate limits** obtained are Lorentzian length spaces (Allen–B. 2022)
- our results in the context of warped products (properties of null distance and GH convergence) can be **extended to LLS** with suitable time functions, and are compatible with synthetic sectional curvature bounds (Kunzinger–Steinbauer 2022)
- suitable **time functions** (with various classical properties) actually **exist on LLS** (B.–García-Heveling 2021)

Summary

- null distance $\hat{d}_\tau$ is a way to obtain a meaningful **intrinsic, conformally invariant metric** on most spacetimes
- closely related to **causality**, but generally weaker ($J \subseteq \hat{d}_\tau$-relation)
- if restricted to a nice class: the choice of **time function** τ may have global impact but at least produces equivalent metrics on compacta
- **warped products** with associated null distances converge in uniform/GH/SWIF sense to the same limit if warping functions converge uniformly (otherwise more complicated)

Thank you for your attention!