Coordinates are messy

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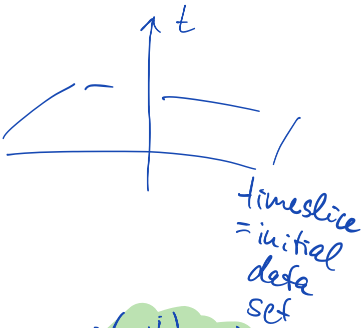
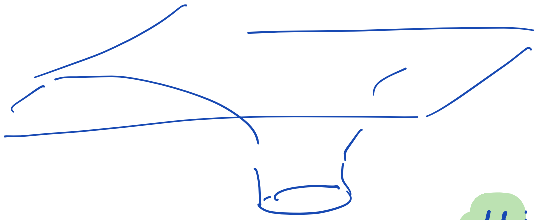
Setup: initial data sets

$$(\underbrace{M^3, g}_{\text{Riem.}}, \underset{(0,2)}{K})$$



asymptotically Euclidean (AE) if

$M^3 \setminus$ compact set $\approx$ $\mathbb{R}^3 \setminus \overline{\text{ball}}$ $\rightsquigarrow (x^i) = \vec{x}$



$$g_{ij} = \delta_{ij} + O_2(|\vec{x}|^{-\frac{1}{2}-\varepsilon})$$

$$K_{ij} = O_1(|\vec{x}|^{-\frac{3}{2}-\varepsilon})$$

$\varepsilon > 0$ as $|\vec{x}| \to \infty$

matter/energy density $\rightarrow$ $\mu := R_g - |K|_g^2 + (\operatorname{tr}_g K)^2 \in L^1$ $= O(|\vec{x}|^{-3-2\varepsilon})$

momentum density $J := \operatorname{div}_g (K - (\operatorname{tr}_g K) g) \in L^1$ $= O(|\vec{x}|^{-3-2\varepsilon})$

Energy $\underset{\substack{\uparrow \\ g_{ij,k}}}{E_{ADM}}$, linear momentum $\underset{\substack{\uparrow \\ K_{ij}}}{\vec{P}_{ADM}}$, mass m_{ADM}

$$m_{ADM} = \sqrt{E_{ADM}^2 - |\vec{P}|_\delta^2}$$

* Why $\frac{1}{2} + \varepsilon$?

Denisov – Solovyev '83

$g_{ij} = \delta_{ij} : \vec{x}$, $K = 0$

$\vec{y} := \vec{x} + \frac{c}{\sqrt{|\vec{x}|}} \vec{x}$

$E_{ADM}(g, \vec{y}) \sim c$

not desirable.

* Why $\mu \in L^1$:

analogy to Newtonian gravity:

$$\rho \in L^1$$

$$m := \int_{\mathbb{R}^3} \rho \, dV$$

$\Delta_g x^i = 0$

in the proof by Bartnik '86: E_{ADM} is well-def

$$E_{ADM} = \text{surf. integral over } S^2_\infty(0)$$

$$\overset{\text{div. thm}}{\approx} \text{volume integral} + \underbrace{\text{bdry at } \partial\Omega}_{\text{are finite}}$$

$$\wr$$

$$\underbrace{\mu}_{R_g} + \underbrace{\text{l.o.t.}}$$

$\Delta_g x^i = 0$

* AE - conditions are Poincaré covariant

$$\vec{x} \rightsquigarrow \underline{\vec{y} = Q\vec{x} + \vec{a}}$$

* E_{ADM} is invariant under changes of AE coordinates on some (M^3, g, K). (Bartnik)

Some possible coord. transfo's that preserve AE-conditions:

(M^3, g, K), $\vec{x}$ AE

$$\vec{x} =: \vec{y} + \sin(\ln|\vec{y}|)\vec{a} \qquad \text{for } \vec{a} \in \mathbb{R}^3$$

or alternatively

$$\vec{x} =: \vec{y} + f(|\vec{y}|)\vec{a}$$

$$\text{if } f = 1 + O_3(|\vec{y}|^{-\alpha}) \qquad \alpha > 0$$

"asymptotic translation"

AHISTORICAL :

$$\Delta U = 4\pi \rho$$

$$U \to 0 \qquad \text{as} \qquad |\vec{x}| \to \infty$$

$$m_{total} = \int_{\mathbb{R}^3} \rho \, dV = \frac{1}{4\pi} \int_{\mathbb{R}^3} \Delta_\delta U \, dV_\delta$$

div. thm

$$= \frac{1}{4\pi} \int_{S^2_\infty(\vec{0})} \nu(U) \, dA_\delta$$

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if $\vec{x}$ Cartesian

$$\vec{x} =: \vec{y} + \sin(\ln|\vec{y}|)\vec{a} \qquad \vec{a} \neq \vec{0}$$

$\longrightarrow$ $\delta^{\vec{y}}_{ij}$ is AE for all $\varepsilon > 0$

$$\boxed{m^{\vec{y}}_{total} = m^{\vec{x}}_{total}}$$

The mess begins when looking at the center of mass (CoM):

Newtonian picture:

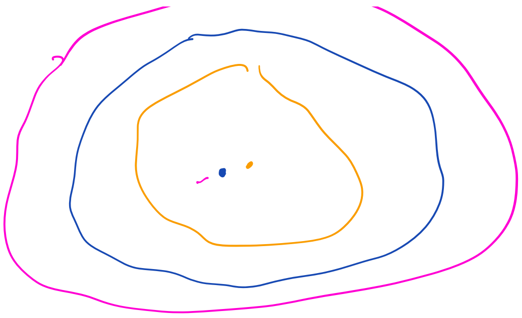
$$\vec{c}^{\,\vec{x}}_{total} = \frac{1}{m_{total}} \int_{\mathbb{R}^3} \underbrace{\rho \vec{x}}_{\in L^1} dV_\delta \qquad \vec{x} \text{ Cartesian}$$

↑ transforms equivariantly under Euclidean motions

but $\vec{x} =: \vec{y} + \sin(\ln|\vec{y}|)\vec{a}$, $\vec{a} \neq \vec{0}$

$\vec{c}^{\,\vec{y}}_{total}$ diverges ! if $U = -\frac{m}{r}$ (or more complicated)

in the surface integral version.



level sets of U

$$\vec{C}^{\vec{x}}_{total} \sim \lim_{\alpha\to 0} \int_{\{U=\alpha\}} \vec{x}\, dA_\delta$$

What do we learn for GR?

↳ should expect $\mu\vec{x} \in L^1$

$\mu = O(|\vec{x}|^{-4-2\varepsilon})$

↳ expect $\vec{C}_{total}$ to diverge w.r.t. $\vec{y}$

$\vec{x} =: \vec{y} + \sin(\ln|\vec{y}|)\vec{a}$, $\vec{a} \neq \vec{0}$

if it converges for $\vec{x}$

several definitions of $\vec{C}_{total}$ in GR

no reference to K

Regge – Teitelboim '74

Beig – Ó Murchadha '87

analogous ADM

$\int_{S^2_\infty(\vec{0})} \cdots g_{ij,k}, g_{ij}$

$\vec{C}_{BÓRT}$

this doesn't always converge

Huisken – Yau '96

via CMC foliations

Under suitable asympt. conditions:

if $E_{ADM} > 0$

then $(\mathbb{R}^3, g, K)$ has a (unique) foliation near ∞ by constant mean curvature surfaces (CMC) Σ_H

and $\lim_{H\searrow 0} \int_{\Sigma_H} \vec{x}\, dA_\delta$

$\overset{!!}{\cdot\cdot}$

$\vec{C}_{CMC} = \vec{C}_{HY}$

Metzger, Huang, Nerz

→ AE

C. – Nerz '15
lim does NOT always exist, even under the original decay assumption

How are $\vec{C}_{BÓRT}$ $\vec{C}_{HY}$ related

Nerz '15 Huang " = " provided weak Regge – Teitelboim (RT) conditions

both converge provided strong RT conditions hold

Regge – Teitelboim conditions

$\varepsilon > 0$, $\eta = \begin{cases} \frac{1}{2} & \text{weak} \\ 1 & \text{strong} \end{cases}$

$(g_{ij})^{odd} = O_2(|\vec{x}|^{-\frac{1}{2}-\varepsilon-\eta})$

$(K_{ij})^{even} = O_1(|\vec{x}|^{-\frac{3}{2}-\varepsilon-\eta})$

$\mu^{odd}, (J_i)^{odd} = O(|\vec{x}|^{-3-2\varepsilon-\eta})$

parity / reflections at $\vec{0}$

Thm C. – Graf – Metzger (ongoing)

1) weak / strong RT conditions are NOT translation invariant (loss of derivatives)

2) there exist initial data sets with no $\vec{x}$ satisfying weak RT conditions (even in vacuum)

Indeed $\vec{C}_{HY}$ diverges w.r.t. $\vec{y}$ $\quad \underset{\nearrow}{\vec{x}} = \vec{y} + \sin(\ln|\vec{y}|)\vec{a}$, $\vec{a} \neq \vec{0}$

converges

From special relativity, expect that K should play a role in the def^n of CoM

evidence: $\mu \vec{x} \in L^1$

$$\mu = R_g \underbrace{- |K|_g^2 + (\mathrm{tr}_g K)^2}_{\text{do play a role}}$$

Thm C. – Sakovich '21

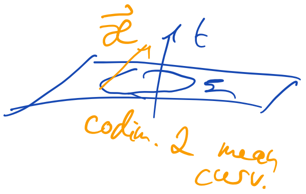
If (M^3, g, K) is AE with $E_{ADM} \neq 0$ then there is a (unique) foliation by "spacetime constant mean curvature" surfaces $\Sigma_{\mathcal{H}}$ near ∞ of M.

$$\underset{ST}{}\quad \underset{CMC}{}$$

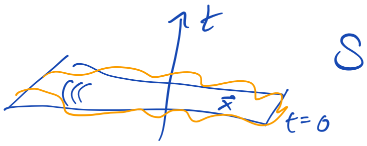
$$\rightsquigarrow \lim_{\mathcal{H}\to 0} \oint_{\Sigma_{\mathcal{H}}} \vec{x}\, dA_g =: \vec{C}_{STCMC} \quad \text{if it converges}$$

$$\mathcal{H} = \text{const} = \sqrt{H^2 - (\mathrm{tr}_\Sigma K)^2} = \sqrt{|\vec{\mathcal{H}}|^2}$$

ST mean curv.



- $K = 0$ $\triangleq$ Huisken – Yau
- if $K \to 0$ very fast: $\vec{C}_{STCMC} = \vec{C}_{HY}$ if they converge or both diverge
- C. – Nerz '15 – example converges to $\vec{C}_{STCMC} = \vec{0}$



Schwarzschild

$$t = \sin(\ln|\vec{x}|) + \frac{\vec{u}\cdot\vec{x}}{|\vec{x}|} \qquad \vec{u} \neq \vec{0}$$

vacuum $\mu = 0$, $J = 0$

AE, even asympt. to Schwarzschild

$\vec{C}_{HY}$ diverges $\sim \vec{u}$

$\vec{C}_{STCMC} = \vec{0}$

- * STCMC approach under Poincaré group
- * surf. integral formular similar to $\vec{C}_{BÓRT}$ (under additional conditions)

+weak RT

$$\vec{C}_{STCMC} = \underset{\vec{C}_{HY}}{\overset{\shortparallel}{\vec{C}_{BÓRT}}} + \underbrace{\text{correction}}_{\text{— } Z \text{ —}}$$

involves K

Conjecture: $\exists$ "natural" atlas, closed under Eucl. motions s.t. $\vec{C}_{STCMC}$ converges for all charts in the atlas for (M^3, g, K) AE with $E_{ADM} \neq 0$, $\mu \vec{x} \in L^1$