# Hype-free Noncommutative Geometry

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Introduction
The best idea we ever had (and why it doesn't work)
The Standard Model (classical aspects)
Noncommutative geometry in a nutshell
NCG in non-Euclidean signature
NCG and general covariance
Solving unimodularity

# Introduction

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- O. Klein (1926): explains the cylinder condition by *compactification*.

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Experimental problems: 1) A "tower" of particles, 2) instability of extra-dimensions[1]

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Theoretical problem: Neither the cylinder condition nor the small radius of the compact dimension are invariant under general diffeo.

$\Rightarrow$ One must restrict diffeos to those preserving a base $\times$ fiber structure.

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$$\begin{array}{rcl} \mathrm{GR} & \rightarrow & \mathrm{GT} \\ \mathrm{manifold} & \rightarrow & G-\mathrm{bundle} \\ \mathrm{metric} & \rightarrow & \mathrm{connection} \end{array}$$

- C.N. Yang and R. Mills (1954): gauge theory with non-abelian group $G$ and Lagrangian $\mathcal{L} = -\frac{1}{2}\mathrm{Tr}(F^2)$, where $F =$ curvature of the $G$-connection.
- Higgs mechanism (1964): adds a scalar field, otherwise the gauge bosons are massless.

[1] R. Penrose, *On the stability of extra space dimensions*, in *The future of Theoretical Physics and Cosmology*, Cambridge 2003

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- $M = 4$-dimensional Lorentzian spin manifold, $\mathcal{S}$ =spinor bundle.
- $G = U(1) \times SU(2) \times SU(3)$ = gauge group.
- Fermion fields (= matter): sections of $\mathcal{S} \otimes V$, where $V = G-$ vector bundle of dim 24 (=number of different elem. fermions).
- Gauge bosons (photon, $W$ and $Z$, gluons): connection $1$-forms.
- Scalar boson (=Higgs field), section of irreducible $SU(2)$-bundle ($\mathbb{C}^2$-valued field).

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+ a bosonic Lagrangian

1. $\mathcal{L}_{\text{gauge}} =$ YM,
2. $\mathcal{L}_{\text{Higgs}} = |D_\mu H|^2 - V(H)$, where $V =$ quartic potential, $H = (\alpha, \beta)$.

+ a Fermionic Lagrangian

1. $\mathcal{L}_{kinetic} = (\Psi, \not{D}\Psi)$,
2. $\mathcal{L}_{minimal} = (\Psi, A^\mu \gamma_\mu \Psi)$,
3. $\mathcal{L}_{Yukawa} = \alpha(\nu_L^i, Y_{ij}^\nu \nu_R^j) + \beta(e_L^i, Y_{ij}^e e_R^j) + \ldots$,
4. $\mathcal{L}_{Majorana} = ((\nu_R^i)^c, M_{ij}\nu_R^j)$.

Constraints: symmetries, anomaly freeness, renormalizability, experiments !

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- NCG was born from an elaboration of QM.

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1. W. Heisenberg (1925): noncommutativity of *phase* space.
2. 1940's: $C^*$-algebras.
3. 1990's: Spectral triples

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We will not talk about noncommutative field theory (deformation of coordinates), derivation-based noncommutativity, . . .

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**Th (Gelfand-Naimark) :**
Compact spaces + continuous maps $\rightleftharpoons$ commutative $C^*$-alg+$C^*$-morphisms

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**Th (Gelfand-Naimark) :**
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Def: A $C^*$-algebra $\mathcal{A}$ is a complete normed $\mathbb{C}$-algebra with $*$ s.t. $||a^*a|| = ||a||^2$.

**Ex:**
1. $\mathcal{A} = \mathcal{C}(X)$, with $||.||_\infty$ and c.c.
2. $\mathcal{A} =$ closed $*$-subalgebra of $B(\mathcal{H})$ with operator norm and Hilbert adjoint.

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Def: A state $\omega$ on a $C^*$-algebra is positive linear functional of norm $1$.
A state is pure if it is not decomposable as a non-trivial convex combination.

On $\mathcal{A} = \mathcal{C}(X)$, $x \in X \rightleftharpoons$ pure state via $x(f) := f(x)$ (Gelfand transform)
If $\mathcal{A}$ is not commutative, one *pretends* $\mathcal{A} = \mathcal{C}(\tilde{X})$ for some (non-existing) $\tilde{X}\ldots$

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Connes' key insight: “$ds = D^{-1}$”.

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# From NC topology to NC geometry

Connes' key insight: "$ds = D^{-1}$".

$$d(\omega, \omega') = \sup_{a \in \mathcal{A}} \{ |\omega(a) - \omega'(a)|, \ \|[D, a]\| \leq 1 \}$$

(Connes' distance formula)

$\longrightarrow$ Gives back the geodesic distance in the case of a manifold.

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**Def:** A (real, even) spectral triple is a multiplet $(\mathcal{A}, \mathcal{H}, \pi, D, J, \chi)$ with $\mathcal{A}$ a $C^*$-algebra, $\mathcal{H}$ a Hilbert space, $\pi$ a rep. of $\mathcal{A}$, $D$, $\chi$ linear and $J$ antilinear s.t.

1. $\chi^2 = 1$, $\chi^* = \chi$, $[\chi, \pi(\mathcal{A})] = 0$, $\{\chi, D\} = 0$,
2. $D^* = D$ (formally)
3. $J^2 = \pm 1$, $J^* J = 1$, $[J, D] = 0$, $J\chi = \pm \chi J$.
4. $[\pi(\mathcal{A}), J\pi(\mathcal{A})J^{-1}] = 0$ (order 0 condition)

signs $\leftrightarrow$ KO dimension

Order 1 condition $\rightarrow$ later

| | |
|---|---|
| $D$ | metric |
| $J$ | spin structure |
| $\chi$ | orientation |

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# The canonical triple of a spin manifold

$(M, g)$ = spin manifold. Spin structure $\sigma = (\mathcal{S}, \rho, H, \chi, J)$ where

1. $\mathcal{S}$ is a complex vector bundle over $M$,
2. $\rho : \mathbb{C}l(TM, g) \to \mathrm{End}(\mathcal{S})$ is a bundle isomorphism,
3. $\chi = i^k \rho(e_1 \dots e_n)$ where $(e_1, \dots, e_n)$ = positive orthonormal basis,
4. $H$ is a *positive spinor metric*: $H(\rho(v)\phi, \psi) = H(\phi, \rho(v)\psi)$ + norm.,
5. $J$ is a bundle map $\mathcal{S} \to \mathcal{S}$, antilinear in the fibres, anticommutes with vectors, satisfies $J^2 = \pm 1$, and $J^\times J = \pm 1$.

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The canonical triple is:

- $\mathcal{A} = \mathcal{C}(M)$,
- $\mathcal{H} = L^2$-completion of compactly supported spinors wrt

$$(\Phi, \Psi) = \int_M H_x(\Phi_x, \Psi_x)\mathrm{vol}_g$$

- $(\pi(f)\Psi)_x = f(x)\Psi_x$, $\chi$ and $J$ as above,
- $D = \not{D} = i \sum_i \rho(e_i)\nabla_{e_i}$ where $(e_i)$ is orthonormal.

$\to$ A ST with $\mathcal{A}$ commutative + smoothness conditions is of this form[2].

[2] A. Connes, J. Noncommut. Geom. 7 (2013) 1-82 arXiv:0810.2088

2 points separated by a distance $\delta$. Which spectral triple ?

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# A discrete example

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2 points separated by a distance $\delta$. Which spectral triple ?

$$\to \mathcal{A} = \mathbb{C}^2, \mathcal{H} = \mathbb{C}^2, D = \frac{1}{\delta}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, [D, \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}] = \frac{a(2)-a(1)}{\delta}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

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3 points with distances $\delta_{ij}$ ?

$$\mathcal{A} = \mathbb{C}^3, \mathcal{H} = \mathbb{C}^3, D = \begin{pmatrix} 0 & \delta_{12}^{-1} & \delta_{13}^{-1} \\ \delta_{12}^{-1} & 0 & \delta_{23}^{-1} \\ \delta_{13}^{-1} & \delta_{23}^{-1} & 0 \end{pmatrix}$$

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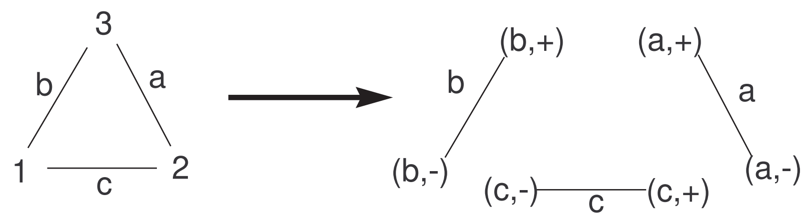
Solution:




$b^+ = a^+ = 3$ (split graph)

$(b,+) \neq (a,+)$

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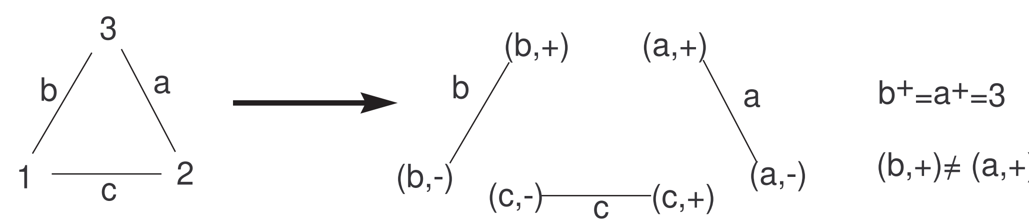
Solution: 

(split graph)

$G = (V, E)$ finite graph, $\delta : E \to \mathbb{R}_+^*$ weight function

$\tilde{E} := E \times \{-;+\}$, $H = L^2(\tilde{E}) = \mathbb{C}^E \otimes \mathbb{C}^2$ + canonical $\langle .,. \rangle$.

$$\pi(a)F(e,\pm) = a(e^\pm)F(e,\pm) = \bigoplus_{e\in E}\begin{pmatrix} a(e^-) & 0 \\ 0 & a(e^+) \end{pmatrix}.$$

$$DF(e,\pm) = \frac{1}{\delta_e}F(e,\mp) = \bigoplus_{e\in E}\frac{1}{\delta_e}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

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Given $\mathcal{S}_1 = (\mathcal{A}_1, \mathcal{H}_1, \ldots)$ and $\mathcal{S}_2 = (\mathcal{A}_2, \mathcal{H}_2, \ldots)$ one can form $\mathcal{S}_1 \hat{\otimes} \mathcal{S}_2 = (\mathcal{A}, \mathcal{H}, \ldots)$ where

- $\mathcal{A} = \mathcal{A}_1 \hat{\otimes} \mathcal{A}_2$,
- $\mathcal{H} = \mathcal{H}_1 \hat{\otimes} \mathcal{H}_2$,
- $\pi = \pi_1 \hat{\otimes} \pi_2$,
- $J = J_1 \hat{\otimes} J_2$,
- $\chi = \chi_1 \hat{\otimes} \chi_2$,
- $D = D_1 \hat{\otimes} 1 + 1 \hat{\otimes} D_2$.

# Almost-commutative spectral triples

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- $\mathcal{H} = \mathcal{H}_1 \hat{\otimes} \mathcal{H}_2$,
- $\pi = \pi_1 \hat{\otimes} \pi_2$,
- $J = J_1 \hat{\otimes} J_2$,
- $\chi = \chi_1 \hat{\otimes} \chi_2$,
- $D = D_1 \hat{\otimes} 1 + 1 \hat{\otimes} D_2$.

$\rightarrow$ If $\mathcal{S}_1 = \mathrm{can}(M_1)$ and $\mathcal{S}_2 = \mathrm{can}(M_2)$, then $\mathcal{S}_1 \hat{\otimes} \mathcal{S}_2 = \mathrm{can}(M_1 \times M_2)$,

$\rightarrow$ KO dimensions add up,

$\rightarrow$ If $\mathcal{S}_1 = \mathrm{can}(M_1)$ and $\mathcal{S}_2$ is finite-dimensional, $\mathcal{S}$ is called *almost-commutative*.

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$$\begin{tikzcd} 1-\text{form} \arrow[r, "\sharp"] \arrow[rrr, bend right, "\gamma"'] & \text{vector field} \arrow[r, "\subset"] & \text{section of } \mathbb{C}l(TM) \arrow[r, "\rho"] & \text{operator on spinor fields} \end{tikzcd}$$

Ex: If $a$ is a smooth function, $[D, \pi(a)] = i\gamma(da)$.

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On a general ST: $\boxed{\Omega^1_D := \{\sum_i \pi(a_i)[D,\pi(b_i)]\} \text{ (noncommutative 1-forms)}}$

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Order 1 condition $(C_1)$ : $[\Omega^1_D, J\pi(\mathcal{A})J^{-1}] = 0$
Let $\omega$ be self-adjoint. Fluctuated Dirac operator:

$$D_\omega = D + \omega + J\omega J^{-1}$$

One can define $d\omega := \sum_i [D,a_i][D,b_i]$ modulo a “junk” ideal and the curvature $F(\omega) = d\omega + \omega^2$.

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Almost-commutative ST based on $\mathrm{can}(M, g)$ and finite dimensional ST $S_F = (\mathcal{A}_F, \mathcal{H}_F, \ldots)$ such that

- $\mathcal{A}_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$,
- $\mathcal{H}_F = \mathcal{H}_R \oplus \mathcal{H}_L \oplus \mathcal{H}_{\bar{R}} \oplus \mathcal{H}_{\bar{L}}$, $H_\sigma = \mathbb{C}^2 \otimes (\mathbb{C} \oplus \mathbb{C}^3_{\text{color}}) \otimes \mathbb{C}^3_{\text{gen}}$,
- $\chi_F = [1_R, -1_L, -1_{\bar{R}}, 1_{\bar{L}}]$,
- $J_F = \begin{pmatrix} 0 & -1_{antipart} \\ 1_{part} & 0 \end{pmatrix} \circ c.c.$,
- $\pi_F(\lambda, q, m) = [\tilde{q}_\lambda, \tilde{q}, \lambda 1_2 \oplus 1_2 \otimes m, \lambda 1_2 \oplus 1_2 \otimes m] \otimes 1_3$, where $q_\lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^* \end{pmatrix}$ and $\tilde{q} = q \oplus q \otimes 1_3 \simeq q \otimes 1_4$.
- $D_F = \left\{ \begin{pmatrix} 0 & Y^\dagger & M^\dagger & 0 \\ Y & 0 & 0 & 0 \\ M & 0 & 0 & Y^T \\ 0 & 0 & Y^* & 0 \end{pmatrix} \right.$, where $Y = \begin{pmatrix} Y_\nu & 0 \\ 0 & Y_e \end{pmatrix} \oplus \begin{pmatrix} 1_3 \otimes Y_u & 0 \\ 0 & 1_3 \otimes Y_d \end{pmatrix}$ and $M = \begin{pmatrix} m_\nu & 0 \\ 0 & 0 \end{pmatrix} \oplus 0$.

Choice of $D_F$ strongly constrained by: 1) odd, 2) commutes with $J$, and 3) order 1 condition

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- Fermionic fields (elements of $\mathcal{H} = \mathcal{H}_F$-valued $L^2$ spinors),
- Bosonic fields $\leftrightarrow$ fluctuated Dirac operators $D_\omega$ $\leftrightarrow$ NC 1-forms $\omega$.

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= gauge bosons $\oplus$ Higgs bosons !

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- Fermionic action = $(J\Psi, D_\omega \Psi)$

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- Bosonic action = $\begin{cases} \text{1) Connes-Lott: } \int_M \mathrm{Tr}(F_\omega^2), \text{ or} \\ \text{2) Spectral action: } \mathrm{Tr}(f(D_\omega^2/m^2)) \text{ with } f \approx 1_{[0;1]} \end{cases}$

  $\rightarrow$ They contain kinetic gauge and Higgs term + Higgs potential.

  $\rightarrow$ They depend on the norm of the mass matrices.

  $\rightarrow$ Less constants than usual, hence makes predictions.

  $\rightarrow$ In particular $g_w = g_Y = g_s$.

  $\rightarrow$ The spectral action also contains Einstein-Hilbert action evaluated at $g$.

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Shortcomings:

1. It is a classical theory.
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1. $4$ times too many fermions (“fermion doubling”) $\to$ comes from $\mathcal{H}_F$ needed to represent $A_F$.

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Lesser known/discussed problems:

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1. Why real $C^*$-algebras ?
2. The massless photon condition.
3. One does not recover GR.
4. Is it possible in principle to obtain a general covariant theory from NCG ? Spin structure & Hilbert space structure (spinor metric) depend on $g$...

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Barrett[3] proposed that physical fields satisfy

$$J\Psi = \Psi, \qquad \chi\Psi = \Psi$$

(Majorana-Weyl conditions)

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- requires $J^2 = 1$ and $J\chi = \chi J$ which constrains the total KO-dimension.

# Solving the fermion doubling problem

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- Not compatible if $M$ is a Euclidean $4$-manifold !

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- Usual see-saw needs a symmetric $M \Rightarrow$ constrains the finite KO-dim.
- Not compatible if $M$ is a Euclidean $4$-manifold !
- (Formally) works if $M$ is a Lorentzian $4$-manifold.
- Proper treatment needs a non-Euclidean finite part

Constraints on four signs ($J^\times = \pm J$, $\chi^\times = \pm\chi$):

1. $\mathcal{M}$ is a Lorentzian four-manifold,
2. MW conditions,
3. symmetry of $M$,
4. non-vanishing of the fermionic kinetic terms,
5. non-vanishing of the Majorana mass term.

There exists a unique choice of signs for the finite part which solves all this[4] !

[3] J. W. Barrett, J. Math. Phys., **48** (2007)
[4] N. Bizi, thesis, abs/1812.00038 (2018)

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Convention: Clifford algebra of $(V, g)$ generated by $V$ with relations
$$vv' + v'v = 2g(v, v')$$

Canonical antiautomorphism: $(v_1 \dots v_k)^T = v_k \dots v_1$.

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$$\begin{tikzcd}
(V, g) \arrow[r, leftrightarrow] \arrow[d, dashed] & Cl(V, g) \arrow[r, leftrightarrow] & (\mathbb{C}l(V), c) \arrow[d, leftrightarrow] \\
(S, H_S) \arrow[r, leftrightarrow] & (\mathrm{End}(S), \times) \arrow[r, phantom, "\cong"] & (\mathbb{C}l(V), \times)
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up to $\lambda \neq 0$ $\qquad$ $S$ a spinor module $\qquad$ $a^\times := c(a^T)$

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**Th:** $\exists H_S$ unique up to scalar multiple such that all $v \in V$ are selfajoint

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$g$ positive definite $\Leftrightarrow$ $H_S$ definite,

$g$ non-positive definite $\Leftrightarrow$ $H_S$ neutral

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up to $\lambda \neq 0$     $S$ a spinor module     $a^\times := c(a^T)$

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$g$ positive definite $\Leftrightarrow$ $H_S$ definite,
$g$ non-positive definite $\Leftrightarrow$ $H_S$ neutral
$g$ anti-Lorentzian $\Leftrightarrow \exists v \in V, H_S(., v \cdot .)$ is definite
and in this case $v$ is timelike

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# pre-Krein vs Krein

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**Th:** If $\exists \eta$ s.t. $(\mathcal{K}, \langle .,. \rangle)_\eta$ is a Hilbert space, then it is the case for all $\eta$ and the norms are all equivalent. $\mathcal{K}$ is then called a *Krein space*.

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Let $(M, g)$ be a spin semi-Riemannian manifold.

- On compactly supported spinor fields, one defines $(\Psi, \Phi) = \int_M H_x(\Psi_x, \Phi_x)\mathrm{vol}_g \to$ *pre*-Krein space
- Given a future-directed timelike vector fields $v$ (congruence of observers), one defines the scalar product $\langle \Psi, \Phi\rangle_v = \int_M H_x(\Psi_x, v_x \cdot \Phi_x)\mathrm{vol}_g$.

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**Th:** $v_1, v_2$ defines equivalent norms iff the hyperbolic angle $(v_1(x), v_2(x))$ is bounded on $M$.

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- Physical interpretation: the Doppler shift between $v_1$ and $v_2$ is bounded.
- Generalization to all signatures using semi-Riemannian angle between maximal negative definite subspaces[5].

[5] FB, N. Bizi, *Doppler shift in semi-Riemannian signature and the non-uniqueness of the Krein space of spinors*, J. Math. Phys. **60** (2019)

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In indefinite signature, $\langle .,.\rangle \to (.,.)$, $\dagger \to \times$.

$J^\times J$ and $\chi^\times \chi$ can be negative: two new signs : $J^\times J = \epsilon\kappa,\ \chi^\times\chi = \epsilon''\kappa''$

$\epsilon, \epsilon'', \kappa, \kappa''$ depend on two integers $[8]$:

| $\nu$ | 0 | 2 | 4 | 6 |
|---|---|---|---|---|
| $\epsilon$ | 1 | -1 | -1 | 1 |
| $\epsilon''$ | 1 | -1 | 1 | -1 |

| $\mu$ | 0 | 2 | 4 | 6 |
|---|---|---|---|---|
| $\kappa$ | 1 | -1 | -1 | 1 |
| $\kappa''$ | 1 | -1 | 1 | -1 |

For a manifold of signature $(p,q)$, $\mu = p+q\ [8]$, $\nu = p-q\ [8]$.

In any case, $\mu, \nu$ are additive wrt $\hat{\otimes}$.

For more on this: N. Bizi, C. Brouder, FB, J. Math. Phys. **59**, 062303 (2018)

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An IST is a ST *except*:

1. $\mathcal{K}$ pre-Krein space,
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The SM IST is the same as the SM ST *except*:

1. $\mathcal{M}$ is a Lorentz $4$-manifold,
2. $\mathcal{K}_F$ is equipped with $(\psi, \phi) := \psi^\dagger \chi_F \phi$,
3. $D_F = \{ \begin{pmatrix} 0 & -Y^\dagger & -M^\dagger & 0 \\ Y & 0 & 0 & 0 \\ M & 0 & 0 & -Y^T \\ 0 & 0 & Y^* & 0 \end{pmatrix},$

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- . . . but it works for the SM IST !

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(FB and C. Brouder, Phys. Rev. D **103**, 035003 (2021)).

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- . . . but it works for the SM IST !
- . . . but it predicts a Higgs mass of 170 GeV (FB and C. Brouder, Phys. Rev. D **103**, 035003 (2021)).

$\Rightarrow$ a consistent and predictive NCG model exists which unifies gauge and Higgs bosons, contains the see-saw mechanism (except if $M = 0$), has the right number of fermions and the correct physical signature (but it has an extra-gauge field and makes an incorrect prediction)

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# NCG and general covariance

# Does NCG really include gravity ?

- The spectral action contains the EH action, but...
- the configuration space $\{D_\omega|\omega^\times = \omega\}$ defines a unique $g$ on $M$ !

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- The spectral action contains the EH action, but...
- the configuration space $\{D_\omega | \omega^\times = \omega\}$ defines a unique $g$ on $M$ !

Natural idea: define

1. a *pre-spectral triple* $\mathcal{T}$ = a ST without Dirac op.,
2. the config space $\mathcal{C}_\mathcal{T} = \{D | (\mathcal{T}, D) = \text{ST}\}$.

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Automorphisms:

1. $\mathrm{Aut}(\mathcal{T}) = \{U | UU^\times = 1, U\pi(\mathcal{A})U^{-1} = \pi(\mathcal{A}), U\chi = \chi U, UJ = JU\}$,
2. $\mathrm{Aut}(\mathcal{T}, D) = \{U \in \mathrm{Aut}(\mathcal{T}) | UD = DU\}$.

If $\pi$ faithful, we have a morphism $\alpha : \mathrm{Aut}(\mathcal{T}) \to \mathrm{Aut}(\mathcal{A})$
Let

1. $\ker(\alpha) = \mathrm{Vert}(\mathcal{T})$,
2. $\mathrm{Im}(\alpha) = \mathrm{Hor}\mathcal{T}$.

If $\alpha$ has a section, then $\mathrm{Aut}(\mathcal{T}) = \mathrm{Hor}(T) \ltimes \mathrm{Vert}(T)$.
Let's test the idea…

# Testing pre-ST

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Consider a *parallelizable* manifold, and (for $n = 4$):

1. A trivial bundle $M \times S$, $S = \mathbb{C}^4$,
2. gamma matrices $\gamma_a \in \mathrm{End}(S)$ (in Dirac or Weyl representation),
3. $\chi = \gamma_5$,
4. $J = \gamma_2 \circ c.c$,
5. spinor metric $H_S(\psi, \psi') = \psi^\dagger \gamma_0 \psi'$.

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Each tetrad $e = (e_a)$ defines 1) a metric $g_e$ such that $e$ is pseudo-orthonormal, 2) a $g_e$-spin structure with rep $\rho_e : \mathbb{C}\ell TM \to \mathrm{End}(S)$ s.t $\rho_e(e_a) = \gamma_a$, and 3) a Dirac operator $D(e) = i \sum(\pm\gamma_a)\nabla^e_{e_a}$.

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Symmetry group of GR $:= G = \mathrm{Diff}(M) \ltimes \mathrm{Spin}(1, n-1)$. Then

1. $\mathrm{Hor}(\mathcal{T}) = \mathrm{Diff}(M)$ for all $n$,
2. $\mathrm{Aut}(\mathcal{T}) \supset G$ for all $n$, with $=$ iff $n < 6$.

Let $\Gamma = \mathrm{Span}(\gamma_a | a = 0, \ldots, 3)$. Then:

1. $\Omega^1_{D(e)} := \Omega^1$ is independent of $e$ and is the space of $\Gamma$-valued fields.
2. This space is invariant under $\mathrm{Diff}(M)$ and $\mathrm{Spin}(1, n-1)$.

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$\Rightarrow \Omega^1$ should be a background structure

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# Another clue

The Clifford, spin-c and spin group are defined by

1. $\Gamma(V,g) = \{g \in \mathbb{C}l(V,g) | \mathrm{Ad}_g(V^{\mathbb{C}}) = V^{\mathbb{C}}\}$,
2. $\mathrm{Spin}^c(V,g) = \{g \in \Gamma(V,g) | \chi g = g\chi,\ gg^{\times} = \pm 1\}$,
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The image of $T^*M$ under Clifford multiplication is $\Omega^1$ !

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- An Algebraic Background $\mathcal{B}$ is a pair $(\mathcal{T}, \Omega^1)$ where $\mathcal{T}$ is a pre-ST and $\Omega^1$ is an odd bimodule $\subset \mathrm{End}(\mathcal{K})$.
- A *compatible Dirac operator* on $\mathcal{B}$ is an operator $D$ such that $(\mathcal{T}, D)$ is a ST and $\Omega^1_D \subset \Omega^1$. (It is *regular* if $\Omega^1_D = \Omega^1$.)
- An automorphism of $\mathcal{B}$ is a Krein unitary $U$ which commutes with $\chi$ and $J$ and stabilizes $\mathcal{A}$ and $\Omega^1$.

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Let $\mathcal{B}(M)$ be the canonical background of a parallelizable semi-riemannian manifold. Then the automorphism group of $\mathcal{B}(M)$ is:

$$\mathrm{Aut}(\mathcal{B}(M)) = \mathrm{Diff}(M) \ltimes \mathrm{Spin}(p,q) \text{ for all } p,q$$

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The regular Dirac operators on $\mathcal{B}(M)$ are

$$D = S_r D(r \cdot e_0) S_r^{-1} + \zeta$$

- $r$ is a field of invertible matrices: acts on tetrads $e \mapsto r \cdot e$,
- $S_r \in \mathrm{End}(\mathcal{K})$ is defined by $\Psi \mapsto |\det r|^{-1/2}\Psi$,
- $D(r \cdot e_0)$ = canonical Dirac operator defined by $r \cdot e_0$,
- $(\zeta\Psi)_x = \zeta_x \Psi_x$, s.t. $\zeta_x^\times = \zeta_x$, $[\zeta, J] = \{\zeta, \chi\} = 0$.

$\Rightarrow$ There are additional *centralizing fields*

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Separately invariant under automorphisms $\Rightarrow$ can be removed.

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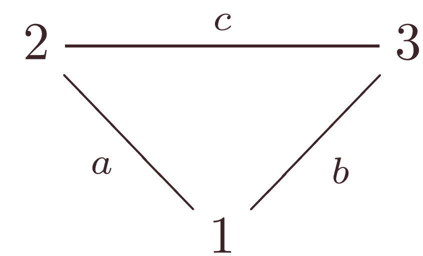
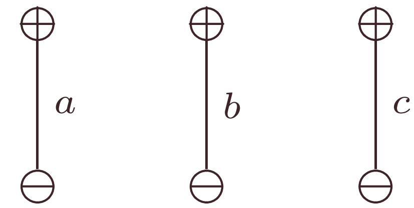
# The case of a finite graph

$G = (V, E)$ finite graph with weight function $\delta : E \to \mathbb{R}_+^*$



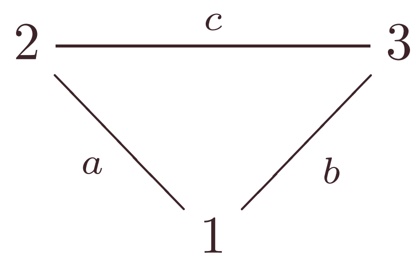

Connes' distance reproduces the geodesic distance with the "split graph" ST:

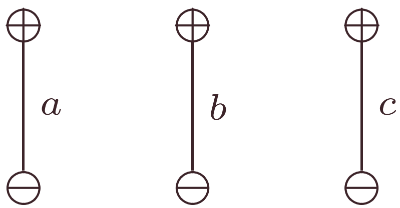
- $\mathcal{A} = \mathbb{R}^V$, $\tilde{E} := E \times \{-;+\}$, $H = L^2(\tilde{E}) = \mathbb{C}^E \otimes \mathbb{C}^2$ + canonical $\langle .,. \rangle$.
- $\pi(a)F(e,\pm) = a(e^\pm)F(e,\pm) = \bigoplus_{e\in E} \begin{pmatrix} a(e^-) & 0 \\ 0 & a(e^+) \end{pmatrix}$.
- $D = \bigoplus_{e\in E} \frac{1}{\delta_e} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\Omega^1 = \Omega^1_D = \{\bigoplus_e \begin{pmatrix} 0 & \omega_e^+ \\ \omega_e^- & 0 \end{pmatrix}\}$
- $\chi = \bigoplus_{e\in E} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $J = c.c.$, KO dim 0

# The case of a finite graph



$G = (V, E)$ finite graph with weight function $\delta : E \to \mathbb{R}_+^*$




Connes' distance reproduces the geodesic distance with the "split graph" ST:

- $\mathcal{A} = \mathbb{R}^V$, $\tilde{E} := E \times \{-;+\}$, $H = L^2(\tilde{E}) = \mathbb{C}^E \otimes \mathbb{C}^2$ + canonical $\langle .,. \rangle$.
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- $\chi = \bigoplus_{e\in E} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $J = c.c.$, KO dim 0

$T(G) = (\mathcal{A}, H, \pi, \chi, J)$, $B(G) = (T(G), \Omega^1)$, $S(G) = (T(G), D)$.

1. $\mathrm{Hor}(T(G)) = \mathrm{Perm}_d(V)$, + large config space,
2. $\mathrm{Hor}(B(G)) = \mathrm{Aut}(G)$, config space $\simeq \{w : E \to \mathbb{R} \cup \{\infty\}\}$,
3. $\mathrm{Hor}(S(G)) = \mathrm{Isom}(G)$, no config space.

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$\mathcal{B}_{SM} = \mathcal{B}(M)\hat{\otimes}\mathcal{B}_F$ where $\mathcal{B}_F$ is the same as the SM IST except $D_F$ is replaced with $\Omega^1_F = \{ \begin{pmatrix} 0 & -Y_0^\dagger \tilde{q}_1 & 0 & 0 \\ \tilde{q}_2 Y_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, q_1, q_2 \in \mathbb{H}\}$.

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For $T \in \mathrm{End}(\mathcal{K})$ define $T^o = JT^\times J^{-1}$. Then for $u \in U(\mathcal{A})$, let $\Upsilon(u) = uJuJ^{-1} = u(u^{-1})^o$.

$\Upsilon$ is a group homomorphism from $U(\mathcal{A})$ into $\mathrm{Aut}(\mathcal{B}_M)$.

$\Upsilon(U(\mathcal{A})) =$ group of local gauge transf. $M \to U(1) \times SU(2) \times U(3)$.

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**Th:** If $\pi_1(M) = \{1\}$, $Y_0$ is invertible and $Y_\nu, Y_e$ (resp. $Y_u, Y_d$) have no common eigenvector, then $\mathrm{Aut}(\mathcal{B}_{SM})$ is generated by

1. diffeo-spino-morphisms $U_\theta \otimes 1$, $U_\Sigma \otimes 1$ coming from the base manifold,

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$\Rightarrow$ AB formalism applied to SM is yelling at us that we need another $U(1)$-gauge field !

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Just replace $\mathcal{A}_F$ by $\mathcal{A}_F^{\text{ext}} = \mathbb{C} \oplus \mathcal{A}_F = \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$, with

$$\pi_F^{\text{ext}}(\lambda, \mu, q, m) = [\tilde{q}_\lambda, \tilde{q}, \mu 1_2 \oplus 1_2 \otimes m, \mu 1_2 \oplus 1_2 \otimes m] \otimes 1_3$$

# The $U(1)_{B-L}$-extension


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and $\Omega_F^1$ by

$$(\Omega_F^1)^{\text{ext}} \ni \begin{pmatrix} 0 & Y_0^\dagger \tilde{q}_1 & z_1 p_\nu \otimes M_0^\dagger & 0 \\ \tilde{q}_2 Y_0 & 0 & 0 & 0 \\ z_2 p_\nu \otimes M_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, z_1, z_2 \in \mathbb{C}, q_1, q_2 \in \mathbb{H}$$

- Only satisfies *weak* order $1$ condition.

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- Only satisfies *weak* order $1$ condition.
- The compatible finite Dirac are $\Phi(q) + \Phi(q)^o + \sigma(zM_0)$.

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- Its configuration space contains: SM fields $+$ anomalous $X + Z'_{B-L} +$ 1 complex scalar $\sigma(zM_0)$, $+$ flavour changing $\zeta_{\text{other}}$.
- When $\zeta_{\text{other}}$ are thrown away and gravity is frozen, all fields are now fluctuations.

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- When $\zeta_{\text{other}}$ are thrown away and gravity is frozen, all fields are now fluctuations.

$\Rightarrow$ The Connes-Lott action can be used on this model.

$\Rightarrow$ It is compatible with the experimental $m_{\text{Higgs}}$ !

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# Solving unimodularity

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Def: A Jordan algebra is a real commutative algebra $(A, \circ)$ such that

$$\forall a, b \in A,\ (a^2 \circ b) \circ a = a^2 \circ (b \circ a) \tag{1}$$

Ex: an associative algebra equipped with the product

$$a \circ b = \frac{1}{2}(ab + ba) \tag{2}$$

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A Jordan algebra is *special* if isomorphic to one this kind.

Def: A JB algebra is a normed Jordan algebra $A$ which is complete in the norm and s.t.

1. $\|a \circ b\| \leq \|a\|\|b\|$,
2. $\|a^2\| = \|a\|^2$,
3. $\|a^2\| \leq \|a^2 + b^2\|$.

Ex: selfadjoint part of $C^*$-algebra (special).

- JB algebras admit a continuous functional calculus.
- A (unital) JB algebra is associative iff it is $\approx \mathcal{C}(X, \mathbb{R})$, $X$ (compact) Hausdorff.

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**Th:** Every finite-dimensional JB-algebra is a direct sum of ones on this list:

1. $H_n(\mathbb{K})$ for $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$,
2. $\mathrm{JSpin}(n)$,
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An *inner derivation* of $A$ is a Jordan derivation of the form

$$\delta = \sum_i [L_{a_i}, L_{b_i}].$$

The Lie algebra of inner derivation will be denoted by $\mathrm{Der}_{Inn}(A)$.

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If $J = H_n(\mathbb{R}), H_n(\mathbb{C}), H_n(\mathbb{H})$ or $\mathrm{JSpin(n)}$, then $\mathrm{Der_{Inn}}(J) = so(n), su(n), sp(n)$ or $so(n)$ respectively.

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Thanks to the magical formula $[L_x, L_y] = \frac{1}{4} ad_{[x,y]}$ one can prove:

**Th:** Let $A = \mathcal{C}(M, A_F)$ with $A_F$ finite-dimensional and $M$ is a Hausdorff space. Then $ad : [\pi(A), \pi(A)] \to \mathrm{Der}_{\mathrm{Inn}}(\pi(A))$ is an isomorphism.

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Def: A (real, even) *special Jordan background* is a gadget $\mathcal{B} = (A, H, \pi, \Omega^1, \chi, J)$ such that

1. $A$ is a special Jordan algebra,
2. $H$ is a Hilbert space,
3. $\pi$ is a faithful associative representation of $A$,
4. $\overline{\pi(A)}$ is a JB algebra,
5. $J$ and $\chi$ as usual,
6. $\Omega^1$ is an odd $\pi(A)$-module for the Jordan multiplication.

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$\rightarrow$ there is a well-defined theory of $1$-forms & fluctuations

$\rightarrow$ the fluctuation space is gauge-invariant under milder assumption than in the associative case

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Def: A (real, even) *special Jordan background* is a gadget $\mathcal{B} = (A, H, \pi, \Omega^1, \chi, J)$ such that

1. $A$ is a special Jordan algebra,
2. $H$ is a Hilbert space,
3. $\pi$ is a faithful associative representation of $A$,
4. $\overline{\pi(A)}$ is a JB algebra,
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$\to$ there is a well-defined theory of $1$-forms & fluctuations

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$\to$ gauge + Higgs fields decomposition works for almost-associative Jordan backgrounds

$\to$ particle models are naturally unimodular

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$\rightarrow$ the fluctuation space is gauge-invariant under milder assumption than in the associative case

$\rightarrow$ gauge + Higgs fields decomposition works for almost-associative Jordan backgrounds

$\rightarrow$ particle models are naturally unimodular

$\rightarrow$ no chiral $U(1)$-gauge group (but one can have Pati-Salam)

$\rightarrow$ only in Euclidean signature for the moment

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Symmetries of known physics: $\mathrm{Diff}(M) \ltimes \mathcal{G}$, where $\mathcal{G}$=gauge group

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Symmetries of known physics: $\mathrm{Diff}(M) \ltimes \mathcal{G}$, where $\mathcal{G}$=gauge group
key insight (Connes):
$\mathrm{Aut}(\mathcal{C}(M, A_F)) = \mathrm{Out} \ltimes \mathrm{Inn} = \mathrm{Diff}(M) \ltimes \Gamma(\mathrm{Aut}(A_F))$

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key insight (Connes):

$\mathrm{Aut}(\mathcal{C}(M, A_F)) = \mathrm{Out} \ltimes \mathrm{Inn} = \mathrm{Diff}(M) \ltimes \Gamma(\mathrm{Aut}(A_F))$

Let $\mathcal{B}$ be an AB with algebra $A = \mathcal{C}(M, A_F)$.

If $A_F$ is associative:

$$\begin{tikzcd} Aut_{\mathrm{Inn}}(A) & U(A) \arrow[l, "\mathrm{Ad}"'] \arrow[r, "\Upsilon"] & Aut(\mathcal{B}) \\ Der_{\mathrm{Inn}}(A) \arrow[u, "\exp"] & \mathrm{Skew}(A) \arrow[l, "\mathrm{ad}"'] \arrow[r, "d_e\Upsilon"] \arrow[u, "\exp"] & \mathcal{G}(A) \arrow[u, "\exp"] \end{tikzcd} \tag{3}$$

If $A_F$ is Jordan:

$$\begin{tikzcd} Aut_{\mathrm{Inn}}(\pi(A)) \arrow[r, squiggly] & U(A) \arrow[r, "\Upsilon"] & Aut(\mathcal{B}) \\ Der_{\mathrm{Inn}}(\pi(A)) \arrow[u, "\exp"] \arrow[r, "\approx"] & {[\pi(A), \pi(A)]} \arrow[r, "d_e\Upsilon"] \arrow[u, "\exp"] & \mathcal{G}(A) \arrow[u, "\exp"] \end{tikzcd} \tag{4}$$

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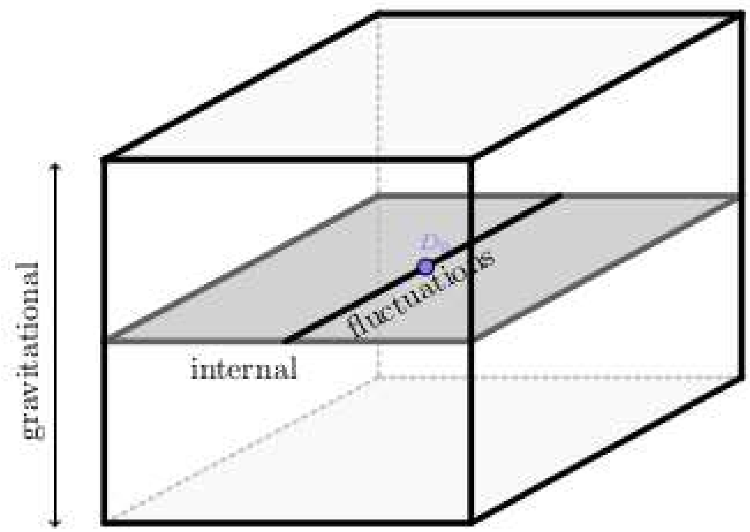
- The 3 main tools of NCG: $D$, AC algebras, Spectral action provide a unified understanding of all the bosonic fields.
- But it faced a number of technical and conceptual problems.
- SA + Jordan background solves all known problems (but has the wrong signature. . . )
- CL + associative background of Lorentzian B-L extended SM more limited in scope but compatible with known physics (except for unimodularity. . . ).




Locks to break:

1. Adapt the Jordan backgrounds to Lorentz signature (seems feasible).
2. Define the SA in the Lorentz signature: still the major challenge despite recent progress.
3. Possible worry even if this is adressed: all Higgs tend to have the same mass with the SA.

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Complete bosonic action:

$$\begin{aligned}\mathcal{L}_b &= -160\frac{N}{3}\mathbb{F}^Y_{\mu\nu}\mathbb{F}^{Y\mu\nu} - 32N\mathbb{F}^W_{\mu\nu a}\mathbb{F}^{W\mu\nu a} - 32N\mathbb{F}^C_{\mu\nu a}\mathbb{F}^{C\mu\nu a} \\ &\quad -\frac{64}{3}NF^{Z'}_{\mu\nu}F^{Z'\mu\nu} - \frac{128}{3}N\mathbb{F}^Y_{\mu\nu}F^{Z'\mu\nu} + 16a|D_\mu H|^2 - 8bs|D_\mu z|^2 \\ &\quad -8V_0(|H|^2-1)^2 - 8W_0(|z|^2-1)^2 + 16sK(|H|^2-1)(|z|^2-1)\end{aligned}$$

Normalization of kinetic terms: $\mathbb{B}^Y_\mu = \frac{1}{2}g_Y Y_\mu$, $\mathbb{B}^{Wa}_\mu = \frac{1}{2}g_w W^a_\mu$, $\mathbb{B}^{Ca}_\mu = \frac{1}{2}g_s G^a_\mu$, $Z'_\mu = \frac{1}{2}g_{Z'}\hat{Z}'_\mu$, $H = k\tilde{H}$, $z = l\tilde{z}$, with

$$g_w^2 = g_s^2 = \frac{5}{3}g_Y^2 = \frac{2}{3}g_{Z'}^2 = \frac{1}{32N}, \quad \kappa = 64\tfrac{N}{3}g_Y g_{Z'} = \sqrt{\tfrac{2}{5}}$$

$$k^2 = \frac{1}{16a}, \qquad l^2 = \tfrac{1}{8b}$$

$$\begin{aligned}M_W^2 &= \frac{1}{k^2}g_w^2 \\ &= \frac{1}{4}\frac{1}{32N}32\mathrm{Tr}(Y_eY_e^\dagger + Y_\nu Y_\nu^\dagger + 3M_u + 3M_d) \\ &= \frac{1}{4N}\sum \text{squared masses of fermions}\end{aligned}$$

In particular for $N = 3$, one obtains $M_{\mathrm{top}} \leq 2M_W$.

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An automorphism of the canonical ST not coming from a diffeo-spinomorphism is possible as soon as dim $\geq 6$.
Example: multiplication by $\sinh t\gamma_1\gamma_2 + \cosh t\gamma_3 \ldots \gamma_6$.
Many automorphisms of the finite SM triple are not AB automorphism (Krein-unitary commuting with $J$ and $\chi$ but not stabilizing $\Omega^1_F$)
Ex: $U = [A, B, A^*, B^*]$ with arbitrary unitary matrices $A, B$. (need not be block-diagonal, other examples exist)