Outer entropy = Bartnik-Bray quasilocal mass

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The Bekenstein-Hawking entropy of a black hole

$$S_{\mathrm{BH}} = \frac{A[\Sigma]}{4G_N\hbar}$$

- Bekenstein: a black hole must have entropy to be consistent with the second law of thermodynamics.

- Bekenstein-Hawking: it is proportional to the area of the event horizon.

- However, what kind of entropy is it?

- Strominger-Vafa: $S_{\mathrm{BH}} = k \log W$

- Ryu-Takayanagi: $S_{\mathrm{BH}} = -\operatorname{Tr} \rho \log \rho$

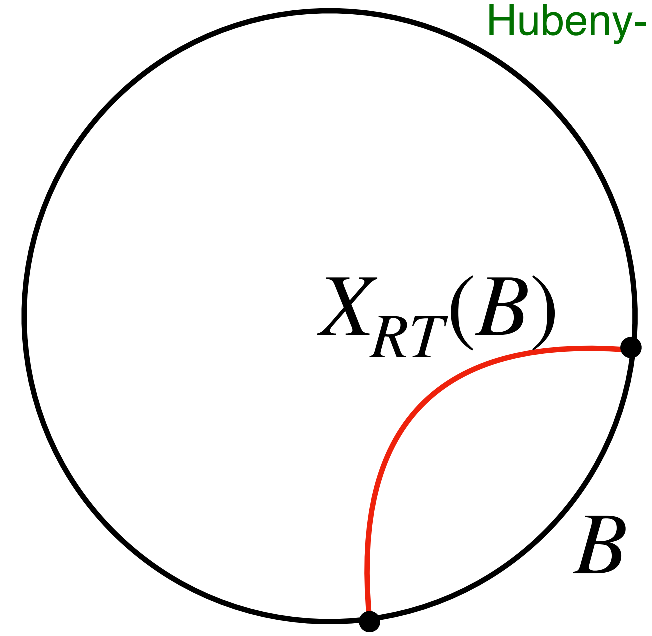
- However, the second law (Hawking's area theorem) suggests that it should it be understood as a coarse-grained entropy.

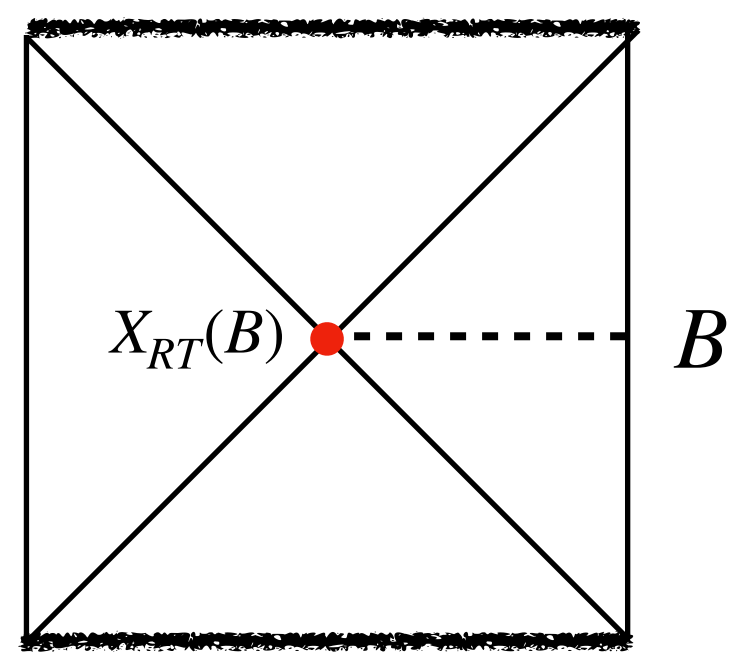
Entanglement Entropy in AdS/CFT

Ryu-Takayanagi, 2006

Hubeny-Rangamani-Takayanagi, 2007

- The von Neumann entropy of a quantum state supported on a boundary subregion is computed **geometrically** using the **Ryu-Takayanagi (RT)** surface X_{RT}, that is the codimension-two extremal surface homologous to the boundary.
- $S(B) = \mathrm{Area}[\mathrm{X_{RT}(B)}]/4G$
- In the Riemannian setting, RT looks for the **minimal surface**; and in the spacetime setting, RT looks for the **extremal surface** (with mean curvatures) with minimal area.
- For a two-sided black hole, the RT surface is the bifurcation surface (wormhole throat) that measures the entanglement entropy of a thermofield double state. It coincides with the BH entropy.
- However, For a one-sided black hole formed from collapse, the RT surface is empty indicating a zero entropy for a pure black hole state. How should one make sense of the BH entropy then?

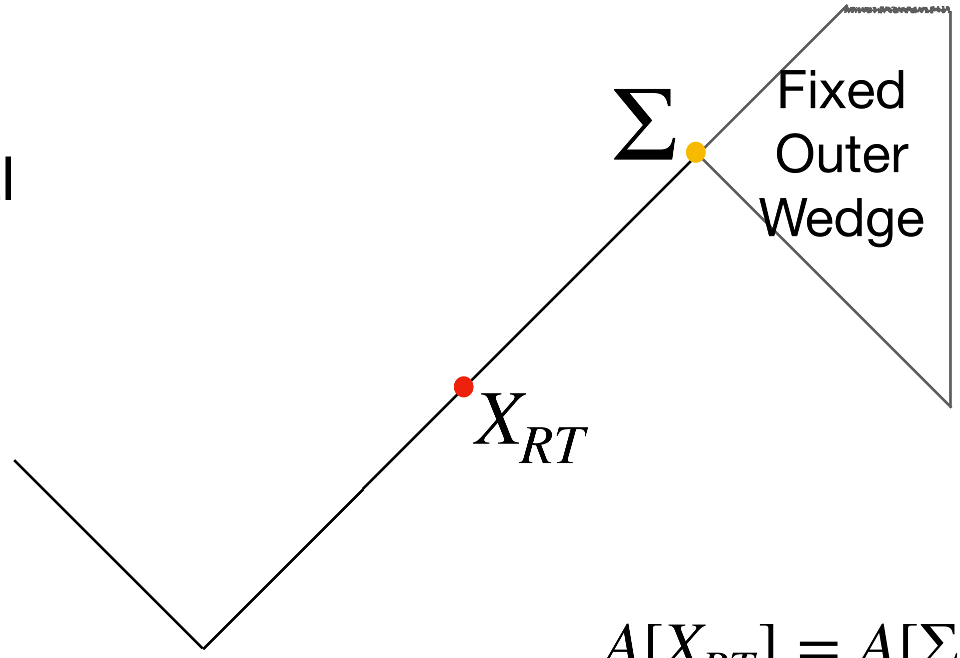




The outer entropy

Engelhardt, Wall, 2018
Nomura, Remmen, 2018
Bousso, Nomura, Remmen, 2019

- Let us assume the validity of the RT formula as a general prescription for the fine-grained gravitational entropy even in asym. flat. spacetimes. (e.g. path integral derivation, recent advances in island formulas.)



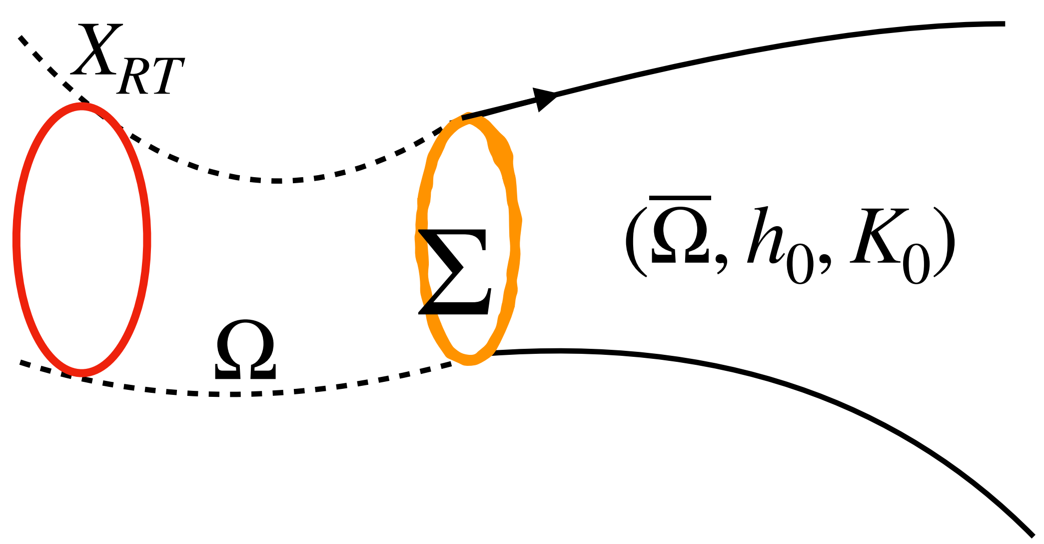
$A[X_{RT}] = A[\Sigma]$

- The outer entropy is introduced by Engelhardt & Wall (**EW**) as a **coarse-grained** Black Hole entropy associated with an apparent horizon Σ.

$$\mathcal{S}(\Sigma) := \sup_{\rho} S_{\mathrm{vN}}(\rho) : D(\overline{\Omega}) \text{ fixed}$$

- EW shows that for an apparent horizon Σ, the maximiser always exists.

$$\mathcal{S}(\Sigma) = \mathrm{Area}[\Sigma]/4G\hbar.$$



- Statistical interpretation of the BH entropy and area law. Built-in area laws associated with trapping horizons.

Outer entropy

The outer entropy of the outer wedge data $(\overline{\Omega}, h_0, K_0)$ *bounded by* $\Sigma = \partial\overline{\Omega}$ *with the asymptotic end* B *is*

Wall, 2013

$$\boxed{\mathcal{S}(\overline{\Omega}, h_0, K_0) := \sup_{(\Omega,h,K)} \frac{A[X_{RT}(B)]}{4G_N\hbar}} = \sup_{(\Omega,h,K)} \max_{N\subset D(\Omega\cup_\Sigma\overline{\Omega})} \min_{\substack{\sigma\subset N\\ \sigma\in[B]}} \frac{A[\sigma]}{4G_N\hbar}$$

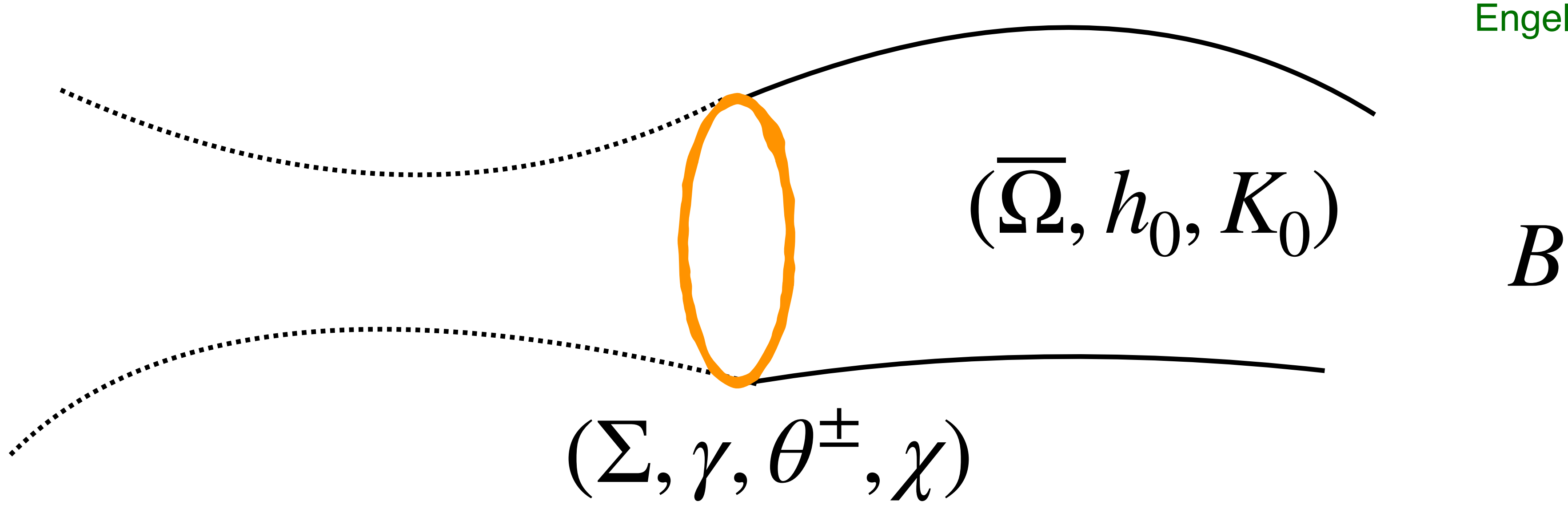
where (Ω, h, K) *is the fill-in data that joins the fixed* $(\overline{\Omega}, h_0, K_0)$ *at* Σ *satisfying DEC and the following constraints:*

$$\gamma|_{\Sigma_{in}} = \gamma|_{\Sigma_{out}}; \;\; \theta^\pm|_{\Sigma_{in}} = \theta^\pm|_{\Sigma_{out}}; \;\; \chi|_{\Sigma_{in}} = \chi|_{\Sigma_{out}}$$

gluing conditions

where $\chi := K(\cdot, \ell^-)$ *is the twist or anholonomicity 1-form and* ℓ^- *is the ingoing null vector normal to* Σ.

Engelhardt, Wall, 2018



Bartnik mass

- The Bartnik (outer) mass $M_B(\Sigma)$ is defined as the **infimum** ADM mass over all horizon-free extensions of the given surface Σ.

$$M_{\text{Bartnik}}(\Sigma) := \inf_{(\overline{\Omega},h,K)} M(\overline{\Omega}, h, K)\,.$$

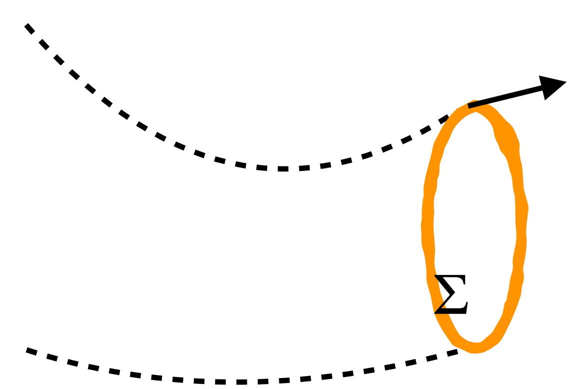
Bartnik, 1989

- Bray proposed a **dual/inner** version of the Bartnik mass in his seminal paper proving the Riemannian Penrose Inequality (RPI) :

Bray, 2001

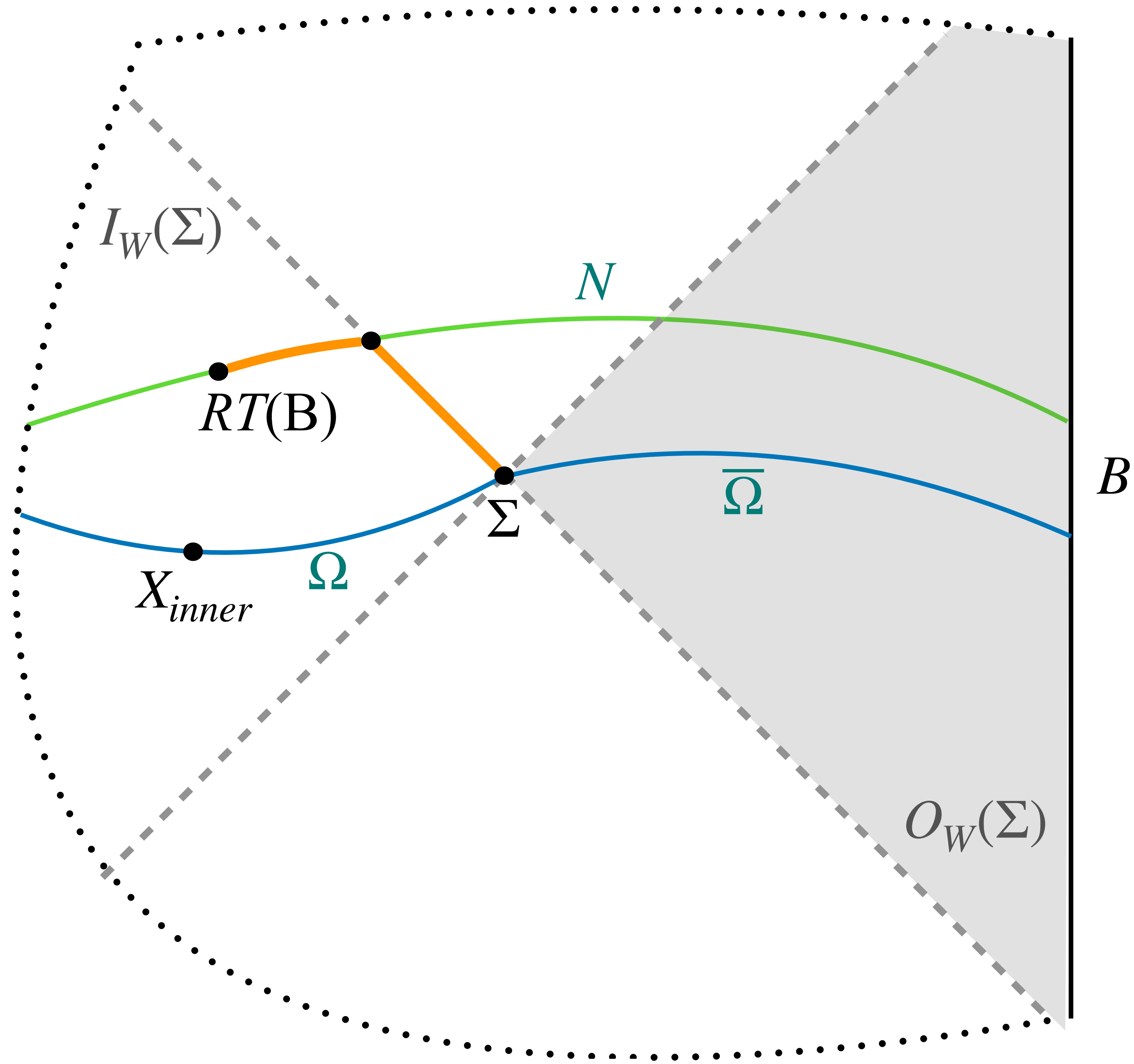
$M(N,h,K) \geq M_{\text{irr}}(A[\Sigma])$, where $M_{\text{irr}}(A) = \frac{1}{2}\left(\frac{A}{\Omega_{n-2}}\right)^{\frac{n-3}{n-2}}$ in AF, or $\frac{1}{2}\left(\frac{A}{\Omega_{n-2}}\right)^{\frac{n-3}{n-2}} + \frac{1}{2}\left(\frac{A}{\Omega_{n-2}}\right)^{\frac{n-1}{n-2}}$ in AH is the **irreducible mass**.

$$M_{\text{inner}}(\Sigma) := \sup_{(\Omega,h,K)} \min_{\substack{\sigma \subset \Omega, \\ \sigma \in [\Sigma]}} M_{\text{irr}}(A[\sigma])\,.$$



Outer entropy = inner mass ?

- The outer entropy is a functional of the entire outer wedge data, so the RT surface is not necessarily inside the fill-in wedge.



$$M_{\text{inner}}(\Sigma) := \sup_{(\Omega,h,K)} \min_{\substack{\sigma\subset\Omega,\\ \sigma\in[\Sigma]}} M_{\text{irr}}(A[\sigma])$$

$= ?$

$$\mathcal{S}(\overline{\Omega},h_0,K_0) := \sup_{(\Omega,h,K)} \frac{A[\ RT(B)]}{4G_N\hbar} = \sup_{(\Omega,h,K)} \max_{N\subset D(\Omega\cup_\Sigma\overline{\Omega})} \min_{\substack{\sigma\subset N\\ \sigma\in[B]}} \frac{A[\sigma]}{4G_N\hbar}$$

Outer-minimising mean-convex surfaces

- Σ is **outer-minimising** means that for any Σ' enclosing Σ, $A[\Sigma] \leq A[\Sigma']$.
- Σ is **mean-convex** (normal) means that $\pm\theta^{\pm} \geq 0$.
- Both **(1)** outer-minimisation and **(2)** mean-convexity are "necessary". We need them to "**quasilocalise**" the outer entropy
- Bartnik: (1) is used to avoid "bag of gold"-like extensions trivialising the Bartnik mass.

  EW: (1), as part of their "minimar" condition, is used to ensure the HRT surface can be found following their procedure.
- (2) and its Riemannian version is common. e.g. Bartnik mass, Weyl problem, positivity of Brown-York mass, Liu-Yau mass, etc.

Equivalence

Our main result is that the outer entropy $\mathcal{S}(\Sigma)$ is **equivalent** to the Bartnik-Bray inner mass $M_{\text{inner}}(\Sigma)$ assuming the DEC:

$$M_{\text{inner}}(\Sigma) = M_{\text{irr}}(4\hbar G_N \mathcal{S}(\Sigma))$$

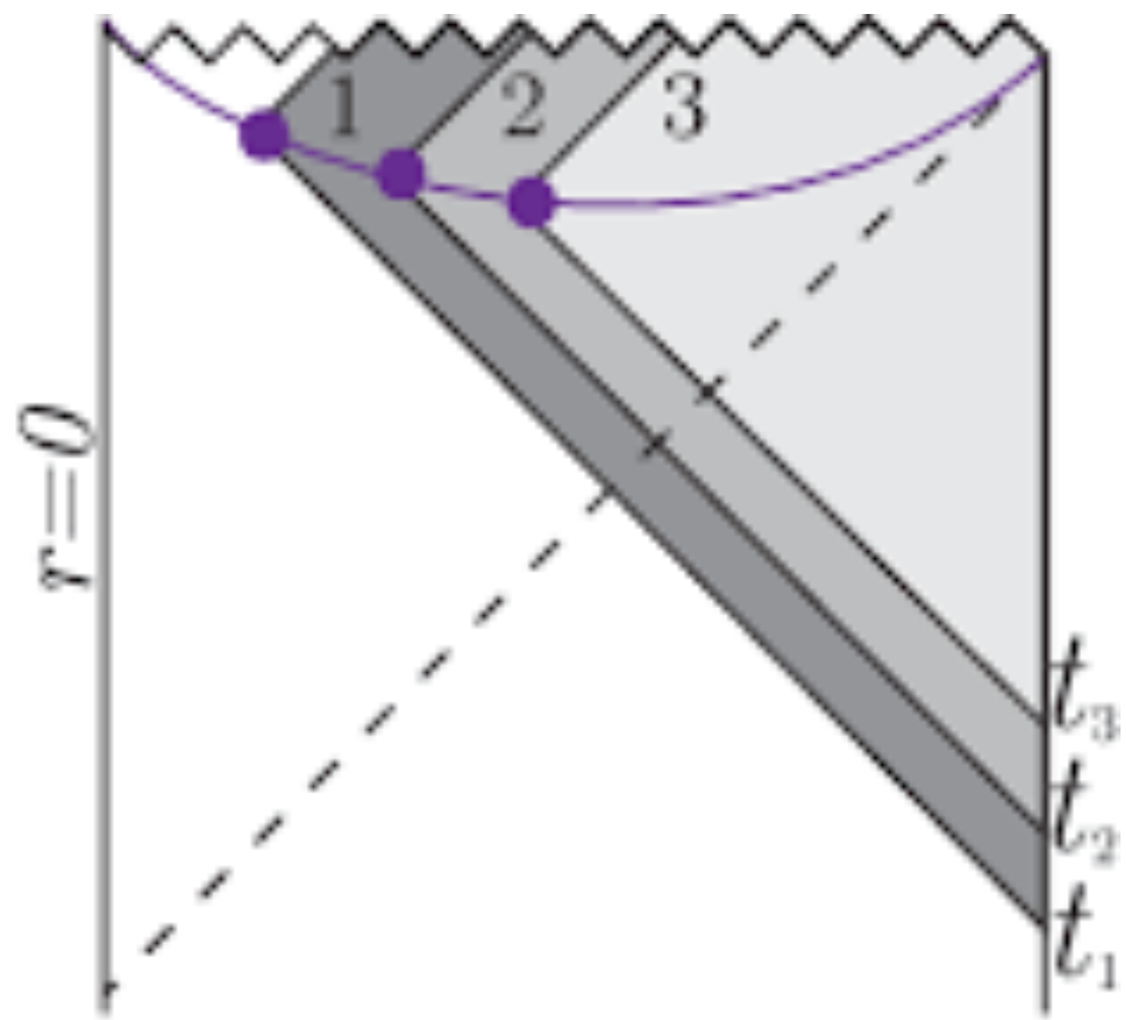
for an **outer-minimising, mean-convex** Σ.

*Though motivated in AdS, this geometric statement also makes sense in flat space.

The proof uses standard focusing arguments.
Check out the details in the paper arXiv: 2007.00030

Implications: area laws

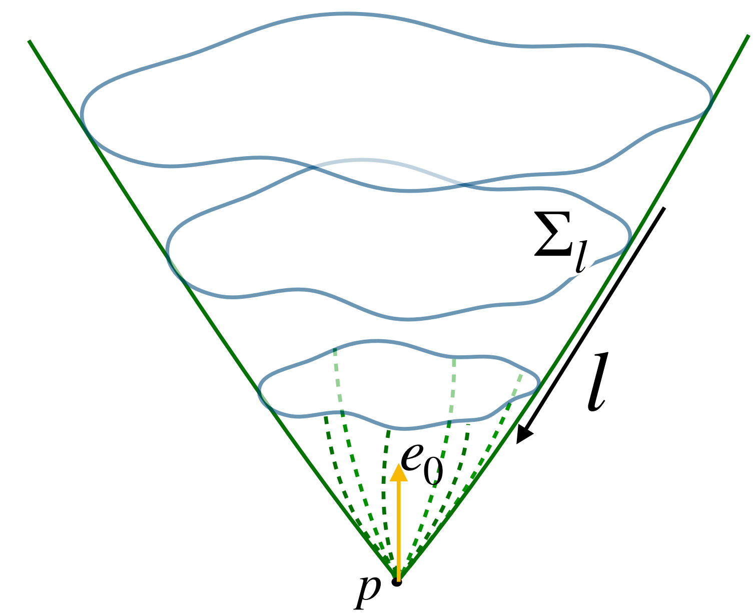
The area laws of trapping/dynamical horizons = monotonicity of the quasilocal mass. Both are the consequences of the variational definitions.



Engelhardt, Wall, 2018

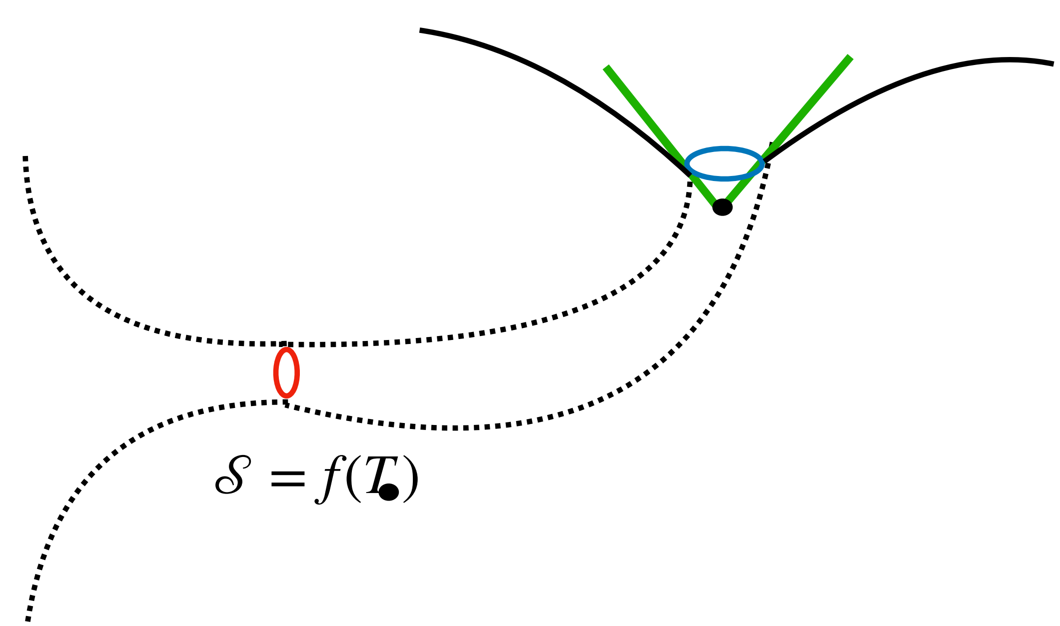
Implications: local mass density

In the **small sphere limit**, any quasilocal mass should reduce to the **stress tensor**, so should the outer entropy. Using an algorithm to construct the fill-in developed by EWBNR, which is not provably optimal, we obtain



$$\mathcal{S}(\Sigma_l) = \frac{\Omega_{n-2} l^{n-2}}{4 G_N \hbar} \left(\frac{2 l^2 \Omega_{n-2} G_N T(e_0, e_0)\,|_p}{n-1} \right)^{\frac{n-2}{n-3}}$$

$$\lim_{l \to 0} l^{-(n-1)} M_{inner}(\Sigma_l) = \frac{\Omega_{n-2}}{n-1} T(e_0, e_0)\,|_p$$



A quasilocal Penrose inequality

- The Penrose Inequality implies

$$\inf_{(\overline{\Omega},h,K)} M(\overline{\Omega},h,K) \geq M_{\text{irr}}(4\hbar G_N \mathcal{S}(\Sigma)).$$

- It's a quasilocal version of the Penrose inequality. It's more interesting to ask when does the **equality** hold? i.e. Bartnik (outer) mass = inner mass.

$$\inf_{(\overline{\Omega},h,K)} M(\overline{\Omega},h,K) = M_{\text{irr}}(4\hbar G_N \mathcal{S}(\Sigma)).$$

- This is a **quasilocal** statement that concerns the entropy and the energy of a closed surface. In words, it says the Bartnik mass of some closed surface is given by the irreducible mass of the largest black hole that can be fit behind it.

- The equality trivially holds if the Bartnik data can be isometrically embedded into Schwarzschild. However, generally the equality needs not to be realised on a particular spacetime. (Note that we have inf/sup in LHS/RHS.)

When does the equality hold?

$$\inf_{(\overline{\Omega},h,K)} M(\overline{\Omega}, h, K) = M_{\text{irr}}(4\hbar G_N \mathcal{S}(\Sigma)) \quad ?$$

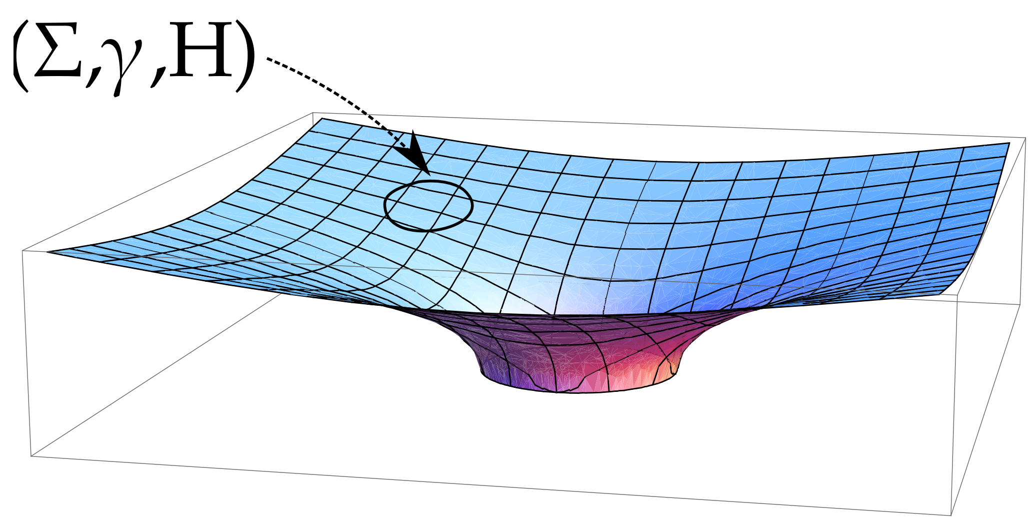
- Extension constructions by Mantoulidis-Schoen for Σ being an apparent horizon (MOTS) give non-trivial examples to the equality.

Mantoulidis-Schoen, 15
Cabrera Pacheco-Cederbaum-McCormick, 18

- Evidence from the **matching** of small sphere limits for any small balls.

Wiygul, 16

- On the other hand, we know that the equality certainly doesn't hold in general. For example, an off-centred ball in Schwarzschild has a **positive** Bartnik mass but **zero inner mass** as it admits **no valid fill-in**.



Jauregui, 11

- Question: **under what conditions** does the equality hold?

Conclusion

- We've shown that the Bartnik-Bray quasilocal mass is equivalent to the Outer Entropy.
- Can we find a procedure to construct the optimal fill-in that computes the inner mass/outer entropy for more general surfaces?
- Under what conditions does the following equality hold?

$$\inf_{(\overline{\Omega},h,K)} M(\overline{\Omega}, h, K) = M_{\mathrm{irr}}(4\hbar G_N \mathcal{S}(\Sigma)).$$

Thank You !