An Introduction to Causal Fermion Systems and the Causal Action Principle

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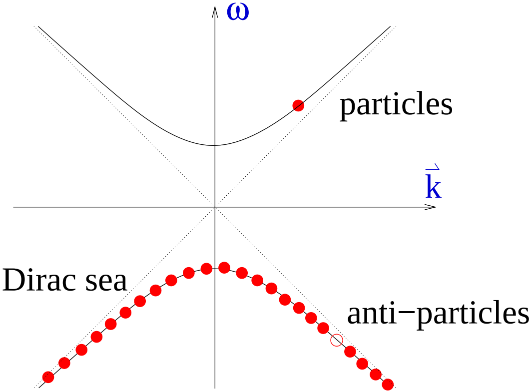
What is a causal fermion system?

- approach to fundamental physics
- novel mathematical model of spacetime
- physical equations are formulated in generalized spacetimes
- Different limiting cases:
  - Continuum limit: Quantized fermionic fields interacting via classical bosonic fields
  - QFT limit: fermionic and bosonic quantum fields (ongoing, more towards the end of the talk)

How causal fermion systems developed (≈ 1989-90)

starting point: Course on relativistic QM and QFT
(following Bjorken-Drell / Itzykson-Zuber)

- Dirac's hole theory (Dirac 1932)



How causal fermion systems developed (≈ 1989-90)

- Problems of the naive Dirac sea picture:
  - infinite charge density
  - infinite negative energy density
- Therefore, we were told in lecture:
  - Dirac sea is not visible due to symmetries (homogeneous, isotropic)
  - Only “deviations” of the sea are observed as particles and anti-particles
  - Forget about the Dirac sea, no longer needed.
- This procedure is implemented in the formalism:
  - Reinterpretation of creation as annihilation operators
  - Wick ordering of field operators in Hamiltonian

I was not convinced by this procedure:

- The interacting Dirac sea should be visible,
  for example in presence of external potential

$$(i\partial\!\!\!/ + A\!\!\!/(x) - m)\psi = 0$$

- Pair creation seems an evidence that the Dirac sea is real.

How causal fermion systems developed ($\approx$ 1989-90)

What is the way out?

- Take all the sea states into account.
- In order to avoid the problems of naive Dirac sea, formulate new type of equations, different structure of the physical equations

Goal in general terms:

Formulate a variational principle
directly for the family of wave functions

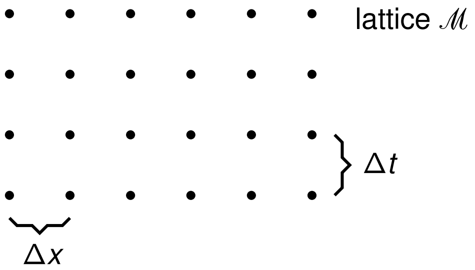
- Intuitive picture: wave functions "organize themselves" in such a way that the Dirac sea configuration is a minimizer.
- In interacting situation the wave functions organize to solutions of the Dirac equation

$$(i\gamma^j \partial_j + e\gamma^j A(x) - m)\psi = 0$$

This should serve as the definition of A.

Formulate a variational principle
directly for a family of wave functions

- For simplicity begin with a discrete spacetime, for example 2d-lattice



- Do not make use of nearest neighbor relation and lattice spacing.
  Better and simpler: spacetime $\mathscr{M}$ is a discrete set of points.

Motivating example: A discrete spacetime

- Consider wave functions $\psi_1, \ldots, \psi_f : \mathscr{M} \to \mathbb{C}$ (with $f < \infty$)
- Introduce scalar product; orthonormalize,

$$\langle \psi_k | \psi_l \rangle = \delta_{kl} \,,$$

gives f-dim Hilbert space $(\mathcal{H}, \langle .|. \rangle_{\mathcal{H}})$.

important object: for any lattice point (t, x) introduce

local correlation operator $F(t,x) : \mathcal{H} \to \mathcal{H}$

- define matrix elements by

$$(F(t,x))^j_k = \overline{\psi_j(t,x)} \psi_k(t,x)$$

basis invariant:

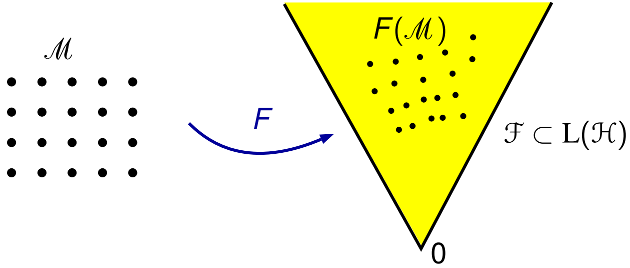
$$\langle \psi, F(t,x)\, \phi \rangle_{\mathcal{H}} = \overline{\psi(t,x)} \phi(t,x) \qquad \text{for all } \psi, \phi \in \mathcal{H}$$

- Hermitian matrix
- Has rank at most one, is positive semi-definite

$$F(t,x) = e^* e \qquad \text{with} \qquad e : \mathcal{H} \to \mathbb{C} \,, \quad \psi \mapsto \psi(x)$$

Motivating example: A discrete spacetime

$$\mathcal{F} := \{ F \text{ Hermitian, rank one, positive semi-definite} \}$$



general idea:

- disregard objects on the left
  (nearest neighbors, lattice spacing)
- work instead with the objects on the right
  (only local correlation operators)

Motivating example: A discrete spacetime

How to set up equations in this setting?
Explain idea in simple example:

- local correlation operators $F_1, \ldots, F_f \in \mathcal{F}$
- product $F_i \, F_j$ tells about correlation of wave functions at different space-time points
- $\mathrm{Tr}(F_i F_j)$ is real number
- minimize

$$\mathcal{S} = \sum_{i,j=1}^{f} \mathrm{Tr}(F_i F_j)^2$$

under suitable constraints.

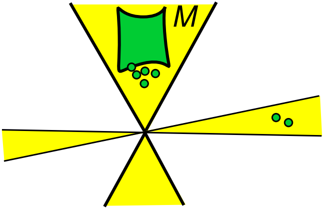
Definition (Causal fermion system)

Let $(\mathcal{H}, \langle .|. \rangle_{\mathcal{H}})$ be Hilbert space
Given parameter $n \in \mathbb{N}$ ("spin dimension")

$$\mathcal{F} := \Big\{ x \in \mathrm{L}(\mathcal{H}) \text{ with the properties:}$$

- x is self-adjoint and has finite rank
- x has at most n positive and at most n negative eigenvalues $\Big\}$

ρ a measure on $\mathcal{F}$



Causal action principle

Let $x, y \in \mathcal{F}$. Then x and y are linear operators.

$x \cdot y \in \mathrm{L}(H)$:

- rank $\leq 2n$
- in general not self-adjoint: $(x \cdot y)^* = y \cdot x \neq x \cdot y$

thus non-trivial complex eigenvalues $\lambda_1^{xy}, \ldots, \lambda_{2n}^{xy}$

Causal action principle

Nontrivial eigenvalues of xy: $\lambda_1^{xy}, \ldots, \lambda_{2n}^{xy} \in \mathbb{C}$

Lagrangian $\qquad \mathcal{L}[A_{xy}] = \frac{1}{4n} \sum_{i,j=1}^{2n} \left(|\lambda_i^{xy}| - |\lambda_j^{xy}| \right)^2 \;\geq\; 0$

action $\qquad \mathcal{S} = \iint_{\mathcal{F}\times\mathcal{F}} \mathcal{L}[A_{xy}]\, d\rho(x)\, d\rho(y) \;\in\; [0, \infty]$

Minimize $\mathcal{S}$ under variations of ρ, with constraints

volume constraint: $\qquad \rho(\mathcal{F}) = \text{const}$

trace constraint: $\qquad \int_{\mathcal{F}} \text{tr}(x)\, d\rho(x) = \text{const}$

boundedness constraint: $\qquad \iint_{\mathcal{F}\times\mathcal{F}} \sum_{i=1}^{2n} |\lambda_i^{xy}|^2\, d\rho(x)\, d\rho(y) \leq C$

- F.F., "Causal variational principles on measure spaces," *J. Reine Angew. Math.* **646** (2010) 141–194

Example: Dirac spinors in Lorentzian space-time

Let $(\mathscr{M}, g)$ be a Lorentzian space-time,
for simplicity 4-dimensional, globally hyperbolic,
then automatically spin,

$$(S\mathscr{M}, \prec .|. \succ) \qquad \text{spinor bundle}$$

- $S_p\mathscr{M} \simeq \mathbb{C}^4$
- spin scalar product

$$\prec .|. \succ_p \;:\; S_p\mathscr{M} \times S_p\mathscr{M} \to \mathbb{C}$$

is indefinite of signature (2,2)

$$(\mathcal{D} - m)\psi_m = 0 \qquad \text{Dirac equation}$$

Example: Dirac spinors in Lorentzian space-time

- Cauchy problem well-posed, global smooth solutions (for example symmetric hyperbolic systems)
- finite propagation speed

$C^\infty_{\text{sc}}(\mathcal{M}, S\mathcal{M})$ spatially compact solutions

$$(\psi_m | \phi_m)_m := 2\pi \int_{\mathcal{N}} \prec \psi_m | \nu\!\!\!/ \, \phi_m \succ_x \, d\mu_{\mathcal{N}}(x) \quad \text{scalar product}$$

completion gives Hilbert space $(\mathcal{H}_m, (.|.)_m)$

Example: Dirac spinors in Lorentzian space-time

- Choose $\mathcal{H}$ as a subspace of the solution space,

$$\mathcal{H} = \overline{\text{span}(\psi_1, \ldots, \psi_f)}$$

- To $x \in \mathbb{R}^4$ associate a local correlation operator

$$\langle \psi | F(x) \phi \rangle = -\prec \psi(x) | \phi(x) \succ_x \qquad \forall \psi, \phi \in \mathcal{H}$$

  Is self-adjoint, rank ≤ 4
  at most two positive and at most two negative eigenvalues
- Here ultraviolet regularization may be necessary:

$$\langle \psi | F(x) \phi \rangle = -\prec (\mathfrak{R}_\varepsilon \psi)(x) | (\mathfrak{R}_\varepsilon \phi)(x) \succ_x \qquad \forall \psi, \phi \in \mathcal{H}$$

$$\mathfrak{R}_\varepsilon \;:\; \mathcal{H} \to C^0(\mathscr{M}, S\mathscr{M}) \qquad \text{regularization operators}$$

$$\varepsilon > 0 : \text{regularization scale (Planck length)}$$

Example: Dirac spinors in Lorentzian space-time

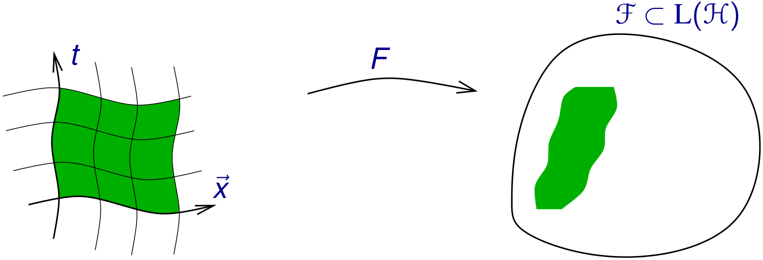
- Thus $F(x) \in \mathcal{F}$ where

$$\mathcal{F} := \Big\{ F \in \mathrm{L}(\mathcal{H}) \text{ with the properties:}$$

  - ▷ F is self-adjoint and has rank ≤ 4
  - ▷ F has at most 2 positive and at most 2 negative eigenvalues $\Big\}$

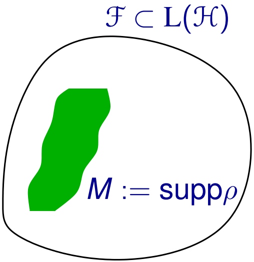
Example: Dirac spinors in Lorentzian space-time

We obtain mapping $\quad x \mapsto F(x) \in \mathcal{F} \subset \mathrm{L}(\mathcal{H})$



Take push-forward measure

$$\rho := F_*(\mu_{\mathcal{M}}) \qquad \text{(i.e. } \rho(\Omega) := \mu_{\mathcal{M}}\big(F^{-1}(\Omega)\big)\text{)}$$



We thus obtain a causal fermion system of spin dimension two.

A few general remarks

One basic object: measure ρ on set $\mathcal{F}$ of linear operators on $\mathcal{H}$, describes spacetime as well as all objects therein

- Underlying structure: family of fermionic wave functions
- Geometric structures encoded in these wave functions

Matter encodes geometry
Quantum spacetime

- Causal action principle describes spacetime as a whole (similar to Einstein-Hilbert action in GR)
- Causal action principle is a nonlinear variational principle (similar to Einstein-Hilbert action or classical field theory)
- Linear dynamics of quantum theory recovered in limiting case (more details later)

Interpretation in terms of spacetime events

- operators in $\mathcal{F}$ can be interpreted as "possible local correlation operators"
  or simply as possible events
- operators in M are the events realized in spacetime
- spacetime is made up of all the realized events
- the physical equations relate the events to each other

For details on this connection:

- F.F, J. Fröhlich, C. Paganini, C. and M. Oppio, "Causal fermion systems and the ETH approach to quantum theory," arXiv:2004.11785 [math-ph] (2020)

Inherent structures of a causal fermion system

Let $(\rho, \mathcal{F}, \mathcal{H})$ be a causal fermion of spin dimension n, spacetime $M := \text{supp}\rho$.

spacetime points are linear operators on $\mathcal{H}$

- For $x \in M$, consider eigenspaces of x.
- For $x, y \in M$,
  - consider operator products xy
  - project eigenspaces of x to eigenspaces of y

Gives rise to:

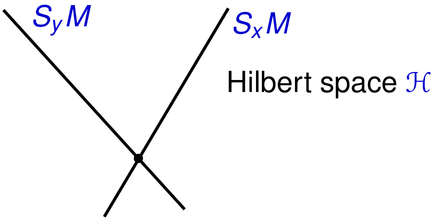
- quantum objects (spinors, wave functions)
- geometric structures (connection, curvature)
- causal structure, analytic structures

- Spinors

$$S_xM := x(\mathcal{H}) \subset \mathcal{H} \quad \text{“spin space”}, \; \dim S_xM \leq 2n$$

$$\prec u|v \succ_x := -\langle u \,|\, x\, v\rangle_{\mathcal{H}} \quad \text{“spin scalar product”},$$

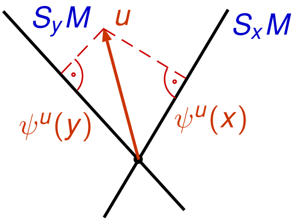
inner product of signature $(\leq n, \leq n)$



- Physical wave functions

$$\psi^u(x) = \pi_x\, u \;\text{ with } u \in \mathcal{H} \qquad \text{physical wave function}$$

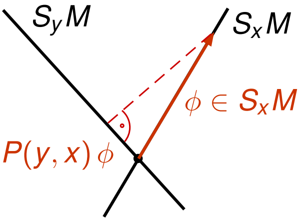
$$\pi_x : \mathcal{H} \to \mathcal{H} \qquad \text{orthogonal projection on } x(\mathcal{H})$$



Inherent structures in spacetime

- The kernel of the fermionic projector:

$$P(y,x) = \pi_y \, x|_{S_x M} \;:\; S_x M \to S_y M$$



$$P(y,x) = -\sum_{i=1}^{f} |\psi^{e_i}(y) \succ \prec \psi^{e_i}(x)| \quad \text{where } (e_i) \text{ ONB of } \mathcal{H}$$

Inherent structures in spacetime

- Geometric structures
  - $P(x,y) \;:\; S_yM \to S_xM$ yields relations between spin spaces.
    Using a polar decomposition $(\ldots,\ldots)$ one gets:
    $$D_{x,y} \;:\; S_yM \to S_xM \text{ unitary} \qquad \text{“spin connection”}$$
  - tangent space T_x, carries Lorentzian metric,
    $$\nabla_{x,y} \;:\; T_y \to T_x \qquad \text{corresponding “metric connection”}$$
  - holonomy of connection gives curvature
    $$R(x,y,z) = \nabla_{x,y}\,\nabla_{y,z}\,\nabla_{z,x} \;:\; T_x \to T_x$$

Causal structure

Let $x, y \in M$. Then

$x \cdot y \in \mathrm{L}(H)$ has non-trivial complex eigenvalues $\lambda_1^{xy}, \ldots, \lambda_{2n}^{xy}$

Definition (causal structure)

The points $x, y \in \mathcal{F}$ are called

$$\begin{cases} \text{spacelike separated} & \text{if } |\lambda_j^{xy}| = |\lambda_k^{xy}| \text{ for all } j, k = 1, \ldots, 2n \\ \text{timelike separated} & \text{if } \lambda_1^{xy}, \ldots, \lambda_{2n}^{xy} \text{ are all real} \\ & \text{and } |\lambda_j^{xy}| \neq |\lambda_k^{xy}| \text{ for some } j, k \\ \text{lightlike separated} & \text{otherwise} \end{cases}$$

- Lagrangian is compatible with causal structure:

$$\text{Lagrangian} \quad \mathcal{L}(x, y) = \frac{1}{4n} \sum_{i,j=1}^{2n} \left(|\lambda_i^{xy}| - |\lambda_j^{xy}| \right)^2 \geq 0$$

thus x, y spacelike separated $\quad \Rightarrow \quad \mathcal{L}(x, y) = 0$

"points with spacelike separation do not interact"

Underlying physical principles

- local gauge principle:
  freedom to perform local unitary transformations of the spinors
- Pauli exclusion principle:
  Choose orthonormal basis $\psi_1, \ldots, \psi_f$ of $\mathcal{H}$. Set

$$\Psi = \psi_1 \wedge \cdots \wedge \psi_f \, ,$$

  gives equivalent description by Hartree-Fock state.
- the "equivalence principle":
  symmetry under "diffeomorphisms" of M
  (note: M merely is a topological measure space)

Spacetime and causal structure are emergent

The continuum limit

Causal fermion system

- abstract mathematical framework
- quantum geometry, causal action

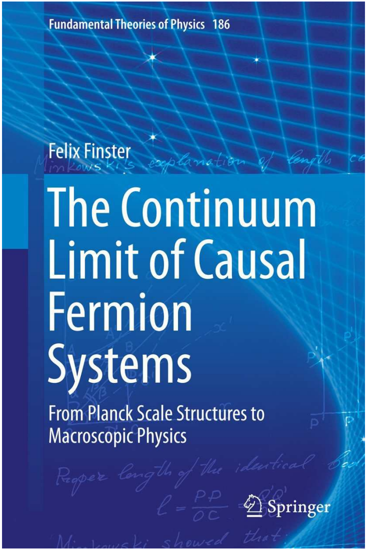
$$\Big\Downarrow \quad \text{continuum limit}$$

description in the continuum limit

- Dirac fields
- strong and electroweak gauge fields
- gravitational field

- fermion field: second-quantized
- bosonic field: classical

The continuum limit



Fundamental Theories of Physics **186**
Springer, 2016
548+xi pages

arXiv:1605.04742 [math-ph]

The causal action principle in the continuum limit

- specify vacuum as sum of Dirac seas,

$$P(x,y) = \sum_{\beta=1}^{g} P^{\text{sea}}_{m_\beta}(x,y)$$

$$P^{\text{sea}}_{m_\beta}(x,y) = \int \frac{d^4k}{(2\pi)^4}\, (\slashed{k} + m_\beta)\, \delta(k^2 - m_\beta^2)\, \Theta(-k^0)\, e^{-ik(x-y)}$$

β labels “generations” of elementary particles

$\Longrightarrow$ Dynamical equations only if three generations ($g = 3$)

The causal action principle in the continuum limit

- Model involving neutrinos and quarks:

$$P(x,y) = \sum_{\beta=1}^{3} \underbrace{P^{\text{sea}}_{m_\beta}(x,y) \oplus \cdots \oplus P^{\text{sea}}_{m_\beta}(x,y)}_{\text{7 identical direct summands}} \oplus P^{\text{sea}}_{\tilde{m}_\beta}(x,y)$$

  - again three generations
  - $4 \times 8 = 32$-component wave functions
  - spin dimension 16
- Regularize on the scale ε (Planck scale), regularization of neutrinos breaks chiral symmetry

The causal action principle in the continuum limit

Remarks on methods for analyzing the continuum limit:

- Consider the Dirac equation in an external potential

$$(i\partial\!\!\!/ + \mathcal{B} - mY)\psi = 0\,.$$

- Question: Are the EL equations of causal action principle satisfied in the limit $\varepsilon \searrow 0$?
- Answer: Yes if and only if $\mathcal{B}$ has a certain structure and satisfies the classical field equations.

Going beyond the continuum limit

Basic question: What about objects in space (densities, probabilities, etc.)

- The scalar product $\langle .|. \rangle_{\mathcal{H}}$ is given abstractly
- A-priori missing: Representation as spatial integral.
  Is there an analog of the relation

$$\langle \psi | \phi \rangle_{\mathcal{H}} = \int_{\mathcal{N}} \prec \psi | \phi \succ_x \, d\mu_{\mathcal{N}}(x)$$

  for Dirac spinors?
- Related question: What about current conservation? Are there conservation laws?
- Ultimately: What is the quantum state?
  What are quantum probabilities?

General ideas for the construction of a quantum state

- F.F. and Kamran, N., "Fermionic Fock spaces and quantum states for causal fermion systems,"
  arXiv:2101.10793 [math-ph], *Ann. Henri Poincaré* **23** (2022) 1359–1398

General setting:

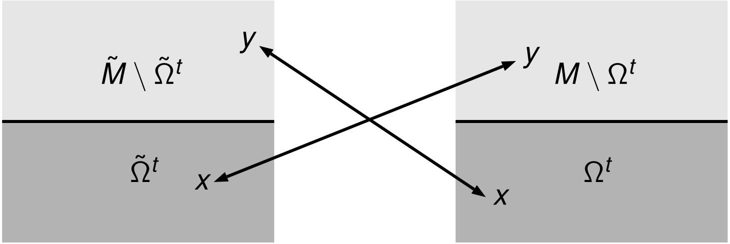
- Two minimizing causal fermion systems
  - $(\mathcal{H}, \mathcal{F}, \rho)$ describing vacuum
  - $(\tilde{\mathcal{H}}, \tilde{\mathcal{F}}, \tilde{\rho})$ describing the interacting spacetime
  - corresponding spacetimes:

$$M := \operatorname{supp} \rho \,, \quad \tilde{M} := \operatorname{supp} \tilde{\rho}$$

- Goal: Compare $\tilde{\rho}$ and ρ at time t.

- Basic object: Nonlinear surface layer integral
  - identify Hilbert spaces by choosing $V : \mathcal{H} \to \tilde{\mathcal{H}}$ unitary

$$\begin{aligned}\gamma^{\tilde{\Omega},\Omega}(\tilde{\rho},\rho) := \int_{\tilde{\Omega}} d\tilde{\rho}(x) \int_{M\setminus\Omega} d\rho(y)\, \mathcal{L}(x,y) \\ - \int_{\tilde{M}\setminus\tilde{\Omega}} d\tilde{\rho}(x) \int_{\Omega} d\rho(y)\, \mathcal{L}(x,y)\end{aligned}$$



Freedom in identifying the Hilbert spaces

- identification of Hilbert spaces:
  - Choose $V : \mathcal{H} \to \tilde{\mathcal{H}}$ unitary
  - Work exclusively in $\mathcal{H}$
  - But: identification is not canonical, gives freedom

$$\rho \to \mathcal{U}\rho\,, \quad (\mathcal{U}\rho)(\Omega) := \rho(\mathcal{U}^{-1}\Omega\mathcal{U})$$

- This freedom is treated by integrating over $\mathcal{U}$
  - Let $\mathcal{G} \subset \mathrm{U}(\mathcal{H})$ be compact subgroup
  - $\mu_{\mathcal{G}}$ normalized Haar measure on $\mathcal{G}$

The partition function

- partition function

$$Z(\beta, \tilde{\rho}) = \fint_{\mathcal{G}} \exp\Big(\beta \, \gamma^{\tilde{\Omega}, \Omega}(\tilde{\rho}, \mathcal{U}\rho) \Big) \, d\mu_{\mathcal{G}}(\mathcal{U})$$

where β free parameter (maybe discuss at the end)

How to “test” the interacting spacetime?

- Interacting spacetime can be arbitrarily complicated (interacting quantum fields, entanglement, collapse)
- describe by objects in the vacuum spacetime: free fields, wave functions, ...
- use insertions:

$$\frac{1}{Z^t} \int_{\mathcal{G}} (\cdots) \exp\Big(\beta\, \gamma^t(\tilde{\rho}, \mathcal{U}\rho)\Big)\, d\mu_{\mathcal{G}}(\mathcal{U})$$

  - formal analogy to path integral formalism
- For the insertions we need more structures of causal fermion system $(\mathcal{H}, \mathcal{F}, \rho)$:
  - linearized fields
  - conserved surface layer integrals

The Euler-Lagrange equations

For clarity of presentation: leave out trace and boundedness constraints

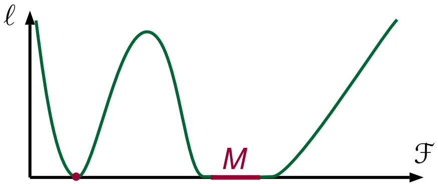
$$\ell(x) := \int_{\mathcal{F}} \mathcal{L}(x,y)\, d\rho(y) - \mathfrak{s}$$

($\mathfrak{s} > 0$ Lagrange multiplier for volume constraint)

Lemma

Let ρ be a minimizer of the causal action. Then

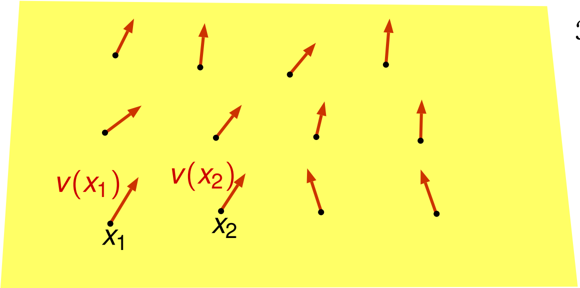
$$\ell|_M \equiv \inf_{\mathcal{F}} \ell = 0$$



Linear perturbations

To simplify presentation assume that:
ρ discrete minimizing measure describing the vacuum.

- What are linear perturbations of the measure?



$\mathcal{F} \subset \mathrm{L}(\mathcal{H})$

Also a scalar weight function $b(x)$ comes into play

- jet $\mathfrak{v} := (b, v) \in \mathfrak{J}$

Jet dynamics

The jet $\mathfrak{v} = (b, v)$ satisfies the linearized field equations

$$\begin{aligned} 0 &= \langle \mathfrak{u}, \Delta \mathfrak{v} \rangle(x) \\ &:= \nabla_{\mathfrak{u}} \bigg(\int_M \big(\nabla_{1,\mathfrak{v}} + \nabla_{2,\mathfrak{v}} \big) \mathcal{L}(x,y)\, d\rho(y) - \nabla_{\mathfrak{v}}\, \mathfrak{s} \bigg) \end{aligned}$$

for all test jets $\mathfrak{u}$, where

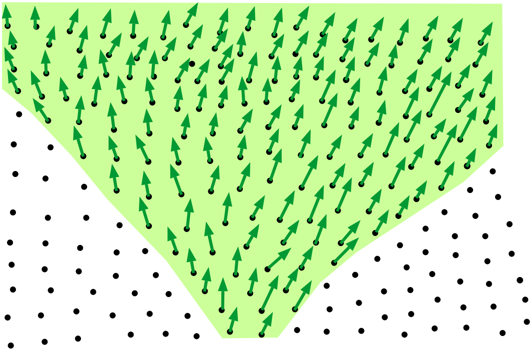
$$\nabla_{\mathfrak{v}} g(x) := a(x)\, g(x) + \big(D_v g\big)(x)$$

There are also corresponding nonlinear field equations.

- F.F., J. Kleiner, "A Hamiltonian formulation of causal variational principles," arXiv:1612.07192 [math-ph], *Calc. Var. Partial Differential Equations* **56:73** (2017)
- F.F., "Perturbation theory for critical points of causal variational principles," arXiv:1703.05059 [math-ph] (2017), *Adv. Theor. Math. Phys.* **24** (2020) 563–619

Existence, Uniqueness, Finite Propagation Speed

for linearized fields



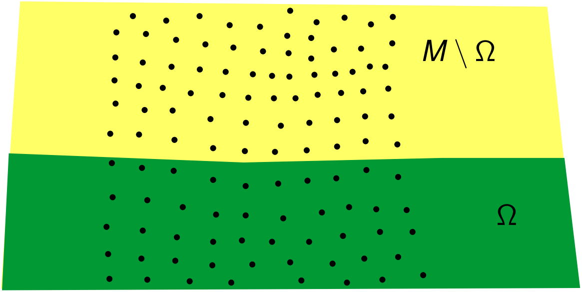
This holds "on the macroscopic scale"

- C. Dappiaggi, F.F., "The Cauchy problem and the causal structure of linearized fields for causal variational principles," arXiv:1811.10587 [math-ph], *Methods Appl. Anal.* **27** (2020) 1–56

based on energy estimates

Surface Layer Integrals

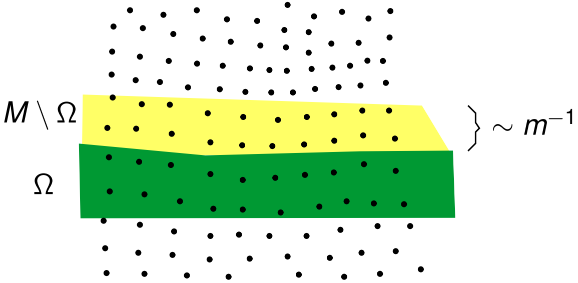
General structure of a surface layer integral:



$$\int_\Omega d\rho(x) \int_{M\setminus\Omega} d\rho(y)\, (\cdots)\mathcal{L}(x,y)$$

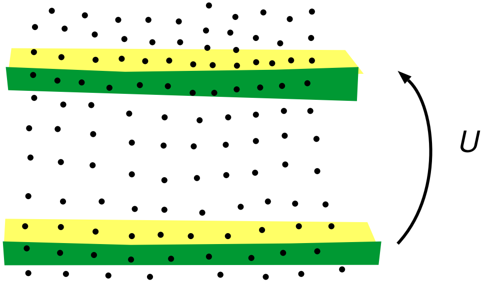
Surface Layer Integrals

Typically: $\mathcal{L}(x,y)$ very small if x and y far apart (Compton scale)



- F.F., J. Kleiner, "Noether-like theorems for causal variational principles," arXiv:1506.09076 [math-ph], *Calc. Var. Partial Differential Equations* **55:35** (2016)

Conservation laws for surface layer integrals



- F.F., J. Kleiner, “A class of conserved surface layer integrals for causal variational principles,” arXiv:1801.08715 [math-ph], *Calc. Var. Partial Differential Equations* **58:38** (2019)
- F.F., N. Kamran, “Complex structures on jet spaces and bosonic Fock space dynamics for causal variational principles,” arXiv:1808.03177 [math-ph], *Pure Appl. Math. Q.* **17** (2021) 55–140

Surface layer integrals for linearized fields

- conserved surface layer integrals:

$$\gamma^\Omega_\rho \;:\; \mathfrak{J} \to \mathbb{R} \qquad \text{(conserved one-form)}$$

$$\gamma^\Omega_\rho(\mathfrak{u}) = \int_\Omega d\rho(x) \int_{M\setminus\Omega} d\rho(y)\, \big(\nabla_{1,\mathfrak{u}} - \nabla_{2,\mathfrak{u}}\big)\mathcal{L}(x,y)$$

$$\sigma^\Omega_\rho \;:\; \mathfrak{J} \times \mathfrak{J} \to \mathbb{R} \qquad \text{(symplectic form)}$$

$$\sigma^\Omega_\rho(\mathfrak{u},\mathfrak{v}) = \int_\Omega d\rho(x) \int_{M\setminus\Omega} d\rho(y)\, \big(\nabla_{1,\mathfrak{u}}\nabla_{2,\mathfrak{v}} - \nabla_{2,\mathfrak{u}}\nabla_{1,\mathfrak{v}}\big)\mathcal{L}(x,y)$$

- other useful surface layer integral
(conserved in non-interacting case)

$$(.,.)^\Omega_\rho \;:\; \mathfrak{J} \times \mathfrak{J} \to \mathbb{R} \qquad \text{(surface layer inner product)}$$

$$(\mathfrak{u},\mathfrak{v})^\Omega_\rho = \int_\Omega d\rho(x) \int_{M\setminus\Omega} d\rho(y)\, \big(\nabla_{1,\mathfrak{u}}\nabla_{1,\mathfrak{v}} - \nabla_{2,\mathfrak{u}}\nabla_{2,\mathfrak{v}}\big)\mathcal{L}(x,y)$$

give rise to complex structure

Surface layer integral for wave functions

Unitary invariance of causal action principle,

$$\mathcal{U}_\tau := \exp(i\tau\mathcal{A}) \qquad \text{with} \quad \mathcal{A}\psi := \langle u|\psi\rangle_{\mathcal{H}}\, u$$

described infinitesimally by commutator jets

$$\mathfrak{C} := (0, \mathcal{C}) \qquad \text{with} \qquad \mathcal{C}(x) := i\big[\mathcal{A}, x\big]$$

$$\gamma^\Omega_\rho(\mathfrak{C}) = \langle u|u\rangle^\Omega_\rho \quad \text{extended to}$$

$$\langle .|.\rangle^\Omega_\rho \;:\; \mathcal{W}^\Omega_\rho \times \mathcal{W}^\Omega_\rho \to \mathbb{C} \qquad \text{(commutator inner product)}$$

$$\langle \psi|\phi\rangle^\Omega_\rho = -2i \bigg(\int_\Omega d\rho(x) \int_{M\setminus\Omega} d\rho(y) - \int_{M\setminus\Omega} d\rho(x) \int_\Omega d\rho(y) \bigg) \times \prec\!\psi(x) \,|\, Q^{\text{dyn}}(x,y)\, \psi(y)\!\succ_x \; .$$

- conserved if ψ, ϕ satisfy dynamical wave equation
- F.F., N. Kamran, M. Oppio, "The linear dynamics of wave functions in causal fermion systems," arXiv:2101.08673 [math-ph], *J. Differential Equations* **293** (2021) 115–187

Field Operators in the Vacuum

- Canonical commutation/anti-commutation relations for $z, z' \in \mathfrak{h}$ and $\psi, \psi' \in \mathcal{H}^{\mathrm{f}}_\rho \subset \mathcal{H}_\rho$

$$
\begin{aligned}
\big[a(\overline{z}), a^\dagger(z') \big] &= (z|z')^\Omega_\rho \\
\big[a(\overline{z}), a(\overline{z'}) \big] &= 0 = \big[a^\dagger(z), a^\dagger(z') \big] \\
\big\{ \Psi(\overline{\phi}), \Psi^\dagger(\phi') \big\} &= \langle \phi | \phi' \rangle^\Omega_\rho \\
\big\{ \Psi(\overline{\phi}), \Psi(\overline{\phi'}) \big\} &= 0 = \big\{ \Psi^\dagger(\phi), \Psi^\dagger(\phi') \big\}
\end{aligned}
$$

  - independent of time
  - generate unital $*$-algebra $\mathscr{A}$

Construction of the quantum state

- Quantum state ω^t at time t:

$$\begin{aligned} \omega^t : \mathscr{A} \to \mathbb{C} \quad & \text{linear and positive, i.e.} \\ \omega^t(A^*A) \geq 0 \quad & \text{for all } A \in \mathscr{A} \end{aligned}$$

- More concretely, represented on Fock space:
  - With a density operator:

$$\omega^t(A) = \mathrm{tr}_{\mathcal{F}}\big(\sigma^t\, A\big)$$

  - As an expectation value (pure state):

$$\omega^t(A) = \langle \Psi | A | \Psi \rangle_{\mathcal{F}}$$

- General structure:

$$\omega^t(\cdots) := \frac{1}{Z} \int_{\mathcal{G}} (\cdots)\, e^{\beta\, \gamma^{\tilde{\Omega},\Omega}\left(\tilde{\rho}, \mathcal{U}\rho\right)}\, d\mu_{\mathcal{G}}(\mathcal{U})$$

How do the insertions look like?

Definition of the quantum state

DEFINITION

The state ω^t at time t is defined by

$$\omega^t\Big(a^\dagger(z'_1)\cdots a^\dagger(z'_p)\,\Psi^\dagger(\phi'_1)\cdots\Psi^\dagger(\phi'_{r'}) \times a(\overline{z_1})\cdots a(\overline{z_q})\,\Psi(\overline{\phi_1})\cdots\Psi(\overline{\phi_r})\Big)$$

$$:= \frac{1}{Z(\beta,\tilde\rho)}\,\delta_{r'r}\,\frac{1}{p!}\sum_{\sigma,\sigma'\in S_r}(-1)^{\mathrm{sign}(\sigma)+\mathrm{sign}(\sigma')}$$

$$\times\int_{\mathcal{G}}\langle\tilde\phi_{\sigma(1)}\,|\,\pi^t_{\mathcal{U}}\,\tilde\phi'_{\sigma'(1)}\rangle^\Omega_\rho\cdots\langle\tilde\phi_{\sigma(r)}\,|\,\pi^t_{\mathcal{U}}\,\tilde\phi'_{\sigma'(r)}\rangle^\Omega_\rho$$

$$\times D_{\tilde z'_1}\gamma^{\tilde\Omega,\Omega}(\tilde\rho,\mathcal{U}\rho)\cdots D_{\tilde z'_p}\gamma^{\tilde\Omega,\Omega}(\tilde\rho,\mathcal{U}\rho)$$

$$\times D_{\overline{\tilde z}_1}\gamma^{\tilde\Omega,\Omega}(\tilde\rho,\mathcal{U}\rho)\cdots D_{\overline{\tilde z}_q}\gamma^{\tilde\Omega,\Omega}(\tilde\rho,\mathcal{U}\rho)\;e^{\beta\,\gamma^{\tilde\Omega,\Omega}(\tilde\rho,\mathcal{U}\rho)}\;d\mu_{\mathcal{G}}(\mathcal{U})$$

Positivity of the quantum state

THEOREM

The state ω^t is positive, i.e.

$$\omega^t\big(A^*A\big) \geq 0 \qquad \text{for all } t \in \mathbb{R} \text{ and } A \in \mathscr{A}$$

The proof makes use of

- Canonical commutation/anti-commutation relations
- Positivity of $(.|.)^\Omega_\rho$ and $\langle .|. \rangle^\Omega_\rho$
- Positivity of insertions:

$$D_{\tilde{z}} \gamma^{\tilde{\Omega},\Omega}(\tilde{\rho}, \mathfrak{U}\rho) \cdot D_{\overline{\tilde{z}}} \gamma^{\tilde{\Omega},\Omega}(\tilde{\rho}, \mathfrak{U}\rho) = \big| D_{\tilde{z}} \gamma^{\tilde{\Omega},\Omega}(\tilde{\rho}, \mathfrak{U}\rho) \big|^2 \geq 0$$

$$\langle \psi \,|\, \pi^t_{\mathfrak{U}}\, \psi \rangle^\Omega_\rho \geq 0 \quad \text{and} \quad \langle \psi \,|\, (\mathbb{1} - \pi^t_{\mathfrak{U}})\, \psi \rangle^\Omega_\rho \geq 0$$

- F.F. and Kamran, N., “Fermionic Fock spaces and quantum states for causal fermion systems,” arXiv:2101.10793 [math-ph], *Ann. Henri Poincaré* **23** (2022) 1359–1398

Outlook: Computation of the quantum state

- Main task: Compute the group integral

$$\int_{\mathcal{G}} (\cdots)\, e^{\beta\, \gamma^{\tilde{\Omega},\Omega}\left(\tilde{\rho},\,\mathcal{U}\rho\right)}\, d\mu_{\mathcal{G}}(\mathcal{U})$$

  - Consider asymptotics $\dim \mathcal{H} \to \infty$ and $\varepsilon \searrow 0$.
  - Explicit analysis possible (Gaussian integrals, saddle point methods).
- Main findings:
  - Refined localized state is also positive
  - Allows for the description of general entangled states
- Is ongoing work with N. Kamran and M. Reintjes

Outlook: Dynamics of the quantum state

- Construction so far gives ω^t for all t
- Next steps:
  - Construct time evolution for the density operator

$$\mathfrak{L}^t_{t_0} \;:\; \sigma^{t_0} \to \sigma^t$$

  - Is there a unitary time evolution on the Fock space?

$$\omega^t = U^t_{t_0}\, \omega^{t_0}\, \big(U^t_{t_0}\big)^{-1}$$

  - Answer: Yes. In this limiting case one gets QFT including loop diagrams, but with intrinsic regularization on scale ε.
  - There are nonlinear corrections. Connection to collapse phenomena?
- Is ongoing work with N. Kamran and M. Reintjes

www.causal-fermion-system.com

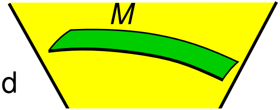
Thank you for your attention!

Representations of the quantum state

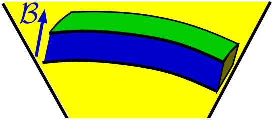
- GNS representation
  - Introduce scalar product on $\mathscr{A}$ by
  $$\langle A|A'\rangle := \omega^t(A^* A') \;:\; \mathscr{A} \times \mathscr{A} \to \mathbb{C}$$
  Forming the completion gives a Hilbert space.
  - $\mathscr{A}$ has a natural representation on this Hilbert space.
  - Setting $\Phi = \mathbb{1}$,
  $$\langle \Phi \,|\, A\, \Phi \rangle = \omega^t(\mathbb{1}^*\, A\, \mathbb{1}) = \omega^t(A)$$
  - always exists, but in general not a Fock representation
- Representation on the Fock space of vacuum
  - choose $\mathcal{F}$ as the Fock space generated by acting with $\mathscr{A}$ on vacuum state (Dirac sea vacuum)
  - construct density operator σ^t on $\mathcal{F}$ with
  $$\omega^t(A) = \mathrm{tr}_{\mathcal{F}}\big(\sigma^t\, A\big)$$
  - inductive construction for states involving *finite number of particles and anti-particles*
  - in general diverges (inequivalent Fock vacua, . . .)
  - makes connection to perturbative description

Outlook: a quantum spacetime

a classical space-time:
M diffeomorphic to manifold



a quantum space-time:
$M \simeq \mathscr{M} \times \mathcal{B}$



- microscopic mixing, holographic mixing
- integrating over additional "degrees of freedom" $\mathcal{B}$ resembles path integral

  … …

- F.F., "Perturbative Quantum Field Theory in the Framework of the Fermionic Projector"
  arXiv:1310.4121 [math-ph], J. Math. Phys. **55** (2014) 042301

Outlook: Connection to collapse models

General structure:

- Nonlinear dynamics of $\tilde{\rho}$ (from causal action principle)
- Conservation laws hold
  (current conservation, conserved symplectic form, ...)
- Causality holds in the sense
  "pairs of points with spacelike separation do not interact"
  in particular: no superluminal signalling
- In approximation ("approximation of inhomogeneous fluctuating fields") one gets linear and unitary time evolution

$$U_{t_0}^t \;:\; \mathcal{F} \to \mathcal{F}$$

Outlook: Connection to collapse models

As observed by J. Kleiner, this seems to indicate that causal fermion systems are an effective collapse theory.

A. Bassi, D. Dürr, G. Hinrichs, *"Uniqueness of the equation for quantum state vector collapse,"* Phys. Rev. Lett. **111**, 210401 (2013)

- No faster-than-light signalling
- Time evolution Markovian and homogeneous in time

$\Longrightarrow$ collapse theory

Can this be adapted to causal fermion systems?

Summary and Outlook

- No deterministic laws:
  not possible to proceed in time steps
- No strong causation
- But causal propagation on macroscopic scales
  - based on positivity properties of surface layer integrals (energy estimates, . . .)
- Causal fermion system approach is background-free
- correct limiting cases:
  - classical field theory: strong, electroweak forces and gravity
  - quantum field theory (work in progress)

What does causality mean?

- There are causal relations:
  - distinction spacelike, timelike
  - direction of time
- Locality holds:

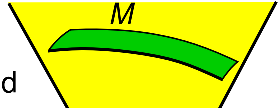
  Spacetime regions with spacelike separation
  have independent dynamics

BUT

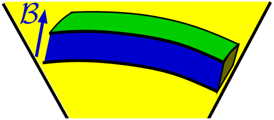
- relation “lies in the future of” not necessarily transitive
- no causation

Outlook: microscopic mixing

a classical space-time:
M diffeomorphic to manifold



a quantum space-time:
$M \simeq \mathscr{M} \times \mathcal{B}$



right now: effectively described by random matrices
microscopic mixing

F.F., “Perturbative Quantum Field Theory in the Framework of the Fermionic Projector”
arXiv:1310.4121 [math-ph], J. Math. Phys. **55** (2014) 042301

Outlook: Holographic Mixing

- $\Psi : \mathcal{H} \to C^0(M, SM)$ wave evaluation operator describing Minkowski vacuum,

$$(i\partial\!\!\!/ - m)\,\Psi = 0$$

- Decompose into holographic components:

$$\Psi_\alpha(x) := \Psi(x)\, B_\alpha \qquad \text{with} \qquad B_\alpha \in \mathrm{L}(\mathcal{H})$$

- Perturb each holographic component by electromagnetic potential A_α,

$$\Delta\Psi_\alpha = s_m\, A\!\!\!/_\alpha\, \Psi\, B_\alpha$$

- Gives rise to *microscopic fluctuations*
  - scaling behavior can be computed explicitly
- *Approximation of inhomogeneous fluctuating fields* gives bosonic loop diagrams

Quantum entanglement

- Holographic components can be decoherent
- Choosing different $\mathcal{U}$ makes different holographic components "visible"

$$\omega^t(\cdots) := \frac{1}{Z^t} \int_{\mathcal{G}} (\cdots)\, e^{\beta\, \gamma^t\left(\tilde{\rho}, \mathcal{U}\rho\right)}\, d\mu_{\mathcal{G}}(\mathcal{U})$$

- $\mathcal{U}$-dependence gives correlations between insertions
- This gives rise to entangled state.

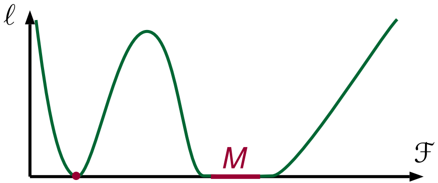
$$\ell(x) := \int_{\mathcal{F}} \mathcal{L}(x, y)\, d\rho(y)$$

(for clarity of presentation:
leave out Lagrange multipliers for constraints)

Lemma

Let ρ be a minimizer of the causal action. Then

$$\ell|_M \equiv \inf_{\mathcal{F}} \ell$$



Proof.

Consider variation $\tilde{\rho}_\tau = (1-\tau)\,\rho + \tau\,\delta_y$

(where δ_y is the Dirac measure supported at y).

$$S(\tilde{\rho}_\tau) = \iint_{\mathcal{F}\times\mathcal{F}} \mathcal{L}(x,y)\, d\tilde{\rho}_\tau(x)\, d\tilde{\rho}_\tau(y)$$

$$\begin{aligned} 0 \leq \frac{d}{d\tau}\mathcal{S}(\tilde{\rho}_\tau)\big|_{\tau=0} &= 2\int_{\mathcal{F}} d\dot{\tilde{\rho}}_\tau\big|_{\tau=0} \int_{\mathcal{F}} d\rho\, \mathcal{L}(x,y) \\ &= 2\left(\ell(y) - \int_M \ell(x)\, d\rho(x)\right) \end{aligned}$$

As a consequence,

$$\ell(y) \geq \int_M \ell(x)\, d\rho(x)\,. \qquad \square$$

Gauß-like theorem

For simplicity leave out scalar components of jets.

$$(\Delta u)(x) = \int_M \big(D_{1,u} + D_{2,u}\big) \mathcal{L}(x,y)\, d\rho(y)$$

$$0 = D_u \ell = \int_M D_{1,u} \mathcal{L}(x,y)\, d\rho(y) \qquad \text{(EL eqns)}$$

Hence

$$(\Delta u)(x) = -\int_M \big(D_{1,u} - D_{2,u}\big) \mathcal{L}(x,y)\, d\rho(y)$$

$$\int_\Omega (\Delta u)(x)\, d\rho(x) = -\int_\Omega d\rho(x) \int_M d\rho(y)\, \big(D_{1,u} - D_{2,u}\big) \mathcal{L}(x,y)$$

$$= -\int_\Omega d\rho(x) \int_{M \setminus \Omega} d\rho(y)\, \big(D_{1,u} - D_{2,u}\big) \mathcal{L}(x,y)$$

(volume integral) = (surface layer integral)

Bosonic Fields in the Vacuum

- give rise to complex structure:

$$\sigma(u, v) = (u, \mathcal{T} v)$$

$$J := -(-\mathcal{T}^2)^{-\frac{1}{2}} \mathcal{T}\,, \qquad J^* = -J,\; J^2 = -\mathbb{1}$$

Complexify and decompose:

$$\mathbf{v} = \mathbf{v}^{\mathrm{hol}} + \mathbf{v}^{\mathrm{ah}}$$

On holomorphic jets introduce scalar product

$$(.|.)^t_\rho := \sigma^t_\rho(\,.\,, J\,.\,) \;:\; \Gamma^{\mathrm{hol}}_\rho \times \Gamma^{\mathrm{hol}}_\rho \to \mathbb{C}$$

Completion gives complex Hilbert space $(\mathfrak{h}, (.|.)^t_\rho)$.

- Cauchy problem: Existence and uniqueness proven.

F.F. and N. Kamran, *"Complex Structures on Jet Spaces and Bosonic Fock Space Dynamics for Causal Variational Principles,"* arXiv:1808.03177 [math-ph], to appear in Pure Appl. Math. Q. (2021)

C. Dappiaggi and F.F., *"Linearized Fields for Causal Variational Principles: Existence Theory and Causal Structure,"* arXiv:1811.10587 [math-ph], Methods Appl. Anal. **27** 1–56 (2020)

Fermionic Fields in the Vacuum

- dynamical wave equation:

$$\int_M Q^{\mathrm{dyn}}(x,y)\,\psi(y) = 0$$

- scalar product defined as surface layer integral:

$$\langle\psi|\phi\rangle_\rho^t = -2i\left(\int_{\Omega^t} d\rho(x)\int_{M\setminus\Omega^t} d\rho(y) - \int_{M\setminus\Omega^t} d\rho(x)\int_{\Omega^t} d\rho(y)\right) \times \prec\psi(x)\,|\,Q^{\mathrm{dyn}}(x,y)\,\phi(y)\succ_x$$

  is conserved in time,
  gives *extended Hilbert space* $\mathcal{H}_\rho \supset \mathcal{H}$.

- Cauchy problem: Existence and uniqueness proven.

  F.F., N. Kamran and M. Oppio, *"The Linear Dynamics of Wave Functions in Causal Fermion Systems,"* arXiv:2101.08673 [math-ph]

Bosonic insertions

- physical picture:

"Measurement" in $\tilde{M}$ with objects in M,
using the identification given by $\mathcal{U}$

- associate z to a linearized field $\tilde{z}$ in $\tilde{M}$:

$$\mathcal{P}_\rho : U \subset \mathfrak{J}^{\text{lin}}_\rho \to \mathcal{B} \qquad \text{perturbation map}$$

$$D\mathcal{P}_\rho|_w : \mathfrak{J}^{\text{lin}}_{\rho,} \to \mathfrak{J}^{\text{lin}}_{\tilde{\rho},}$$

$$\tilde{z} := D\mathcal{P}_\rho|_w \, z \,, \qquad \overline{\tilde{z}} := D\mathcal{P}_\rho|_w \, \overline{z}$$

- perturb nonlinear surface layer integral:

$$D_{\tilde{z}} \gamma^t(\tilde{\rho}, \mathcal{U}\rho) \,, \qquad D_{\overline{\tilde{z}}} \gamma^t(\tilde{\rho}, \mathcal{U}\rho)$$

Fermionic insertions

- Work with scalar product $\langle .|.\rangle^t_\rho$ in vacuum.
- Map wave functions from $\tilde{M}$ to M:

$$\psi = \pi_{\rho,\tilde{\rho}}\,\tilde{\psi}\,, \quad \psi(x) := \frac{1}{\tilde{\mathfrak{t}}(x)} \int_{\tilde{M}} \pi_x\, \mathfrak{U}^{-1}\, \tilde{\psi}(y)\, |xy|^2\, d\tilde{\rho}(y)$$

$$\tilde{\mathfrak{t}}(x) := \int_{\tilde{M}} |xy|^2\, d\tilde{\rho}(y)$$

- Gives subspace $\pi^t_{\rho,\tilde{\rho}}\,\mathcal{H} \subset \mathcal{H}_\rho$,

$$\pi^t_{\mathfrak{U}} : \mathcal{H}^\rho \to \pi^t_{\rho,\tilde{\rho}}\,\mathcal{H} \quad \text{orthogonal projection}$$

- one-particle measurement: $\langle \psi \,|\, \pi^t_{\mathfrak{U}}\, \phi\rangle^t_\rho$
- multi-particle measurement:

$$\frac{1}{p!} \sum_{\sigma,\sigma' \in S_p} (-1)^{\mathrm{sign}(\sigma)+\mathrm{sign}(\sigma')}$$
$$\times \langle \tilde{f}_{\sigma(1)} \,|\, \pi^t_{\mathfrak{U}}\, \tilde{f}_{\sigma'(1)}\rangle^t_\rho \cdots \langle \tilde{f}_{\sigma(p)} \,|\, \pi^t_{\mathfrak{U}}\, \tilde{f}_{\sigma'(p)}\rangle^t_\rho$$

Pauli exclusion principle arises

- Can the state be written as follows?

$$\omega^t(\cdots) = \frac{1}{\beta^k\, Z^t(\beta, \tilde{\rho})} \underbrace{D \cdots D}_{k \text{ derivatives}} Z^t(\beta, \tilde{\rho})$$

Short answer: Yes, up to rather subtle technical issues.

$$Z^t(\beta, \tilde{\rho}) = \int_{\mathcal{G}} \exp\Big(\beta\, \gamma^t(\tilde{\rho}, \mathcal{U}\rho)\Big)\, d\mu_{\mathcal{G}}(\mathcal{U})$$